2.a. Yes.

Let $r: \mathbb{R} \to [0,1]$ be defined by

$$r(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Then r is continuous by calculus/the pasting lemma (on $A = (-\infty, 0]$, $B = [0,1]$, $C = [1, +\infty)$), and clearly fixed on $B = [0,1] \subset X = [0,1]$.

2b. No.

Suppose $\exists r: \mathbb{R} \to (0,1)$ retraction.

Then, in particular, $r(x) = x$ $\forall x \in (0,1)$. (*)

Take any sequence eg. $x_n = \frac{1}{n} \to 0$ from the right.

Then $r(1/n) = 1/n$ by (*).

Also, since r is continuous, $\frac{1}{n} \to 0$ implies $r(1/n) = 1/n \to r(0)$.

Thus $r(0) = 0$.

But we are supposed to have $r(0) \in (0,1)$ so $r(0) \neq 0$. $\lightning$.

2.c. Yes.

Let $r : \mathbb{R}^2 \to \overline{D}^2$ be defined as

$$r(\underline{x}) = \begin{cases} \dfrac{\underline{x}}{\|\underline{x}\|} & \text{if } \|\underline{x}\| \geq 1 \\ \underline{x} & \text{if } \|\underline{x}\| \leq 1 \text{ i.e. } \underline{x} \in \overline{D}^2 \end{cases}$$

Then r is onto $\overline{D}^2$, $r(\underline{x}) = \underline{x}$ $\forall \underline{x} \in \overline{D}^2$.
r is also continuous by calculus/the Pasting Lemma (applied to $A = \overline{D}^2$ and $B = \{\underline{x} \mid \|\underline{x}\| \geq 1\}$ closed subsets of $\mathbb{R}^2$).

2.d No.

Assume $\exists\, r : \mathbb{R}^2 \to S^1$ retraction
i.e. r is continuous, onto and $r(\underline{x}) = \underline{x}$ $\forall \underline{x} \in S^1$.

Then, considering $i : S^1 \to \mathbb{R}^2$, $\underline{x} \mapsto \underline{x}$ inclusion map, we get
$r \circ i = \mathrm{Id}_{S^1}$. So for the induced

homomorphisms we must have

$$(r \circ i)_* = (\mathrm{Id}_{S^1})_*$$

By properties of induced homomorphisms this means

$$r_* \circ i_* = 1_{\pi_1(S^1)}$$ ← the identity isomorphism on $\pi_1(S^1)$

where $i_* : \pi_1(S^1) \to \pi_1(\mathbb{R}^2)$

$r_* : \pi_1(\mathbb{R}^2) \to \pi_1(S^1)$

so we have the following commutative diagram:

$$\begin{tikzcd} \pi_1(S^1) \arrow[r, "i_*"] \arrow[rr, bend right, "1_{\pi_1(S^1)}"'] \arrow[rr, phantom, bend right=20, "G"] & \pi_1(\mathbb{R}^2) \arrow[r, "r_*"] & \pi_1(S^1) \end{tikzcd}$$

that is mappings

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i_*"] \arrow[rr, bend right, "1_{\mathbb{Z}}"'] & 0 \arrow[r, "r_*"] & \mathbb{Z} \end{tikzcd}$$

But this is not possible, since i_* cannot be 1-1 & r_* cannot be onto.