We have $X=\mathbb{R}^2$ and given $\underline{x}=(x_1,y_1)$, $\underline{y}=(x_2,y_2)$, $d(\underline{x},\underline{y})=\max\{|x_1-x_2|, |y_1-y_2|\}$.

a.) Claim: d is a metric.

Clearly, $d(\underline{x},\underline{y})\geq 0$ $\forall \underline{x},\underline{y}\in\mathbb{R}^2$, since absolute values are positive.

If $\underline{x}=\underline{y}$, then in particular $x_1=x_2$, $y_1=y_2$

so $d(\underline{x},\underline{y})=0$.

On the other hand, if $d(\underline{x},\underline{y})=0$ for some $\underline{x},\underline{y}\in\mathbb{R}^2$, then the only way this is possible is $|x_1-x_2|=0$ and $|y_1-y_2|=0$.

$\Rightarrow x_1=x_2,\ y_1=y_2 \Rightarrow \underline{x}=\underline{y}$

So property 1 of a metric holds.

Since, $|x_1-x_2|=|x_2-x_1|$

and $|y_1-y_2|=|y_2-y_1|$

we have $d(\underline{x},\underline{y})=d(\underline{y},\underline{x})$

So property 2 of a metric holds, too.

Claim: $\forall \underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^2$

$d(\underline{x}, \underline{y}) \leq d(\underline{x}, \underline{z}) + d(\underline{y}, \underline{z})$

Pf: Let $\underline{x} = (x_1, y_1)$

$\underline{y} = (x_2, y_2)$

$\underline{z} = (x_3, y_3)$

By the regular triangle inequality

$$|x_1 - x_2| \leq |x_1 - x_3| + |x_3 - x_2| \quad (1)$$

$$|y_1 - y_2| \leq |y_1 - y_3| + |y_3 - y_2| . \quad (2)$$

Now, consider wlog

$$d(\underline{x}, \underline{y}) = \max\{|x_1 - x_2|, |y_1 - y_2|\} = |x_1 - x_2|$$

$\Rightarrow$ using (1), we have

$$d(\underline{x}, \underline{y}) \leq |x_1 - x_3| + |x_3 - x_2| .$$

Since clearly.

$$|x_1 - x_3| \leq \max\{|x_1 - x_3|, |y_1 - y_3|\} = d(\underline{x}, \underline{z})$$

$$|x_3 - x_2| \leq \max\{|x_2 - x_3|, |y_2 - y_3|\} = d(\underline{z}, \underline{y})$$

we have $d(\underline{x}, \underline{y}) \leq d(\underline{x}, \underline{z}) + d(\underline{z}, \underline{y})$

B.) The circle of radius 1 around the origin

$$S^1 = \{ \underline{y} \in \mathbb{R}^2 \mid d(\underline{0}, \underline{y}) = 1 \}$$

$$= \{ \underline{y} \in \mathbb{R}^2 \mid \max\{|x_2|, |y_2|\} = 1 \}$$

To simplify let $x = x_2$, $y = y_2$

then $\max\{|x|, |y|\} = 1$

$\iff |x| = 1$ and $|y| \leq 1$

i.e. $x = 1$ and $-1 \leq y \leq 1$ (3)

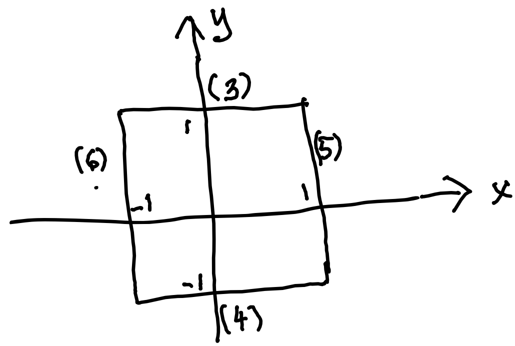
OR $x = -1$ and $-1 \leq y \leq 1$ (4)

OR $|x| \leq 1$ and $|y| = 1$

i.e. $-1 \leq x \leq 1$ and $y = 1$ (5)

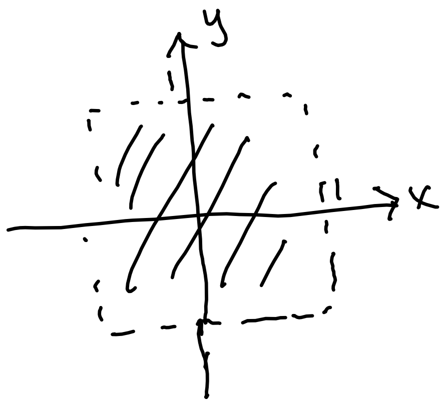
OR $-1 \leq x \leq 1$ and $y = -1$ (6)

$\Rightarrow$



By a similar analysis

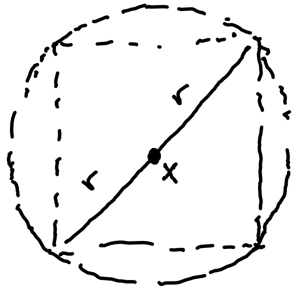
$\underline{B_0}(1) = \{ \underline{y} \in \mathbb{R}^2 \mid \max\{|x|, |y|\} < 1 \}$



Open balls are open squares (of various centers and sizes).

c.) Suppose $U \subset \mathbb{R}^2$ is open with respect to the usual metric. Let $x \in U$. Then $\exists r > 0$ s.t. for $B^u_x(r)$ = usual open ball = open disc we have $B^u_x(r) \subset U$.

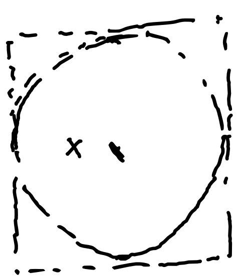
But every open disc has an open square with the same center in it. (In fact, e.g. of sidelength $\sqrt{2}\, r$.) $\Rightarrow$



and so that open square $\subset \mathcal{U}$.

This shows $\mathcal{U}$ is open in the new metric. (Since x was arbitrary.)

Vice-versa, if $\mathcal{U}$ is open in the new metric, then $\forall x \in \mathcal{U}$ $\exists$ open square $B_x^{new}(r) \subset \mathcal{U}$.



$2r$

But every open square has a usual open ball ie a disc in it, with the same center.

Eg. if the square has sidelength $2r$, then a disc of radius r works.

So $B_x^{u}(r) \subset B_x^{new}(r) \subset \mathcal{U} \Rightarrow$ $\mathcal{U}$ is open in the usual sense.