2. a. Assume $K \subset \mathbb{R}$ is compact and connected.

Since K is connected, $\forall a, b \in K$ s.t. $a < b$ and $\forall c \in (a, b)$ we must have $c \in K$. If not, i.e. if $c \notin K$, then

for $U = K \cap (-\infty, c)$

and $V = K \cap (c, +\infty)$

we have $U, V \in \tau_K$, $U \neq \phi$ since $a \in U$,

$V \neq \phi$ since $b \in V$.

$U \cap V \subset (-\infty, c) \cap (c, +\infty) = \phi$

so $U \cap V = \phi$.

Also,

$$U \cup V = [K \cap (-\infty, c)] \cup [K \cap (c, +\infty)]$$
$$= K \cap [(-\infty, c) \cup (c, +\infty)]$$
$$= K \cap \mathbb{R} \setminus \{c\} = K$$

since we assumed $c \notin K$.

$\Rightarrow$ K would be disconnected.

So $\forall a, b \in K$, $a < b$, $\forall c \in (a, b)$ we have $c \in K$. So K is an interval.

Since K is compact, it is closed and bounded, since in $\mathbb{R}$ the Heine-Borel thm is applied. So $K = [a, b]$ for some $a, b \in \mathbb{R}$, $a \leq b$.

2b. <u>Claim</u>: $\nexists K \subset \mathbb{R}$ s.t. $K \sim S^1$

<u>Pf</u>: Assume $\exists K \subset \mathbb{R}$ s.t. $K \sim S^1$.

Then, since S^1 is compact, connected K must be compact, connected, too.

By part a), $K = [a,b]$ for some $a \leq b$, $a, b \in \mathbb{R}$.

Clearly, $a = b$ is not possible, since S^1 has more than one element.

So $K = [a,b] \sim S^1$ where $a < b$.

i.e. $\exists f : [a,b] \longrightarrow S^1$ homeomorphism.

Let $c = \frac{a+b}{2}$ so $a < c < b$.

Then the restriction onto $B = [a,b] \setminus \{c\}$

$$f\big|_B : B = [a,b] \setminus \{c\} \longrightarrow S^1 \setminus \{f(c)\}$$

is also a homeomorphism.

However, $B = ((-\infty, c) \cap B) \cup ((c, +\infty) \cap B)$ is a way to disconnect B, while

$S^1 \setminus \{f(c)\}$ is path-connected so connected.

Thus

$B = [a,b] \setminus \{c\}$ cannot be homeomorphic to $S^1 \setminus \{f(c)\}$

and so no homeomorphism from $[a,b]$ to S^1 exists.