Hydrodynamic behavior of hyperbolic two-component systems

Outline of PhD thesis

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In the subsequent sections we give an overview of the results of the PhD thesis "Hydrodynamic behavior of hyperbolic two-component systems". We start with a short introduction of the considered topics and the main motivation of the thesis. Then we describe the investigated family of models and the results.

1 Introduction

The main problem of non-equilibrium statistical physics is the study of the dynamics of interacting particle systems, their behavior in space and time. We can think, for example about gas molecules in a room, or particles of a flowing fluid. Generally, the size of these systems is enormous (of order 10^{26}), thus the task of tracking every single particle is hopeless, even if we know everything about the microscopic dynamics. There is another, much more effective approach to this problem: we have to look at the 'big picture', i.e. the macroscopic evolution. This means that we characterize the state of our system with the local densities of certain physically relevant conserved quantities (particle number, momentum, energy) and the time-evolution of these functions, which are usually driven by coupled partial differential equations, gives us the needed description.

Hydrodynamic limit (hdl) is the device to get these pde-systems from the microscopic dynamics via some rescaling of space and time. In the physics literature there are a number of well-known phenomenological derivations of the hydrodynamic limit for several systems, starting with the classical work of Euler, Navier, Stokes, etc. See e.g. [11], [6]. It is a challenging and important program of mathematical physics to give mathematically rigorous versions of these derivations. For completely deterministic systems (e.g. which are governed by Newtonian mechanics) this is still an unsolved problem. However, if we add some stochastic elements to the evolution, the problem becomes more treatable (but far from trivial!). In the last couple of decades considerable advances have been made in the theory of hydrodynamic limits for stochastic systems (see the monographs [12, 5, 2]). Much effort has been made in the analysis of lattice gas models with conserved quantities (e.g. simple exclusion, zero range, Ginzburg-Landau models). These can be viewed as an approximation for the deterministic systems, but they also turn up as models for numerous phenomena in biology, chemistry and physics (e.g. deposition and growth models, biological chemotaxis).

2 Motivation

In the thesis we prove results for the hydrodynamic behavior of certain one dimensional lattice models motivated by a conjecture of B. Tóth and W. Werner. This section serves to describe this conjecture and to state the main objective of the thesis.

In [13] B. Tóth proved limit theorems about the so-called 'true self-avoiding walk' which is a discrete-time random process on $\mathbb{Z}$. The process is a negatively reinforced nearest-neighbor random walk: if the random walker is at a given lattice point then he chooses to go left or

right in the next step with probabilities depending on the difference of the local times of the respective neighboring edges, giving more weight to the edge he has visited fewer times. Suppose the walker puts down a unit brick in each step on the edge he has just jumped through, building a wall during the course of its walk. Then the height of this wall at a given edge is equal to the respective value of the local time function and the movement of the walker is driven by the negative gradient of the wall (he rather goes 'downhill' than 'uphill', going through edges he has visited rarer).

In [17] the authors constructed a continuous process as a scaling limit of this discrete-valued process, we can view the process as the continuous-time movement of a particle on $\mathbb{R}$ which is building a wall (its local time) following similar rules as the discrete version. They also proved that the process obeys some dynamical driving mechanism corresponding to these rules. If we denote its position at time t by X_t and its local time (or the height of the wall) at a time t and position x by $h(t,x)$ then the movement of the particle is driven by the slope of the wall:

$$'dX_t = -\partial_x h(t, X_t)dt', \tag{1}$$

and the wall is 'built up' by the presence of the particle:

$$'\partial_t h(t,x) = \delta(X_t - x)'. \tag{2}$$

Of course, these equations do not make sense in this form (hence the inverted commas), but they can be made rigorous. For the details of the construction and primary properties of the process we refer the reader to the original paper. We only remark one interesting and unusual feature: the process X_t has the 2/3 scaling: $(\alpha^{-2/3}X_{\alpha t}, t \geq 0)$ has the same law as $(X_t, t \geq 0)$. In fact $((\alpha^{-2/3}X_{\alpha t}, \alpha^{-1/3}h(\alpha^{2/3}x), \alpha t), t \geq 0, x \in \mathbb{R})$ has the same joint law as $((X_t, h(x,t)), t \geq 0, x \in \mathbb{R})$.

It is natural to consider the case when instead of one particle we have many building the same wall. Corresponding to the discrete case, in [18] a 1 dimensional particle system with 2 conserved quantities was introduced (the 'bricklayer' model): we have several particles (bricklayers) positioned on the lattice sites of $\mathbb{Z}$ who are building a wall from unit-bricks piled above the edges of the lattice. Each bricklayer jumps to a neighboring site with rates depending on the negative gradient of the wall at its position (with the 'downhill' jump getting bigger rate than the 'uphill' jump) and at each jump a unit brick is deposited to the column above the respective edge. This way holes in our walls are tend to be filled quickly by the bricklayers. The two conserved quantities are the particle number and discrete negative gradient of the wall. In the next section we will discuss a whole family of similar models in more detail.

In the continuous setting we would get the following picture: a continuously distributed cloud of particles is building a wall with their movement driven by the slope of the wall. If we denote the density of the particles at x and time t by $\rho(t,x)$ and $u(t,x) := -\partial_x h(t,x)$, then from (1), (2) and some formal computations we get the following partial differential equation

system:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t u + \partial_x \rho &= 0. \end{cases} \tag{3}$$

As noted in [18], this pde-system can also be viewed as a general phenomenological description of a deposition/domain growth – or, in biological term: chemotaxis – mechanism: $\rho(t,x)$ is the density of population performing the deposition $h(t,x)$ is the height of the deposition and $u(t,x) := -\partial_x h(t,x)$. The physics of the phenomenon is contained in the following two rules:

(a) The velocity field of the population is proportional to the *negative gradient of the height* of the deposition. That is, the population is pushed towards the local decrease of the deposition height. This rule, together with the conservation of total mass of the population leads to the continuity equation, the first equation in (3).

(b) The deposition rate is proportional to ρ:

$$\partial_t h \sim \rho, \tag{4}$$

i.e. the deposition is done additively by the population. This, after differentiating with respect to x, gives the second equation of (3).

In [18] from the previously mentioned bricklayer model the pde (3) was derived using formal, non-rigorous hydrodynamic limit and low density perturbation analysis. It was conjectured there that the arguments can be made rigorous and should hold for a large class of models with two conserved quantities. In the paper there were made connections to the Kardar-Parisi-Zhang equation (KPZ equation) which is one of the most famous models in the physical literature for growing interfaces (c.f. [4]). It gives a general phenomenological description of a surface growing to normal direction to its boundary, with a 'tension' that tries to keep the surface together (fills the holes quickly). This resembles properties of our growing wall built by the bricklayers. The KPZ equation itself is (in mathematical sense) an ill-posed non-linear pde with a stochastic term which takes the following form in one dimension:

$$\partial_t h = \nabla h - (\partial_x h)^2 + W \tag{5}$$

where $W = W(t,x)$ is a space-time white noise.

Motivated by the KPZ equation we modify rule (b) in the phenomenological description of the deposition/domain growth mechanism of [18] by adding term proportional to $(\partial_x h)^2$ in (4):

$$\partial_t h \sim \rho + \gamma(\partial_x h)^2. \tag{6}$$

This means that the deposition is not only done solely by the population, but there is also some self-generated deposition (in the spirit of KPZ). Differentiating this with respect to x we get the following pde system instead of (3):

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t u + \partial_x(\rho + \gamma u^2) &= 0, \end{cases} \tag{7}$$

where γ is a real parameter. This is a system of hyperbolic conservation laws with behavior depending largely on the value of γ. Hyperbolicity means that the Jacobian of the pde has two different real eigenvalues. The most important consequence of the hyperbolicity is the following: there is no global strong solution, arbitrary smooth initial conditions yield shocks in finite time (apart from some very specially prepared initial conditions).
Now we are ready to state the main objective of the thesis:
We want to derive (7) as a mathematically rigorous universal low-density hydrodynamic limit for a large class of one dimensional interacting particle systems with two conserved quantities.

3 Models

We consider one dimensional lattice models and in order to keep the state space finite we work with the discrete tori $\mathbb{T}^n := \mathbb{Z}/n\mathbb{Z}$ with $n \in \mathbb{N}$. We will denote the continuous torus $\mathbb{R}/\mathbb{Z}$ by $\mathbb{T}$. We have a finite local spin state denoted by Ω which is the set of the possible observables at a given site. The state space of the interacting particle system is

$$\Omega^n := \Omega^{\mathbb{T}^n},$$

configurations will be denoted by

$$\underline{\omega} := (\omega_j)_{j\in\mathbb{T}^n} \in \Omega^n.$$

The dynamics of our process will be Markovian in continuous time and we only allow elementary jumps (changes of the configuration) which effect two neighboring sites. If at a given time the process is in the configuration $\underline{\omega}$, then at sites $j, j+1$ the spins ω_j, ω_{j+1} can change to ω'_j, ω'_{j+1} with some rate depending only on $\omega_j, \omega_{j+1}, \omega'_j, \omega'_{j+1}$. Thus the dynamics is governed by translation invariant local rules. We denote the rate function by $r : \Omega \times \Omega \times \Omega \times \Omega \to \mathbb{R}_+$, thus $r(\omega_j, \omega_{j+1}; \omega'_j, \omega'_{j+1})$ is the rate of the previously described elementary jump.

We will consider models with two (discrete) conserved quantities which are denoted by

$$\zeta : \Omega \to \mathbb{Z}, \qquad \eta : \Omega \to \mathbb{Z}.$$

We also use the notations $\zeta_j = \zeta(\omega_j)$, $\eta_j = \eta(\omega_j)$. We only allow elementary jumps which conserve the sums $\sum_{j\in\mathbb{T}^n} \zeta_j$, $\sum_{j\in\mathbb{T}^n} \eta_j$. This means that if we have an elementary jump which changes (ω_j, ω_{j+1}) to $(\omega'_j, \omega'_{j+1})$ with a positive rate then

$$\begin{aligned} \zeta_j + \zeta_{j+1} &= \zeta'_j + \zeta'_{j+1}, \\ \eta_j + \eta_{j+1} &= \eta'_j + \eta'_{j+1}. \end{aligned}$$

The conserved quantities have to be different: we assume that ζ, η and the constant 1 function on Ω are linearly independent.

It is easy to see, that (possibly shifting η to be always nonnegative) we can always interpret our model locally as a growing or decaying wall made by unit bricks piled on the edges of the

lattice and built by bricklayers positioned on lattice sites. η_j is the number of bricklayers at site j and ζ_j is the difference between the height of the columns on $(j-1, j)$ and $(j, j+1)$ (see Figure 1). Then an elementary move of the process affecting the sites $(j-1, j)$ corresponds to one of the following changes:

– a couple of particles jump from $j-1$ to j, or vice versa,

– a couple of bricks are deposited on or removed of the top of the column standing on the edge $(j-1, j)$,

– a combination of the two previous things.

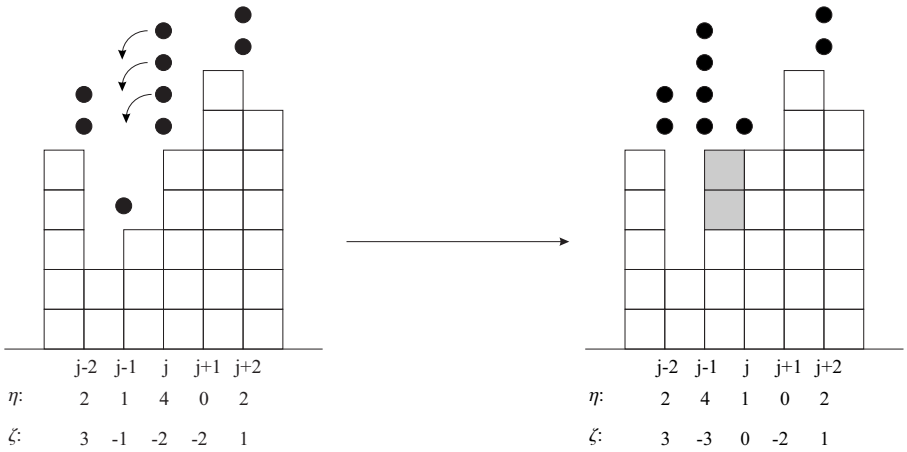

Figure 1: The picture shows a possible elementary move in the bricklayer setting. 3 particles jumped from j to $j-1$ and 2 bricks were deposited on the top of the column standing on $(j-1, j)$.

For a precise formulation of the infinitesimal generator on Ω^n we first define the map $\Theta_j^{\omega',\omega''}$: $\Omega^n \to \Omega^n$ for every $\omega', \omega'' \in \Omega$, $j \in \mathbb{T}^n$:

$$\left(\Theta_j^{\omega',\omega''}\underline{\omega}\right)_i = \begin{cases} \omega' & \text{if} \quad i = j \\ \omega'' & \text{if} \quad i = j+1 \\ \omega_i & \text{if} \quad i \neq j, j+1. \end{cases}$$

The infinitesimal generator of the process defined on Ω^n is

$$L^n f(\underline{\omega}) = \sum_{j \in \mathbb{T}^n} \sum_{\omega',\omega'' \in \Omega} r(\omega_j, \omega_{j+1}; \omega', \omega'')(f(\Theta_j^{\omega',\omega''}\underline{\omega}) - f(\underline{\omega})).$$

We denote by $\mathcal{X}_t^n$ the Markov process on the state space Ω^n with infinitesimal generator L^n.

We impose several technical, mostly combinatorial conditions on the rate function r. Because of the length constraints of this outline we do not give a full description of these conditions, we only list a couple of the fundamental consequences:

(1) There are no hidden conservations besides $\sum \zeta_j$ and $\sum \eta_j$.

(2) There exists a probability measure π on Ω which puts positive weight on every element and for which the product measure $\pi^n := \prod_{j \in \mathbb{T}^n} \pi$ is stationary for $\mathcal{X}_t^n$ (for any n).

(3) The process is not reversible, we do not have the detailed balanced condition.

Actually, we will have a whole family of stationary measures similar to π^n. For every $\theta, \tau \in \mathbb{R}$ let $G(\theta, \tau)$ be the moment generating function defined below:

$$G(\theta, \tau) := \log \sum_{\omega \in \Omega} e^{\theta\zeta(\omega) + \tau\eta(\omega)} \pi(\omega).$$

In thermodynamic terms $G(\theta, \tau)$ corresponds to the Gibbs free energy, see [8]. We define the probability measures

$$\pi_{\theta,\tau}(\omega) := \pi(\omega) \exp(\theta\zeta(\omega) + \tau\eta(\omega) - G(\theta, \tau)) \tag{8}$$

on Ω. Then for any $\theta, \tau \in \mathbb{R}$ and n the measures

$$\pi^n_{\theta,\tau} := \prod_{j \in \mathbb{T}^n} \pi_{\theta,\tau}$$

are stationary for $\mathcal{X}^n_t$. We denote the expectations of the conserved quantities with respect to $\pi_{\theta,\tau}$ by

$$\begin{aligned} u(\theta, \tau) &:= \mathbf{E}_{\pi_{\theta,\tau}}(\zeta) = \sum_{\omega \in \Omega} \zeta(\omega) \pi_{\theta,\tau}(\omega) = G'_\theta(\theta, \tau), \\ v(\theta, \tau) &:= \mathbf{E}_{\pi_{\theta,\tau}}(\eta) = \sum_{\omega \in \Omega} \eta(\omega) \pi_{\theta,\tau}(\omega) = G'_\tau(\theta, \tau). \end{aligned}$$

It is easy to show that the function $(\theta, \tau) \mapsto (u(\theta, \tau), v(\theta, \tau))$ is invertible, we denote the inverse function by $(u, v) \mapsto (\theta(u, v), \tau(u, v))$. The domain of this inverse function is called the *physical domain*, it is denoted by $\mathcal{D}$ and

$$\mathcal{D} = \mathrm{co}\{(\eta(\omega), \zeta(\omega)) : \omega \in \Omega\}$$

where co stands for the convex hull. With slight abuse of notation we shall denote:

$$\pi_{\theta(u,v),\tau(u,v)} =: \pi_{u,v}, \qquad \pi^n_{\theta(u,v),\tau(u,v)} =: \pi^n_{u,v},$$

this gives another natural parametrization for our family of stationary product measures.

Denoting by $(u, v) \mapsto S(u, v)$ the convex conjugate (Legendre transform) of the strictly convex function $(\theta, \tau) \mapsto G(\theta, \tau)$:

$$S(u, v) := \sup_{\theta,\tau} \big(u\theta + v\tau - G(\theta, \tau)\big), \tag{9}$$

we have $\theta(u, v) = S'_u(u, v)$, $\tau(u, v) = S'_v(u, v)$. In probabilistic terms: $S(u, v)$ is the rate function for joint large deviations of $(\sum_j \zeta_j, \sum_j \eta_j)$, in thermodynamic terms: $S(u, v)$ corresponds to the equilibrium thermodynamic entropy (see [8]).

Because of the nearest neighbor interactions we get that the infinitesimal generator acts on the conserved quantities as follows:

$$L^n\zeta_i = -\phi(\omega_i, \omega_{i+1}) + \phi(\omega_{i-1}, \omega_i) =: -\phi_i + \phi_{i-1},$$
$$L^n\eta_i = -\psi(\omega_i, \omega_{i+1}) + \psi(\omega_{i-1}, \omega_i) =: -\psi_i + \psi_{i-1},$$

with functions $\phi : \Omega \times \Omega \to \mathbb{R}$, $\psi : \Omega \times \Omega \to \mathbb{R}$. The explicitly computable functions ϕ, ψ are called *microscopic fluxes*, their expectations with respect to $\pi^2_{u,v}$ are called *macroscopic fluxes*:

$$\Phi(u, v) := \mathbf{E}_{\pi^2_{u,v}}\phi(\omega_1, \omega_2),$$
$$\Psi(u, v) := \mathbf{E}_{\pi^2_{u,v}}\psi(\omega_1, \omega_2).$$

As we will see, the macroscopic flux functions $\Phi(u, v), \Psi(u, v)$ will govern the macroscopic evolution of density-profiles of the conserved quantities. These functions depend on the microscopic model.

There are several detailed concrete examples of microscopic models fitting to this framework in Section 2.2 of the thesis (or [15]).

4 Eulerian scaling

Hydrodynamic limit gives the macroscopic behavior of the density-profiles of the conserved quantities via some suitable scaling of space and time. The scaling of space will be the same in all the results of the thesis: we rescale it by n. This means that we represent $\mathbb{T}^n$ with n sites on the continuous torus $\mathbb{T}$, with distance $1/n$ between the neighboring sites.

There are several heuristical derivations which (formally) yield that under Eulerian scaling (which means the rescaling of time and space by n) the macroscopic density-profiles of the conserved quantities ζ, η evolve according to the equation

$$\begin{cases} \partial_t u + \partial_x \Phi(u, v) &= 0, \\ \partial_t v + \partial_x \Psi(u, v) &= 0, \end{cases} \tag{10}$$

which is usually a hyperbolic conservation law.

This means the following. Suppose that $u_0(\cdot), v_0(\cdot)$ are real functions on $\mathbb{T}$ with $(u_0(x), v_0(x)) \in \mathcal{D}$ for $x \in \mathbb{T}$. Fix a microscopic model and take its versions for every n on Ω^n with space rescaled by n. Assume that we have initial (random) configurations of our processes such that the densities of ζ, η approximate the functions $u_0(\cdot), v_0(\cdot)$. Then letting the systems run up to time nt the density-profiles of ζ, η will approximate the functions $u(t, \cdot), v(t, \cdot)$ which are the solutions of (10) with initial conditions $u(0, x) = u_0(x), v(0, x) = v_0(x)$. The approximation of a deterministic function by the density-profile can be defined in several ways. A natural definition is the following weak approximation: for any smooth test function $g : \mathbb{T} \to \mathbb{R}$

$$\frac{1}{N}\sum_{j\in\mathbb{T}^n} g(j/N)\zeta_j(nt) \xrightarrow{\mathbf{P}} \int_{\mathbb{T}} g(x)u(t, x)\,dx,$$
$$\frac{1}{N}\sum_{j\in\mathbb{T}^n} g(j/N)\eta_j(nt) \xrightarrow{\mathbf{P}} \int_{\mathbb{T}} g(x)v(t, x)\,dx. \tag{11}$$

Then the (heuristic) result may be summarized the following way: if the previous limits hold for $t = 0$ for any test function g then they will hold at any $t > 0$.

Theorem 1 will give a rigorous version of this result. There is essentially one robust, model-independent method for proving hydrodynamic limits which works for hyperbolic interacting particle systems with two conserved quantities: H.T. Yau's relative entropy method, introduced in [20]. For attractive one component systems there exist stronger results (c.f. [9]), but these cannot be extended to our case. It does not depend much on the microscopic properties on the model, but this great generality has one drawback: the proof only works for smooth solutions of the limiting pde. (Smooth actually means some finite differentiability conditions.) However, as we already mentioned, hyperbolic conservation laws with generic initial conditions cannot have globally smooth solutions. Thus, the relative entropy method can only apply up to a finite time, before the first appearance of shocks. We note, that [7] gives the first major result about the Eulerian hydrodynamic limit for multi-component hyperbolic systems, namely for Hamiltonian systems perturbed by a weak noise.

Before stating the theorem, we need to take a brief look at the relative entropy method. If μ, π are measures on the same probability space (Ω) then we denote their relative entropy by $H(\mu|\pi)$ and it is defined as

$$H(\mu|\pi) := \sup_{\|f\|_\infty < \infty} \left\{ \mathbf{E}_\mu f - \log \mathbf{E}_\pi e^f \right\}.$$

If the density $\frac{d\mu}{d\pi}$ exists then

$$H(\mu|\pi) = \mathbf{E}_\mu \left(\log \frac{d\mu}{d\pi} \right) = \mathbf{E}_\pi \left(\frac{d\mu}{d\pi} \log \frac{d\mu}{d\pi} \right).$$

Relative entropy gives a way to measure the distance of two probability measures on the same probability space (although it is not a distance in the topological sense). The basic idea of the relative entropy method is the following: instead of comparing the density profiles of the conserved quantities of our process to a pair of deterministic functions, we try to 'guess' the distribution itself and compare the real distribution to this 'educated guess'. We usually look at the system after it ran for a long time (nt in this case, with a large n), thus it is reasonable to believe that at least locally, the distribution of our process looks like the canonical stationary distribution. Therefore, if we think that the density profiles of ζ, η approximate a pair of functions $(u(t,\cdot), v(t,\cdot))$ then the following distribution on Ω^n is a 'good guess' for the real distribution of the process:

$$\nu_t^n := \prod_{j \in \mathbb{T}^n} \pi_{u\left(t,\frac{j}{n}\right), v\left(t,\frac{j}{n}\right)}. \tag{12}$$

The measure ν_t^n is our time-dependent reference measure to which we compare the distribution of the process. We denote the real distribution at *microscopic time* nt by μ_t^n. Then the following theorem is true:

Theorem 1. *Suppose $(u(t,x), v(t,x))$ is a solution of (10) which is smooth for $t \in [0,T]$ and $(u(0,x), v(0,x)) \in \mathcal{D}$ for $x \in \mathbb{T}$. Then if*

$$H(\mu_0^n | \nu_0^n) = o(n),$$

then

$$H(\mu_t^n | \nu_t^n) = o(n),$$

uniformly for $t \in [0,T]$.

The theorem states that if the initial distribution is close to ν_0^n in the relative entropy sense then the distribution at time nt will be close to ν_t^n in the same sense. The fact, that 'close' should mean $o(n)$ is easily justified (see [15]). It might not be transparent if there is any connection at all between this setting and the weak approximation defined in (11). Actually, the following corollary holds:

Corollary 1. *Under the conditions of Theorem 1, for any $t \in [0,T]$, the following limits hold for any smooth test function $g : \mathbb{T} \to \mathbb{R}$ as $n \to \infty$*

$$\frac{1}{n} \sum_{j \in \mathbb{T}^n} g(j/n) \zeta_j(t) \xrightarrow{\mathbf{P}} \int_{\mathbb{T}} g(x) u(t,x)\, dx,$$
$$\frac{1}{n} \sum_{j \in \mathbb{T}^n} g(j/n) \eta_j(t) \xrightarrow{\mathbf{P}} \int_{\mathbb{T}} g(x) v(t,x)\, dx.$$

The proof of Theorem 1 follows the standard steps of the relative entropy method, but there is one novelty. In order to complete the proof, we need the following symmetry relation between the macroscopic fluxes Φ, Ψ, reminiscent of the Onsager reciprocity relations which can be proved using the existence of a stationary measure with product structure.

Lemma 1.

$$\partial_\tau \Psi(u(\theta,\tau), v(\theta,\tau)) = \partial_\theta \Phi(u(\theta,\tau), v(\theta,\tau)).$$

This symmetry relation is an important element of the proof, but also allows us to show some interesting (although not surprising) facts about the pde (10):

Corollary 2. *The system of conservation laws (10) is (weakly) hyperbolic inside the domain $\mathcal{D}$. Furthermore, the equilibrium thermodynamic entropy $(u,v) \mapsto S(u,v)$ is a globally convex Lax entropy for the system (10).*

Weak hyperbolicity means, that the Jacobian

$$\begin{pmatrix} \Phi'_u(u,v) & \Phi'_v(u,v) \\ \Psi'_u(u,v) & \Psi'_v(u,v) \end{pmatrix} \tag{13}$$

can be diagonalized in a real sense. A Lax-entropy is a function $S(u,v)$ for which there exists a flux function $F(u,v)$ such that if $u(t,x), v(t,x)$ are smooth solutions of (10) then

$$\partial_t S(u,v) + \partial_x F(u,v) = 0.$$

Essentially, this is an extra conservation law.

The results of this section are published in [15].

5 Deriving the universal pde (7)

5.1 Perturbation analysis

In this section we use the variable ρ instead of v for the density of the conserved quantity η. Using the results of the Eulerian scaling we can give a formal, non-rigorous derivation the pde (7). Suppose that $\min_{\omega\in\Omega}\eta(\omega)=0$ (this is a natural assumption if we interpret η as the number of particles at a given site) and that our microscopic model has a left-right reflection-symmetry. The latter is implemented the following way. There is an involution

$$R:\Omega\to\Omega,\qquad R\circ R=Id$$

which acts on the conserved quantities as follows:

$$\eta(R\omega)=\eta(\omega),\qquad \zeta(R\omega)=-\zeta(\omega),$$

and for which

$$\pi(R\omega)=\pi(\omega)\qquad\text{and}\qquad r(R\omega_2,R\omega_1;R\omega_2',R\omega_1')=r(\omega_1,\omega_2;\omega_1',\omega_2').$$

This means, that changing the direction of the lattice, our wall evolves with the same dynamics. We note that the pde (7) has this reflection-symmetry: if $(\rho(t,x),u(t,x))$ is a solution then so is $(\rho(t,-x),-u(t,-x))$.

Under the previous assumptions we get the following asymptotics for the macroscopic fluxes:

$$\begin{aligned}\Phi(\rho,u) &= a\,(\rho+\gamma u^2)\left(1+\mathcal{O}(\rho+u^2)\right),\\ \Psi(\rho,u) &= b\,\rho u\left(1+\mathcal{O}(\rho+u^2)\right),\end{aligned}\tag{14}$$

if $\rho,|u|\ll 1$. Using these asymptotics the pde (7) may be derived by perturbing the constant $(0,0)$ solution of the Eulerian pde

$$\begin{cases}\partial_t u+\partial_x\Phi(\rho,u) &= 0,\\ \partial_t\rho+\partial_x\Psi(\rho,u) &= 0.\end{cases}\tag{15}$$

Let $\rho_0(x)$ and $u_0(x)$ be given profiles and assume that $\rho^\varepsilon(t,x)$, $u^\varepsilon(t,x)$ is a solution of the Eulerian pde (15) with initial condition

$$\rho^\varepsilon(0,x)=\varepsilon^2\rho_0(x),\quad u^\varepsilon(0,x)=\varepsilon\,u_0(x).$$

Then, at least formally, if $\varepsilon\to 0$

$$\varepsilon^{-2}\rho^\varepsilon(\varepsilon^{-1}t,x)\to\rho(t,x),\quad \varepsilon^{-1}u^\varepsilon(\varepsilon^{-1}t,x)\to u(t,x),$$

where $\rho(t,x)$, $u(t,x)$ is the solution of the pde (7) with initial conditions

$$\rho(0,x)=\rho_0(x),\quad u(0,x)=u_0(x).$$

Actually, the constants a,b from the asymptotics also appear in the equation, but they can be scaled out by simple linear transformations to get (7). It is important to note that the constant γ cannot be scaled out of the equations and that is the only trace left of the microscopic model in the pde-system.

5.2 The main result

From the perturbation analysis we can guess how we should derive (7) as a 'universal' hydrodynamic limit. Fix a microscopic model with the previous assumptions ($\min \eta = 0$ and reflection-symmetry), a small constant $\beta > 0$ and suppose that $\rho(t,x)$, $u(t,x)$ is the solution of the pde (7) with initial conditions

$$\rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x).$$

If at time $t = 0$ the density-profiles of η, ζ approximate the functions $n^{-2\beta}\rho_0(\cdot), n^{-\beta}u_0(\cdot)$ then at time $n^{1+\beta}t$ they should approximate the functions $n^{-2\beta}\rho(t,\cdot), n^{-\beta}u(t,\cdot)$. *Note that this is not Eulerian scaling.*

For a given $\rho(t,x)$, $u(t,x)$ solution of the pde (7) we define the time-dependent reference measure ν_t^n as

$$\nu_t^n := \prod_{j\in\mathbb{T}^n} \pi_{n^{-2\beta}\rho\left(t,\frac{j}{n}\right),n^{-\beta}u\left(t,\frac{j}{n}\right)}.$$

We compare this to the real distribution of our process at time $n^{1+\beta}t$ which we denote with μ_t^n. In order to enhance convergence to local equilibrium and thus help estimating some error terms in the hydrodynamic limiting procedure we also add some 'extra speed' to the symmetric part of the infinitesimal generator. To be more precise: we also have a *symmetric* rate function s with similar properties as r and the elementary change $(\omega_j, \omega_{j+1}) \to (\omega'_j, \omega'_{j+1})$ is performed with rate $r(\omega_j, \omega_{j+1}; \omega'_j, \omega'_{j+1}) + n^\delta s(\omega_j, \omega_{j+1}; \omega'_j, \omega'_{j+1})$. We also need the logarithmic-Sobolev inequality for s. The parameter δ is positive and less than 1. It is chosen in a way that the effect of the symmetric part is *not seen* in the hydrodynamic limit. Now we are ready to state the theorem.

Theorem 2. *Fix a microscopic model for which the parameter γ is greater than 1. Suppose that $\rho(t,x)$, $u(t,x)$ is a smooth solution of the pde (7) for $t \in [0,T]$. Choose $\beta \in (0,1/2)$ and $\delta \in (1/2,1)$ so that $2\delta - 8\beta > 1$, $\delta + 3\beta < 1$ and define μ_t^n, ν_t^n as before. If*

$$H(\mu_t^n | \nu_t^n) = o(n^{1-2\beta})$$

holds at $t = 0$ then it will hold uniformly for $t \in [0,T]$.

This theorem is of the same structure as Theorem 1 and it has a similar corollary.

Corollary 3. *Assume the conditions of Theorem 2. Let $g : \mathbb{T} \to \mathbb{R}$ be a smooth test function. Then for any $t \in [0,T]$*

$$n^{2\beta-1} \sum_{j\in\mathbb{T}^n} g(\frac{j}{n})\eta_j(t) \quad \xrightarrow{\mathbf{P}} \quad \int_{\mathbb{T}} g(x)\rho(t,x)\,dx,$$

$$n^{\beta-1} \sum_{j\in\mathbb{T}^n} g(\frac{j}{n})\zeta_j(t) \quad \xrightarrow{\mathbf{P}} \quad \int_{\mathbb{T}} g(x)u(t,x)\,dx.$$

The restriction $\gamma > 1$ comes from the fact that the geometrical structure of the pde is different for $\gamma < 1$ and $\gamma > 1$. Interestingly, this change of geometric structure is also responsible for a change in the treatability of (7) from the pde point of view. For $\gamma < 1$ it is not even possible to prove the Lax's maximum principle, i.e. that bounded initial conditions yield a bounded solution. The proof of Theorem 2 is not merely a simple application of the relative entropy method, it also uses nontrivial elements of pde theory. In order to control the fluctuations of some terms with Poissonian (rather than Gaussian) decay coming from the low density approximations we have to apply refined pde estimates, in particular Lax entropies of these pde systems play a key role in the main part of the proof.

The results of this section are also contained in the paper [15] (a submitted preprint).

6 Perturbation of a hyperbolic equilibrium point

The result of Theorem 2 may be interpreted as the description of the hydrodynamic behavior for the perturbation of a *singular* equilibrium point. Indeed, the point $(0, 0)$ (around which we considered the perturbation) is not hyperbolic for the Eulerian pde (15), the Jacobian is a multiple of the matrix

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

However, most of the points in $\mathcal{D}$ are hyperbolic, thus it is a natural question to ask what the behavior of the perturbation will be if we perform it around a 'common' hyperbolic point. This section deals with this question, we will use the notations of Section 4.

Suppose $(u_0, v_0) \in \mathcal{D}$ is a (strongly) hyperbolic point of the pde (10). In order to simplify notations, we assume that the Jacobian (13) at (u_0, v_0) is diagonal:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},$$

$\lambda \neq \mu \in \mathbb{R}$ are the two eigenvalues. We can always reduce the general case to this via some linear transformation of the conserved quantities. Let $u^*(x), v^*(x)$ be given smooth functions and assume, that $u^\varepsilon(t, x), v^\varepsilon(t, x)$ is the solution of (10) with initial conditions

$$u^\varepsilon(0, x) = u_0 + \varepsilon u^*(x), \quad v^\varepsilon(0, x) = v_0 + \varepsilon v^*(x).$$

Applying standard perturbation techniques (e.g. the method of geometric optics, see [1] or [3]) it can be shown that (at least formally):

$$\begin{aligned} u^\varepsilon(t, x) &\approx u_0 + \varepsilon u(\varepsilon t, x - \lambda t) + \mathcal{O}(\varepsilon^2), \\ v^\varepsilon(t, x) &\approx v_0 + \varepsilon v(\varepsilon t, x - \mu t) + \mathcal{O}(\varepsilon^2), \end{aligned}$$

as $\varepsilon \to 0$ where $u(t, x)$ and $v(t, x)$ are the solutions of two *decoupled* partial differential equations. These are Burgers' equations, if $\Phi''_{uu}(u_0, v_0) \neq 0$, respectively, $\Psi''_{vv}(u_0, v_0) \neq 0$, and linear transport equations otherwise.

From the (formal) perturbation result we can guess the hydrodynamic picture. Fix a small positive parameter β. Suppose we have a microscopic model with two conserved quantities and the initial distribution is such that the macroscopic density profiles are close to the functions $u_0 + n^{-\beta}u^*(x), v_0 + n^{-\beta}v^*(x)$. Then at microscopic time $n^{1+\beta}t$ the density profiles should be close to the functions $u_0 + n^{-\beta}u(t, x - n^\beta \lambda t), v_0 + n^{-\beta}v(t, x - n^\beta \mu t)$. The precise formulation of this result in the relative entropy setting will be similar to the previous results. First, we construct a time-dependent reference measure: fix the functions $u(t,x), v(t,x)$ which are smooth in $[0,T] \times \mathbb{T}$ and solutions of the respective partial differential equations we get from the formal perturbation. Then we define:

$$\nu_t^n := \prod_{j \in \mathbb{T}^n} \pi_{u_0 + n^\beta u\left(t, \frac{j}{n} - n^\beta \lambda t\right), v_0 + n^{-\beta} v\left(t, \frac{j}{n} - n^\beta \mu t\right)},$$

and we denote by μ_t^n the real distribution of the process on Ω^n at time $n^{1+\beta}t$. We also need one additional condition on the rate function r: a uniform bound on the inverse of the spectral gap of the generator which is quadratic in the system-size. We do not need speeded up symmetric rates as in Theorem 2. Now we can state our last theorem.

Theorem 3. *Fix a parameter β with $0 < \beta < \frac{1}{5}$ and define μ_t^n and ν_t^n as before. If*

$$H(\mu_t^n | \nu_t^n) = o(n^{1-2\beta})$$

holds at $t = 0$ then it will hold uniformly for $t \in [0,T]$.

If we assume the uniform logarithmic-Sobolev condition on our rate function r (this is stronger then the spectral gap condition) then the result of Theorem 3 holds for $0 < \beta < \frac{1}{3}$. From Theorem 3 we get the following corollary.

Corollary 4. *Assume the conditions of Theorem 3. Let $g : \mathbb{T} \to \mathbb{R}$ be a smooth test function. Then for any $t \in [0,T]$*

$$\left| n^{-1+\beta} \sum_{j \in \mathbb{T}^n} g(\frac{j}{n}) \left(\zeta_j(n^{1+\beta}t) - u_0 \right) - \int_{\mathbb{T}} g(x) \left(u(t, x - \lambda n^\beta t) \right) dx \right| \xrightarrow{\mathbf{P}} 0,$$

$$\left| n^{-1+\beta} \sum_{j \in \mathbb{T}^n} g(\frac{j}{n}) \left(\eta_j(n^{1+\beta}t) - v_0 \right) - \int_{\mathbb{T}} g(x) \left(v(t, x - \mu n^\beta t) \right) dx \right| \xrightarrow{\mathbf{P}} 0.$$

This result is an extension of [10] and [14] where the analogue result was proved for systems with one conservation law: i.e. perturbations of equilibrium of order $n^{-\beta}$ evolve according to the Burgers' equation if time is rescaled by $n^{1+\beta}$. In [10] the respective result is proved for the so-called totally asymmetric stick process (a 1 dimensional system with one conserved quantity) with coupling methods, but in a stronger form: for $0 < \beta < \frac{1}{2}$ and for all t, even after the appearance of shocks. The upper bound $\frac{1}{2}$ for β is sharp, since at $\beta = \frac{1}{2}$ the equilibrium fluctuations of the constant profile could not be distinguished from the perturbation. In [14] it

was shown with the application of the Yau's method, that the previous result holds *universally*, for a wide family of one-component systems, but that result is only valid for $0 < \beta < \frac{1}{5}$ and in the regime of smooth solution (similarly to Theorem 3).

The proof of Theorem 3 is an application of Yau's relative entropy method and it also strongly relies on the Onsager-type relation of Lemma 1. The reason for the decoupling of the resulting pde system is the hyperbolicity, basically, the two different eigenvalues (sound speeds) cause the equations to separate.
The results of this section are contained in the paper [19] (a submitted preprint).

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