A TALL SPACE WITH A SMALL BOTTOM

ISTVÁN JUHÁSZ, SAHARON SHELAH, LAJOS SOUKUP, AND ZOLTÁN SZENTMIKLÓSSY

Abstract. We introduce a general method of constructing locally compact scattered spaces from certain families of sets and then, with the help of this method, we prove that if $\kappa^{<\kappa} = \kappa$ then there is such a space of height κ^+ with only κ many isolated points. This implies that there is a locally compact scattered space of height ω_2 with ω_1 isolated points in ZFC, solving an old problem of the first author.

1. Introduction

Let us start by recalling that a topological space X is called *scattered* if every non-empty subspace of X has an isolated point and that such a space has a natural decomposition into levels, the so called Cantor-Bendixson levels. The α^{th} Cantor-Bendixson level of X will be denoted by $\mathrm{I}_\alpha(X)$. We shall write $\mathrm{I}_{<\lambda}(X) = \bigcup_{\alpha<\lambda} \mathrm{I}_\alpha(X)$. The height of X, $\mathrm{ht}(X)$, is the least α with $\mathrm{I}_\alpha(X) = \emptyset$. The sequence $\langle |I_\alpha(X)| : \alpha \in \mathrm{ht}(X)\rangle$ is said to be the *cardinal sequence* of X. The width of X, $\mathrm{wd}(X)$, is defined by $\mathrm{wd}(X) = \sup\{\, |\mathrm{I}_\alpha(X)| : \alpha < \mathrm{ht}(X)\}$.

The cardinality of a T_3 , in particular of a locally compact, scattered (in short: LCS) space X is at most $2^{|\mathrm{I}(X)|}$, hence clearly $\mathrm{ht}(X) < (2^{|\mathrm{I}(X)|})^+$. Therefore under CH there is no LCS space of height ω_2 with only countably many isolated points. On the other hand, I. Juhász and W. Weiss, [2, theorem 4], proved in ZFC that for every $\alpha < \omega_2$ there is an LCS space X with $\mathrm{ht}(X) = \alpha$ and $\mathrm{wd}(X) = \omega$. The natural question if the existence of an LCS space of height ω_2 with countable width follows from $\neg CH$ was answered in the negative by W. Just, who proved, [3, theorem 2.13], that if one adds Cohen reals to a model of CH then in the generic extension there are no LCS spaces of height ω_2 and $\mathrm{wd}(X) = \omega$. On the other hand, Baumgartner and Shelah proved in [1] that it is consistent (with $\neg CH$) that such an LCS space exists.

The above mentioned estimate $\mathrm{ht}(X) < (2^{|I(X)}|)^+$ is sharp for LCS spaces with countably many isolated points : it is easy to construct an LCS space with countable "bottom" and of height α for each $\alpha < (2^\omega)^+$ (see theorem 2.20). Much less is known about LCS spaces with ω_1 isolated points, for example it is a long standing open problem whether there is, in ZFC, an LCS space of height ω_2 and width ω_1. In

2000 *Mathematics Subject Classification.* 54A25, 06E05, 54G12, 03E20.

Key words and phrases. locally compact scattered space, superatomic Boolean algebra.

The first, third and fourth authors were supported by the Hungarian National Foundation for Scientific Research grant no. 25745 .

The second author was supported by The Israel Science Foundation founded by the Israel Academy of Sciences and Humanities. Publication 714.

The third author was partially supported by Grant-in-Aid for JSPS Fellows No. 98259 of the Ministry of Education, Science, Sports and Culture, Japan.

fact, as was noticed by Juhász in the mid eighties, even the much simpler question if there is a ZFC example of an LCS space of height ω_2 with only ω_1 isolated points, turned out to be surprisingly difficult. On the other hand, Martínez in [5, theorem 1] proved that it is consistent that for each $\alpha < \omega_3$ there is a LCS space of height α and width ω_1. As the main result of the present paper, we shall give an affirmative answer to the above question of Juhász: in section 2 we construct, in ZFC, an LCS space of height ω_2 with ω_1 isolated points. Since the space we construct in theorem 2.21 has width ω_2, the following question remains:

Problem 1. *Is there an LCS space X of height ω_2 and width ω_1 in ZFC?*

The methods used in the proof of theorem 2.21 do not seem to suffice to get LCS spaces with ω_1 isolated points of arbitrary height $< \omega_3$. Thus we have the following problem:

Problem 2. *Is there, in ZFC, an LCS space with ω_1 isolated points and of height α for each $\alpha < \omega_3$?*

Although one of our main results, theorem 2.19, can be applied to higher cardinals, it does not seem to suffice to get the analogous result e.g. for ω_3 instead of ω_2. If $2^\omega \le \omega_2$ but $2^{\omega_1} > \omega_2$ then neither theorem 2.20 nor theorem 2.19 can be applied to get an LCS space of height ω_3 with only ω_2 many isolated points. Thus the following version of Juhász' problem remains open:

Problem 3. *Is there, in ZFC, an LCS space X of height ω_3 having ω_2 isolated points?*

Let us mention here that the problem of the existence of (λ^+, λ)-thin-tall spaces, i. e. LCS spaces of width λ and height λ^+, is mentioned in [7, Problem 6.4, p.53]. However, it is erroneously stated there that the existence of a (λ^+, λ)-thin-tall space follows from $\lambda^{<\lambda} = \lambda$ or from the existence of a λ^+-tree.

The authors would like to express their thanks to the referee for pointing out several errors in a previous version of the paper.

2. Constructing a space of height ω_2 with ω_1 isolated points

Definition 2.1. Given a family of sets $\mathcal{A}$ we define the topological space $X(\mathcal{A}) = \langle \mathcal{A}, \tau_{\mathcal{A}} \rangle$ as follows: $\tau_{\mathcal{A}}$ is the coarsest topology in which the sets $\mathrm{U}_{\mathcal{A}}(A) = \mathcal{A} \cap \mathcal{P}(A)$ are clopen for each $A \in \mathcal{A}$, in other words: $\{\mathrm{U}_{\mathcal{A}}(A), \mathcal{A} \setminus \mathrm{U}_{\mathcal{A}}(A) : A \in \mathcal{A}\}$ is a subbase for $\tau_{\mathcal{A}}$.

We shall write $\mathrm{U}(A)$ instead of $\mathrm{U}_{\mathcal{A}}(A)$ if $\mathcal{A}$ is clear from the context.

Clearly $X(\mathcal{A})$ is a 0-dimensional T_2-space. A family $\mathcal{A}$ is called *well-founded* iff $\langle \mathcal{A}, \subset \rangle$ is well-founded. In this case we can define the rank-function $\mathrm{rk} : \mathcal{A} \longrightarrow On$ as usual:

$$\mathrm{rk}(A) = \sup\{\mathrm{rk}(B) + 1 : B \in \mathcal{A} \ \wedge \ B \subsetneq A\},$$

and write $\mathrm{R}_\alpha(\mathcal{A}) = \{A \in \mathcal{A} : \mathrm{rk}(A) = \alpha\}$.

The family $\mathcal{A}$ is said to be $\cap$-*closed* iff $A \cap B \in \mathcal{A} \cup \{\emptyset\}$ whenever $A, B \in \mathcal{A}$.

It is easy to see that if $\mathcal{A}$ is $\cap$-closed, then a neighbourhood base in $X(\mathcal{A})$ of $A \in \mathcal{A}$ is formed by the sets

$$\mathrm{W}(A; B_1, \dots, B_n) = \mathrm{U}(A) \setminus \bigcup_1^n \mathrm{U}(B_i),$$

where $n \in \omega$ and $B_i \subsetneq A$ for $i = 1, \dots, n$. (For $n = 0$ we have $\mathrm{W}(A) = \mathrm{U}(A)$.)

The following simple result enables us to obtain LCS spaces from certain families of sets. Let us point out, however, that not every LCS space is obtainable in this manner, but we do not dwell upon this because we will not need it.

Lemma 2.2. *Assume that $\mathcal{A}$ is both $\cap$-closed and well-founded. Then $X(\mathcal{A})$ is an LCS space.*

Proof. Given a non-empty subset $\mathcal{Y}$ of $\mathcal{A}$ let A be a $\subset$-minimal element of $\mathcal{Y}$. Then $\mathrm{U}(A) \cap \mathcal{Y} = \{A\}$, i.e. A is isolated in $\mathcal{Y}$. Thus $X(\mathcal{A})$ is scattered.

Next we prove that every $\mathrm{U}(A)$ is compact by well-founded induction on $\langle \mathcal{A}, \subset \rangle$. Assume that $\mathrm{U}(B)$ is compact for each $B \subsetneq A$. By Alexander's subbase lemma it is enough to prove that any cover of $\mathrm{U}(A)$ with subbase elements contains a finite subcover. So let $\mathcal{B}, \mathcal{C} \subset \mathcal{A}$ be such that

$$\mathrm{U}(A) \subset \bigcup_{B \in \mathcal{B}} \mathrm{U}(B) \cup \bigcup_{C \in \mathcal{C}} (\mathcal{A} \setminus \mathrm{U}(C)).$$

If $A \in \mathrm{U}(B)$ for some $B \in \mathcal{B}$ then $A \subset B$ and so $\mathrm{U}(A) \subset \mathrm{U}(B)$.

Hence we can assume that $A \in \mathcal{A} \setminus \mathrm{U}(C)$, i.e. $A \not\subset C$ for some $C \in \mathcal{C}$. If $A \cap C = \emptyset$ then $\mathrm{U}(A) \setminus \{\emptyset\} \subset \mathcal{A} \setminus \mathrm{U}(C)$, and we are clearly done. So we can assume that $A \cap C \neq \emptyset$, and consequently $A \cap C \in \mathcal{A}$. Then $\mathrm{U}(A) \setminus (\mathcal{A} \setminus \mathrm{U}(C)) = \mathrm{U}(A) \cap \mathrm{U}(C) = \mathrm{U}(A \cap C)$. Since $A \cap C \neq A$ the set $\mathrm{U}(A \cap C)$ is compact by the induction hypothesis, hence $\mathrm{U}(A \cap C)$ is covered by a finite subfamily $\mathcal{F}$ of $\{\mathrm{U}(B) : B \in \mathcal{B}\} \cup \{\mathcal{A} \setminus \mathrm{U}(D) : D \in \mathcal{C}\}$. Therefore $\mathcal{F} \cup \{\mathcal{A} \setminus \mathrm{U}(C)\}$ is a finite cover of $\mathrm{U}(A)$. Consequently $\mathrm{U}(A)$ is compact. □

To simplify notation, if $X(\mathcal{A})$ is scattered then we write $\mathrm{I}_\alpha(\mathcal{A}) = \mathrm{I}_\alpha(X(\mathcal{A}))$.

Clearly each minimal element of $\mathcal{A}$ is isolated in $X(\mathcal{A})$; more generally we have $\alpha \leq \mathrm{rk}(A)$ if $A \in \mathrm{I}_\alpha(\mathcal{A})$, as is shown by an easy induction on $\mathrm{rk}(A)$.

Example 2.3. Assume that $\langle T, \prec \rangle$ is a well-ordering, $\mathrm{tp}\,\langle T, \prec \rangle = \alpha$, and let $\mathcal{A}$ be the family of all initial segments of $\langle T, \prec \rangle$, i. e. $\mathcal{A} = \{T\} \cup \{T_x : x \in T\}$, where $T_x = \{t \in T : t \prec x\}$. Then $\mathcal{A}$ is well-founded, $\cap$-closed and it is easy to see that $X(\mathcal{A}) \cong \alpha + 1$, i.e. the space $X(\mathcal{A})$ is homeomorphic to the space of ordinals up to and including α.

Example 2.3 above shows that, in general, $\mathrm{R}_\alpha(\mathcal{A})$ and $\mathrm{I}_\alpha(\mathcal{A})$ may differ even for $\alpha = 0$. Indeed, if x is the successor of y in $\langle T, \prec \rangle$ then T_x is isolated in $X(\mathcal{A})$ because $\{T_x\} = W(T_x; T_y) = \mathrm{U}_{\mathcal{A}}(T_x) \setminus \mathrm{U}_{\mathcal{A}}(T_y)$ is open, but $\mathrm{rk}(T_x) = \mathrm{tp}(T_x) > 0$. However, for a wide class of families, the two kinds of levels do agree. Let us call a well-founded family $\mathcal{A}$ rk-*good* iff the following condition is satisfied:

$$\forall A \in \mathcal{A}\ \forall \alpha < \mathrm{rk}(A)\ |\{A' \in \mathcal{A} : A' \subset A \wedge \mathrm{rk}(A') = \alpha\}| \geq \omega.$$

Then we have the following result.

Lemma 2.4. *If $\mathcal{A}$ is well-founded, $\cap$-closed and* rk-*good then $\mathrm{I}_\alpha(\mathcal{A}) = \mathrm{R}_\alpha(\mathcal{A})$ for each α.*

Proof. We prove this by induction on α. Assume that $\mathrm{I}_\xi(\mathcal{A}) = \mathrm{R}_\xi(\mathcal{A})$ for all $\xi < \alpha$.

If $A \in \mathrm{R}_\alpha(\mathcal{A})$ then $\mathrm{U}(A) \setminus \{A\} \subset \bigcup_{\xi<\alpha} \mathrm{R}_\xi(\mathcal{A}) = \bigcup_{\xi<\alpha} \mathrm{I}_\xi(\mathcal{A})$ and so A is an isolated point of $\mathcal{A} \setminus \bigcup_{\xi<\alpha} \mathrm{I}_\xi(\mathcal{A})$, i.e. $A \in \mathrm{I}_\alpha(\mathcal{A})$. Thus we have $\mathrm{R}_\alpha(\mathcal{A}) \subset \mathrm{I}_\alpha(\mathcal{A})$.

Now assume that $A \in \mathrm{I}_\alpha(\mathcal{A}) \setminus \mathrm{R}_\alpha(\mathcal{A})$. Then by our above remark $\alpha < \mathrm{rk}(A)$, moreover there are $B_1, \dots B_n \in \mathrm{U}_{\mathcal{A}}(A) \setminus \{A\}$ such that

$$(\star) \qquad \mathrm{W}(A; B_1, \dots, B_n) \setminus \{A\} \subset \mathrm{I}_{<\alpha}(\mathcal{A}) = \mathrm{R}_{<\alpha}(\mathcal{A}).$$

Let $\eta = \max\{\alpha, \max_{i=1,\dots,n} \mathrm{rk}\, B_i\}$. Then $\eta < \mathrm{rk}(A)$, moreover we have $\mathrm{U}_{\mathcal{A}}(A) \cap \mathrm{R}_\eta(\mathcal{A}) \subset \{B_1, \dots B_n\}$ by $(\star)$, contradicting $|\,\mathrm{U}_{\mathcal{A}}(A) \cap \mathrm{R}_\eta(\mathcal{A})| \geq \omega$. Note that this argument is valid for $n = 0$ as well. Indeed, in this case we have $\mathrm{U}_{\mathcal{A}}(A) \setminus \{A\} \subset \mathrm{I}_{<\alpha}(\mathcal{A})$, moreover $\eta = \alpha$. Thus we have concluded that $\mathrm{I}_\alpha(\mathcal{A}) = \mathrm{R}_\alpha(\mathcal{A})$. $\square$

Example 2.5. For a fixed cardinal κ and any ordinal $\gamma < \kappa^+$ we define the family $\mathcal{E}_\gamma \subset \mathcal{P}(\kappa^\gamma)$ as follows:

$$\mathcal{E}_\gamma = \Big\{ \big[\kappa^{1+\alpha} \cdot \xi, \kappa^{1+\alpha} \cdot (\xi + 1)\big) : \ \alpha \leq \gamma, \ \kappa^{1+\alpha} \cdot (\xi+1) \leq \kappa^{1+\gamma} \Big\}.$$

Of course, throughout this definition exponentiation means ordinal exponentiation.

$\mathcal{E}_\gamma$ is clearly well-founded, $\cap$-closed, moreover $\mathrm{rk}\Big(\big[\kappa^{1+\alpha} \cdot \xi, \kappa^{1+\alpha} \cdot (\xi+1)\big)\Big) = \alpha$, hence $\mathcal{E}_\gamma$ is also rk-good. Consequently $X(\mathcal{E}_\gamma)$ is an LCS space of height $\gamma + 1$ in which the α^{th} level is $\Big\{\big[\kappa^{1+\alpha} \cdot \xi, \kappa^{1+\alpha} \cdot (\xi+1)\big) \ : \kappa^{1+\alpha} \cdot (\xi+1) \leq \kappa^{1+\gamma}\Big\}$, i. e. all levels except the top one are of size κ.

To get an LCS space of height κ^+ with "*few*" isolated points, our plan is to *amalgamate* the spaces $\{X(\mathcal{E}_\gamma) : \gamma < \kappa^+\}$ into one LCS space X in such a way that $|\,\mathrm{I}_0(X)| \leq \kappa^{<\kappa}$. The following definition describes a situation in which such an amalgamation can be done.

Definition 2.6. A system of families $\{\mathcal{A}_i : i \in I\}$ is called *coherent* iff $A \cap B \in \mathcal{A}_i \cup \{\emptyset\}$ whenever $\{i, j\} \in \big[I\big]^2$, $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$.

To simplify notation, we introduce the following convention. Whenever the system of families $\{\mathcal{A}_i : i \in I\}$ is given, we will write $\mathrm{U}_i(A)$ for $\mathrm{U}_{\mathcal{A}_i}(A)$, and τ_i for $\tau_{\mathcal{A}_i}$. If the family $\mathcal{A}$ is defined then we will write $\mathrm{U}(A)$ for $\mathrm{U}_{\mathcal{A}}(A)$, and τ for $\tau_{\mathcal{A}}$.

Lemma 2.7. *Assume that $\{\mathcal{A}_i : i \in I\}$ is a coherent system of well-founded, $\cap$-closed families and $\mathcal{A} = \cup\{\mathcal{A}_i : i \in I\}$. Then for each $i \in I$ and $A \in \mathcal{A}_i$ we have $\mathrm{U}(A) = \mathrm{U}_i(A)$, $\mathcal{A}$ is also well-founded and $\cap$-closed, moreover $\tau_i \restriction \mathrm{U}(A) = \tau \restriction \mathrm{U}(A)$. Consequently each $X(\mathcal{A}_i)$ is an open subspace of $X(\mathcal{A})$ and thus $\{X(\mathcal{A}_i) : i \in I\}$ forms an open cover of $X(\mathcal{A})$.*

Proof. Let $A \in \mathcal{A}_i$. Then it is clear from coherence that

$$\mathrm{U}_i(A) \subseteq \mathrm{U}(A) = \bigcup_{j \in I} \{B : B \in \mathcal{A}_j \wedge B \subset A\} \subseteq \mathrm{U}_i(A),$$

hence $\mathrm{U}_i(A) = \mathrm{U}(A)$.

Next let $B \in \mathcal{A}_j$. If $A \cap B = \emptyset$ then $\mathrm{U}(A) \cap \mathrm{U}(B) \subset \{\emptyset\}$. Now assume that $A \cap B \neq \emptyset$. Then, again by coherence, $A \cap B \in \mathcal{A}_i$ and we have

$$\begin{aligned} \mathrm{U}(A) \cap \mathrm{U}(B) = \mathrm{U}_i(A) \cap \mathrm{U}(B) &= \{C \in \mathcal{A}_i : C \subset A \wedge C \subset B\} \\ &= \{C \in \mathcal{A}_i : C \subset A \cap B\} = \mathrm{U}_i(A \cap B). \end{aligned}$$

In both cases $\mathrm{U}(A) \cap \mathrm{U}(B)$ is τ_i-open. Similarly we can see that $\mathrm{U}(A) \setminus \mathrm{U}(B) = \mathrm{U}_i(A) \setminus \mathrm{U}(B)$ is τ_i-open, hence the topologies $\tau_i \restriction \mathrm{U}(A)$ and $\tau \restriction \mathrm{U}(A)$ coincide.

To show that $\mathcal{A}$ is well-founded, assume that $\{A_n : n \in \omega\} \subset \mathcal{A}$ and $A_0 \supsetneq A_1 \supsetneq \dots$. If $A_0 \in \mathcal{A}_i$ then $\{A_n : n \in \omega\} \subset \mathcal{A}_i$ because $\mathrm{U}(A_0) = \mathrm{U}_i(A_0)$. Thus there is $n \in \omega$ with $A_m = A_n$ for each $m \geq n$ because $\mathcal{A}_i$ is well-founded. Finally, that $\mathcal{A}$ is $\cap$-closed is an easy consequence of coherence and the $\cap$-closedness of the families $\mathcal{A}_i$. □

Given a system of families $\{\mathcal{A}_i : i \in I\}$ we would like to construct a coherent system of families $\{\widehat{\mathcal{A}_i} : i \in I\}$ such that $\mathcal{A}_i$ and $\widehat{\mathcal{A}_i}$ are isomorphic for all $i \in I$. A sufficient condition for when this can be done will be given in lemma 2.9 below.

First, however, we need a definition. While reading it, one should remember that an ordinal is identified with the family of its proper initial segments.

Definition 2.8. Given a limit ordinal ρ and a family $\mathcal{A}$ with $\rho \subset \mathcal{A} \subset \mathcal{P}(\rho)$, let us define the family $\widehat{\mathcal{A}}$ as follows. Consider first the function $\mathrm{k}_{\mathcal{A}}$ on ρ determined by the formula $\mathrm{k}_{\mathcal{A}}(\eta) = \mathrm{U}_{\mathcal{A}}(\eta + 1)$ for $\eta \in \rho$ and put

$$\widehat{\mathcal{A}} = \{\mathrm{k}''_{\mathcal{A}}\, A : A \in \mathcal{A}\}.$$

Since $\rho \subset \mathcal{A}$, for each $\eta \in \rho$ we clearly have $\cup\, \mathrm{U}_{\mathcal{A}}(\eta) = \eta$ and so $\mathrm{k}_{\mathcal{A}}(\eta) = \mathrm{U}_{\mathcal{A}}(\eta + 1) \neq \mathrm{U}_{\mathcal{A}}(\xi + 1) = \mathrm{k}_{\mathcal{A}}(\xi)$ whenever $\{\eta, \xi\} \in \big[\rho\big]^2$. Consequently, $\mathrm{k}_{\mathcal{A}}$ is a bijection that yields an isomorphism between $\mathcal{A}$ and $\widehat{\mathcal{A}}$ (and so the spaces $X(\mathcal{A})$ and $X(\widehat{\mathcal{A}})$ are homeomorphic).

If the system of families $\{\mathcal{A}_i : i \in I\}$ is given, then we write k_i for $\mathrm{k}_{\mathcal{A}_i}$ for each $i \in I$.

If $\mathcal{A} \subset \mathcal{P}(\rho)$ and $\xi \leq \rho$ then we let

$$\mathcal{A} \restriction \xi = \{A \cap \xi : A \in \mathcal{A}\}.$$

For $\mathcal{A}_0 \neq \mathcal{A}_1 \subset \mathcal{P}(\rho)$ we let

$$\Delta(\mathcal{A}_0, \mathcal{A}_1) = \min\{\delta : \mathcal{A}_0 \restriction \delta \neq \mathcal{A}_1 \restriction \delta\}.$$

Clearly we always have $\Delta(\mathcal{A}_0, \mathcal{A}_1) \leq \rho$. If, in addition, $\rho + 1 \subset \mathcal{A}_0 \cap \mathcal{A}_1$, moreover both $\mathcal{A}_0$ and $\mathcal{A}_1$ are $\cap$-closed then we also have

$$\Delta(\mathcal{A}_0, \mathcal{A}_1) = \min\{\delta : \mathrm{U}_0(\delta) \neq \mathrm{U}_1(\delta)\},$$

because then $\mathcal{A}_i \restriction \delta = \mathrm{U}_i(\delta)$ whenever $i \in 2$ and $\delta \leq \rho$.

Lemma 2.9. *Assume that κ is a cardinal, $\{\mathcal{A}_i : i \in I\} \subset \mathcal{P}\mathcal{P}(\kappa)$ are $\cap$-closed families, $\kappa \dot{+} 1 \subset \mathcal{A}_i$ for each $i \in I$, and $\Delta(\mathcal{A}_i, \mathcal{A}_j)$ is a successor ordinal whenever $\{i, j\} \in \big[I\big]^2$. Then the system $\{\widehat{\mathcal{A}_i} : i \in I\}$ is coherent.*

Proof. Let $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$, where $\Delta(\mathcal{A}_i, \mathcal{A}_j) = \rho + 1$. Then $B \cap \rho \in \mathrm{U}_j(\rho) = \mathrm{U}_i(\rho)$ by the choice of ρ and so $A \cap B \cap \rho \in \mathcal{A}_i \cup \{\emptyset\}$ because $\mathcal{A}_i$ is $\cap$-closed. Consequently we have

$$\begin{aligned}
&\mathrm{k}''_i\, A \cap \mathrm{k}''_j\, B = \\
&\qquad \big\{\mathrm{U}_i(\eta + 1) : \eta \in A \cap B \ \wedge\ \mathrm{U}_i(\eta + 1) = \mathrm{U}_j(\eta + 1)\big\} = \\
&\qquad\quad \big\{\mathrm{U}_i(\eta + 1) : \eta \in A \cap B \ \wedge\ \eta < \rho\big\} = \mathrm{k}''_i\,(A \cap B \cap \rho) \in \widehat{\mathcal{A}_i} \cup \{\emptyset\},
\end{aligned}$$

as required by the definition of coherence. □

More is needed still if we want the "amalgamated" family to provide us a space with a small bottom, i.e. having not too many isolated points. This will be made clear by the following lemma.

Lemma 2.10. *Let κ be a cardinal and $\{\mathcal{A}_i : i \in I\} \subset \mathcal{P}\mathcal{P}(\kappa)$ be a system of families such that*

(i) $\kappa \dot{+} 1 \subset \mathcal{A}_i$ *and $\mathcal{A}_i$ is well-founded and $\cap$-closed for each $i \in I$,*
(ii) $\Delta(\mathcal{A}_i, \mathcal{A}_j)$ *is a successor ordinal for each $\{i, j\} \in \left[I\right]^2$.*

Then

(a) *the system $\{\widehat{\mathcal{A}_i} : i \in I\}$ is coherent and thus $\mathcal{A} = \bigcup\{\widehat{\mathcal{A}_i} : i \in I\}$ is well-founded, $\cap$-closed and $X(\mathcal{A})$ is covered by its open subspaces $\{X(\widehat{\mathcal{A}_i}) : i \in I\}$.*

If, in addition, we also have

(iii) $\mathrm{I}_0(\mathcal{A}_i) \subset \left[\kappa\right]^{<\kappa}$ *for each $i \in I$,*

and

(iv) $|\,\mathrm{U}_i(\xi)| < \kappa$ *for each $i \in I$ and $\xi \in \kappa$,*

then

(b) $\mathrm{I}_0(\mathcal{A}) \subset \left[\left[\left[\kappa\right]^{<\kappa}\right]^{<\kappa}\right]^{<\kappa}$.

Proof of lemma 2.10. The system $\{\widehat{\mathcal{A}_i} : i \in I\}$ is coherent by lemma 2.9, thus (a) holds by lemma 2.7.

Consequently we have

$$\mathrm{I}_0(\mathcal{A}) = \bigcup\{\mathrm{I}_0(\widehat{\mathcal{A}_i}) : i \in I\}.$$

Now if $A \in \mathrm{I}_0(\mathcal{A}_i)$ and $\eta \in \kappa$ then $|A| < \kappa$ by (iii) and $\mathrm{U}_i(\eta) \in \left[\left[\kappa\right]^{<\kappa}\right]^{<\kappa}$ by (iv), hence

$$\mathrm{k}_i''\, A = \{\mathrm{U}_i(\eta + 1) : \eta \in A\} \in \left[\left[\left[\kappa\right]^{<\kappa}\right]^{<\kappa}\right]^{<\kappa}.$$

This, by $\mathrm{I}_0(\widehat{\mathcal{A}_i}) = \{\mathrm{k}_i''\, A : A \in \mathrm{I}_0(\mathcal{A}_i)\}$, proves (b). □

Definition 2.11. If ρ is an ordinal and $\mathcal{A} \subset \mathcal{P}(\rho)$ let us put

$$\mathcal{A}^* = \{A \cap \xi : A \in \mathcal{A} \wedge \xi \le \rho\} = \mathcal{A} \cup \{A \cap \xi : A \in \mathcal{A} \wedge \xi < \rho\}.$$

Definition 2.12. A family $\mathcal{A}$ is called *chain-closed* if for each non-empty $\mathcal{B} \subset \mathcal{A}$ if $\mathcal{B}$ is ordered by $\subset$ (i.e. if $\mathcal{B}$ is a chain) then $\cup\mathcal{B} \in \mathcal{A}$.

Lemma 2.13. *If ρ is an ordinal and $\mathcal{A}_0, \mathcal{A}_1 \subset \mathcal{P}(\rho)$ are chain-closed, $\cap$-closed and well-founded families such that ${\mathcal{A}_0}^* \ne {\mathcal{A}_1}^*$ then $\Delta({\mathcal{A}_0}^*, {\mathcal{A}_1}^*)$ is a successor ordinal.*

Proof. Assume that δ is a limit ordinal and ${\mathcal{A}_0}^* \restriction \gamma = {\mathcal{A}_1}^* \restriction \gamma$ for all $\gamma < \delta$. We want to show that ${\mathcal{A}_0}^* \restriction \delta = {\mathcal{A}_1}^* \restriction \delta$. Since ${\mathcal{A}_i}^* \restriction \delta = \bigcup_{\xi \le \delta} \mathcal{A}_i \restriction \xi$ and $\bigcup_{\xi < \delta} \mathcal{A}_i \restriction \xi = \bigcup_{\xi < \delta} {\mathcal{A}_i}^* \restriction \xi$, moreover $\emptyset \in {\mathcal{A}_0}^* \cap {\mathcal{A}_1}^*$, it is enough to show that $(\mathcal{A}_0 \restriction \delta) \setminus \{\emptyset\} = (\mathcal{A}_1 \restriction \delta) \setminus \{\emptyset\}$.

So assume that $A \in \mathcal{A}_0$ with $\sup(A \cap \delta) = \delta$ and verify that then $A \cap \delta \in \mathcal{A}_1 \restriction \delta$.

For each $\gamma \in A \cap \delta$ let $\mathcal{B}_\gamma = \{B \in \mathcal{A}_1 : B \cap (\gamma + 1) = A \cap (\gamma + 1)\}$. Since $A \cap (\gamma + 1) \in {\mathcal{A}_0}^* \restriction (\gamma + 1) = {\mathcal{A}_1}^* \restriction (\gamma + 1)$, there is $B \in \mathcal{A}_1$ and $\xi \le \kappa$ such that $A \cap (\gamma + 1) = (B \cap \xi) \cap (\gamma + 1)$. Since $\gamma \in A \cap (\gamma + 1)$ it follows that $\xi > \gamma$ and so

$B\cap(\gamma+1)=(B\cap\xi)\cap(\gamma+1)$. Thus $B\in\mathcal{B}_\gamma$, i.e. $\mathcal{B}_\gamma\neq\emptyset$. Let B_γ be the $\subset$-minimal element of $\mathcal{B}_\gamma$, which exists because $\mathcal{A}_1$ is both $\cap$-closed and well-founded.

Then $\{B_\gamma:\gamma\in A\cap\delta\}$ is a chain because $B_{\gamma'}\in\mathcal{B}_\gamma$ for $\gamma<\gamma'<\delta$, $\gamma,\gamma'\in A$. Thus $B=\cup\{B_\gamma:\gamma\in A\cap\delta\}\in\mathcal{A}_1$ and clearly $A\cap\delta=B\cap\delta$ because $\sup(\cap\delta)A=\delta$. □

The last result shows us that the operation * is useful because its application yields us families that satisfy condition (ii) of lemma 2.10. On the other hand, the following result tells us that the LCS spaces associated with certain families modified by * do not differ significantly from the spaces given by the original families, moreover they also satisfy condition (iii) of lemma 2.10.

Lemma 2.14. *Let κ be a cardinal and $\mathcal{A}\subset\left[\kappa\right]^{\kappa}$ be well-founded and $\cap$-closed. Then so is $\mathcal{A}^*$, moreover*

(a) $X(\mathcal{A})$ is a closed subspace of $X(\mathcal{A}^*)$,
(b) $\mathrm{I}_0(\mathcal{A})\subseteq\mathrm{I}_\kappa(\mathcal{A}^*)$.
(c) $\mathrm{ht}(\mathcal{A}^*)\geq\kappa\dot{+}\mathrm{ht}(\mathcal{A})$,
(d) $\mathrm{I}_0(\mathcal{A}^*)\subset\left[\kappa\right]^{<\kappa}$.

Proof of lemma 2.14. We shall write $\mathrm{U}(A)$ for $\mathrm{U}_{\mathcal{A}}(A)$, and $\mathrm{U}_*(A)$ for $\mathrm{U}_{\mathcal{A}^*}(A)$.

First observe that because

$$\mathrm{U}_*(A)\cap\mathcal{A}=\begin{cases}\mathrm{U}(A) & \text{if } A\in\mathcal{A},\\ \emptyset & \text{if } A\in\mathcal{A}^*\setminus\mathcal{A},\end{cases}$$

$X(\mathcal{A})$ is a closed subspace of $X(\mathcal{A}^*)$, hence (a) holds.

Now let $A\in\mathrm{I}_0(\mathcal{A})$. Then there are $B_1,\ldots,B_n\in\mathrm{U}(A)\setminus\{A\}$ such that

$$\{A\}=\mathrm{W}(A;B_1,\ldots,B_n)=\mathrm{U}(A)\setminus\bigcup_{i=1}^{n}\mathrm{U}(B_i).$$

Since here $B_i\subsetneq A$ and $|A|=\kappa$, we can fix $\eta\in A$ such that $(A\cap\eta)\not\subset B_i$ for every $i=1,\ldots,n$.

Now consider the basic neighbourhood

$$\mathcal{Z}=\mathrm{W}_*(A;A\cap\eta,B_1,\ldots,B_n)=\mathrm{U}_*(A)\setminus\mathrm{U}_*(A\cap\eta)\setminus\bigcup_{i=1}^{n}\mathrm{U}_*(B_i)$$

of A in $X(\mathcal{A}^*)$. We claim that $\mathcal{Z}=\{A\cap\xi:\eta<\xi\leq\kappa\}$. The inclusion $\supset$ is clear from the choice of η. On the other hand, if $C\cap\xi\in\mathcal{Z}$ with $C\in\mathcal{A}$ and $\xi\leq\kappa$, then $C\cap\xi\subset A$ hence $C\cap\xi=A\cap C\cap\xi$, so as $\mathcal{A}$ is $\cap$-closed we can assume that $C\subseteq A$. If we had $C\neq A$ then $\{A\}=\mathrm{W}(A;B_1,\ldots,B_n)$ would imply $C\subset B_i$ for some i, hence $C\cap\xi\in\mathrm{U}_*(B_i)$ and so $C\cap\xi\notin\mathcal{Z}$, a contradiction, thus we must have $C=A$. Moreover, since $\mathrm{U}_*(A\cap\eta)\supset\{A\cap\nu:\nu\leq\eta\}$, we must also have $\xi>\eta$.

By example 2.3 we have $X(\mathcal{Z})\cong\kappa+1$. Moreover, the topologies $\tau_{\mathcal{Z}}$ and $\tau_{\mathcal{A}^*}\restriction\mathcal{Z}$ coincide because the above argument also shows that for each $C\in\mathcal{A}$ and $\zeta\leq\kappa$ we have

$$\mathrm{U}_*(C\cap\zeta)\cap\mathcal{Z}=\mathrm{U}_*(A\cap C\cap\zeta)\cap\mathcal{Z}=\begin{cases}\mathrm{U}_{\mathcal{Z}}(A\cap\zeta) & \text{if } A\subset C \text{ and } \zeta>\eta;\\ \emptyset & \text{otherwise.}\end{cases}$$

Hence $X(\mathcal{Z})\cong\kappa\dot{+}1$ is a clopen subspace of $X(\mathcal{A}^*)$ and so $\{A\}=\mathrm{I}_\kappa(\mathcal{Z})=\mathrm{I}_\kappa(\mathcal{A}^*)\cap\mathcal{Z}$, what proves (b).

(c) follows immediately from (a) and (b).

Finally, $\mathrm{I}_0(\mathcal{A}^*) \subset \mathrm{I}_{<\kappa}(\mathcal{A}^*) \subset (\mathcal{A}^* \setminus \mathcal{A}) \subset [\kappa]^{<\kappa}$, as follows immediately from (b), proving (d). $\square$

Before we could apply the amalgamation result to the families $\mathcal{E}_\gamma$, however, we need some further preparation that will be useful in ensuring the fulfillment of condition (iv) in 2.10.

Definition 2.15. A family $\mathcal{A}$ is called *tree-like* iff $A \cap A' \neq \emptyset$ implies that $A \subset A'$ or $A' \subset A$, whenever $A, A' \in \mathcal{A}$.

It is easy to see that the families $\mathcal{E}_\gamma$ given in example 2.5 are both tree-like and chain-closed. Also, tree-like families are clearly $\cap$-closed.

Lemma 2.16. *If δ is an ordinal and $\mathcal{A} \subset \mathcal{P}(\delta)$ is tree-like, well-founded and chain-closed then so is $\mathcal{A}\restriction\xi$ for each $\xi \leq \delta$.*

Proof of lemma 2.16. It is obvious that $\mathcal{A}\restriction\xi$ is tree-like. To show that $\mathcal{A}\restriction\xi$ is chain-closed, let $\emptyset \neq \mathcal{B} \subset \mathcal{A}\restriction\xi$ be ordered by $\subset$. If $\mathcal{B} = \{\emptyset\}$ then $\cup\mathcal{B} = \emptyset \in \mathcal{A}\restriction\xi$. If, however, $\mathcal{B} \neq \{\emptyset\}$ then put $\tilde{\mathcal{B}} = \{A \in \mathcal{A} : A \cap \xi \in \mathcal{B} \setminus \{\emptyset\}\}$. Since $\mathcal{A}$ is tree-like, $\tilde{\mathcal{B}}$ is also ordered by $\subset$ and clearly $\tilde{\mathcal{B}} \neq \emptyset$. So $\cup\tilde{\mathcal{B}} \in \mathcal{A}$ and $\cup\mathcal{B} = \cup\tilde{\mathcal{B}} \cap \xi \in \mathcal{A}\restriction\xi$, which was to be shown.

To show that $\mathcal{A}\restriction\xi$ is well-founded assume that $A_0 \cap \xi \supseteq A_1 \cap \xi \supseteq \dots$, where each $A_n \in \mathcal{A}$. If $A_n \cap \xi = \emptyset$ for some n, then we are done. Otherwise for each $n \in \omega$ we have $A_n \cap \xi = (\cap_{m \leq n} A_m) \cap \xi \neq \emptyset$, hence as $\mathcal{A}$ is $\cap$-closed we can assume that $A_0 \supseteq A_1 \supseteq \dots$. Since $\mathcal{A}$ is well-founded, there is n such that $A_m = A_n$, and so $A_m \cap \xi = A_n \cap \xi$ as well, for each $m \geq n$. $\square$

Definition 2.17. Given a family $\mathcal{A} \subset \mathcal{P}(\delta)$ and $\alpha, \beta \in \delta$ let us put

$$\mathrm{S}^{\mathcal{A}}(\alpha, \beta) = \cup\{A \in \mathcal{A} : \alpha \in A \text{ and } \beta \notin A\}.$$

Lemma 2.18. *Assume that δ is an ordinal and $\mathcal{A} \subset \mathcal{P}(\delta)$ is a tree-like, well-founded and chain-closed family with $\delta \in \mathcal{A}$. Then*

$$\mathcal{A} \setminus \{\emptyset\} = \{\delta\} \cup \{\mathrm{S}^{\mathcal{A}}(\alpha, \beta) : \alpha, \beta \in \delta\} \setminus \{\emptyset\}.$$

Consequently, $|\mathcal{A}| \leq |\delta|^2$.

Proof of lemma 2.18. Given $\alpha, \beta \in \delta$, the family $\mathcal{S} = \{A \in \mathcal{A} : \alpha \in A, \beta \notin A\}$ is ordered by $\subset$ because $\mathcal{A}$ is tree-like. Thus either $\mathcal{S} = \emptyset$ and so $\mathrm{S}^{\mathcal{A}}(\alpha, \beta) = \cup\mathcal{S} = \emptyset$, or if $\mathcal{S} \neq \emptyset$ then $\mathrm{S}^{\mathcal{A}}(\alpha, \beta) = \cup\mathcal{S} \in \mathcal{A}$, for $\mathcal{A}$ is chain-closed.

Assume now that $A \in \mathcal{A} \setminus \{\emptyset, \delta\}$ and let $\mathcal{D} = \{D \in \mathcal{A} : A \subsetneq D\}$. Clearly $\delta \in \mathcal{D}$. Since $\mathcal{A}$ is tree-like, $\mathcal{D}$ is ordered by $\subset$, so it has a $\subset$-least element, say D, because $\langle \mathcal{A}, \subset \rangle$ is also well-founded. Pick $\beta \in D \setminus A$ and let $\alpha \in A$. We claim that $A = \mathrm{S}^{\mathcal{A}}(\alpha, \beta)$. Clearly $A \subset \mathrm{S}^{\mathcal{A}}(\alpha, \beta)$ because $\alpha \in A$ and $\beta \notin A$. On the other hand, if $A' \in \mathcal{A}$, $\alpha \in A'$ and $\beta \notin A'$ then either $A' \subset A$ or $A \subset A'$ because $\mathcal{A}$ is tree-like. But $\beta \notin A'$ implies that $A' \notin \mathcal{D}$, i.e. $A \subsetneq A'$ can not hold. Thus $A' \subset A$ and so $\mathrm{S}^{\mathcal{A}}(\alpha, \beta) = A$ is proved. $\square$

Now we are ready to collect the fruits of all the preparatory work.

Theorem 2.19. *If $\kappa^{<\kappa} = \kappa$ then there is an LCS space X of height κ^+ with $|\mathrm{I}_0(X)| = \kappa$.*

Proof of theorem 2.19. For each $\gamma < \kappa^+$ consider the well-founded, $\cap$-closed, rk-good family $\mathcal{E}_\gamma$ constructed in example 2.5:

$$\mathcal{E}_\gamma = \Big\{ \big[\kappa^{1+\alpha} \cdot \xi, \kappa^{1+\alpha} \cdot (\xi+1)\big) : \ \alpha \le \gamma, \ \kappa^{1+\alpha} \cdot \xi < \kappa^\gamma \Big\}.$$

Fix a bijection $f_\gamma : \kappa^\gamma \longrightarrow \kappa$, and let $\mathcal{F}_\gamma = \{ f_\gamma{}'' E : E \in \mathcal{E}_\gamma\}$, i.e. $\mathcal{F}_\gamma$ is simply an isomorphic copy of $\mathcal{E}_\gamma$ on the underlying set κ. As $\mathcal{E}_\gamma$ is also chain-closed and tree-like, hence so is $\mathcal{F}_\gamma$.

We shall now show that the *-modified families $\{ {\mathcal{F}_\gamma}^* : \gamma < \kappa^+\}$ satisfy conditions (i)-(iv) of lemma 2.10. Since $\kappa \in \mathcal{F}_\gamma$ it follows that $\kappa \dot{+} 1 \subset {\mathcal{F}_\gamma}^*$ and so (i) is true. For $\{\gamma, \delta\} \in \big[\kappa^+\big]^2$, the height of $X(\mathcal{E}_\gamma)$ is $\gamma+1$ and the height of $X(\mathcal{E}_\delta)$ is $\delta+1$, hence $\mathcal{E}_\gamma$ and $\mathcal{E}_\delta$ are not isomorphic. Thus $\mathcal{F}_\gamma \ne \mathcal{F}_\delta$ and so ${\mathcal{F}_\gamma}^* \ne {\mathcal{F}_\delta}^*$ as well because $\mathcal{F}_\gamma = {\mathcal{F}_\gamma}^* \cap \big[\kappa\big]^\kappa$ and $\mathcal{F}_\delta = {\mathcal{F}_\delta}^* \cap \big[\kappa\big]^\kappa$. Hence $\Delta({\mathcal{F}_\gamma}^*, {\mathcal{F}_\delta}^*)$ is a successor ordinal by lemma 2.13, i.e. (ii) is satisfied.

(iii) holds by 2.14.(d.)

To show (iv), let us fix $\xi < \kappa$. Then $\mathrm{U}_{{\mathcal{F}_\gamma}^*}(\xi) = {\mathcal{F}_\gamma}^* \restriction \xi = \cup\{\mathcal{F}_\gamma \restriction \zeta : \zeta \le \xi\}$ where $|\mathcal{F}_\gamma \restriction \zeta| \le |\zeta|^2$ for all $\zeta \le \xi$ by lemmas 2.16 and 2.18, consequently $|{\mathcal{F}_\gamma}^* \restriction \xi| \le |\xi|^3 < \kappa$.

Thus we may apply lemma 2.10 to the family $\mathcal{F} = \cup\{\widehat{{\mathcal{F}_\gamma}^*} : \gamma < \kappa^+\}$ and conclude that the space $X = X(\mathcal{F})$ is LCS, moreover $|\,\mathrm{I}_0(X)| \le \big((\kappa^{<\kappa})^{<\kappa}\big)^{<\kappa} = \kappa$. Since for every $\gamma \in \kappa^+$ the space $X({\mathcal{F}_\gamma}^*)$ is an open subspace of X, we have $\mathrm{ht}(X) \ge \mathrm{ht}(X({\mathcal{F}_\gamma}^*)) > \gamma$, consequently $\mathrm{ht}(X) \ge \kappa^+$. $\square$

In particular, if $2^\omega = \omega_1$ then the above result yields an LCS space X with $\mathrm{ht}(X) = \omega_2$ and $|\,\mathrm{I}_0(X)| = \omega_1$. That such a space also exist under $\neg$CH, hence in ZFC, follows from the following result.

Theorem 2.20. *For each $\alpha < (2^\omega)^+$ there is a locally compact, scattered space X_α with $|X_\alpha| = |\alpha| + \omega$, $\mathrm{ht}(X_\alpha) = \alpha$ and $|\,\mathrm{I}_0(X_\alpha)| = \omega$.*

Proof. We do induction on α. If $\alpha = \beta + 1$ then we let X_α be the 1-point compactification of the disjoint topological sum of countably many copies of X_β.

If α is limit then we first fix an almost disjoint family $\{A_\beta : \beta < \alpha\} \subset \big[\omega\big]^\omega$, which is possible by $|\alpha| \le 2^\omega$. Applying the inductive hypothesis, for each $\beta < \alpha$ we can also fix a locally compact scattered space X_β of cardinality $\omega + |\beta|$ and height β such that $\mathrm{I}_0(X_\beta) = A_\beta$ and $X_\beta \cap X_\gamma = A_\beta \cap A_\gamma$ for $\{\beta, \gamma\} \in \big[\alpha\big]^2$. Now amalgamate the spaces X_β as follows: consider the topological space $X = \langle \cup_{\beta<\alpha} X_\beta, \tau\rangle$ where τ is the topology generated by $\cup_{\beta<\alpha} \tau_{X_\beta}$. Since $A_\beta \cap A_\gamma$ is a finite and open subspace of both X_β and X_γ it follows that each X_β is an open subspace of X. Consequently, X is LCS with countably many isolated points, and $\mathrm{ht}(X) = \sup_{\beta<\alpha} \mathrm{ht}\, X_\beta = \alpha$. $\square$

Corollary 2.21. *There is, in ZFC, a locally compact scattered space of height ω_2 and having ω_1 isolated points.*

Proof. If $2^\omega = \omega_1$, then theorem 2.19 gives such a space.

If $2^\omega > \omega_1$ then $(2^\omega)^+ \ge \omega_3$ and so according to theorem 2.20 for each $\alpha < \omega_3$ there is locally compact, scattered space of height α and with countably many isolated points. $\square$

References

[1] J. E. Baumgartner, S. Shelah, *Remarks on superatomic Boolean algebras*, Ann. Pure Appl. Logic, 33 (l987), no. 2, 109-129.

[2] I. Juhász, W. Weiss, *On thin-tall scattered spaces*, Colloquium Mathematicum, vol XL (1978) 63–68.

[3] W. Just, *Two consistency results concerning thin-tall Boolean algebras* Algebra Universalis 20(1985) no.2, 135–142.

[4] K. Kunen, *Set Theory*, North-Holland, New York, 1980.

[5] J. C. Martínez, *A forcing construction of thin-tall Boolean algebras*, Fundamenta Mathematicae, 159 (1999), no 2, 99-113.

[6] Judy Roitman, *Height and width of superatomic Boolean algebras*, Proc. Amer. Math. Soc. 94(1985), no 1, 9–14.

[7] S. Shelah *On what I do not understand (and have something to say) Part I*, Fundamenta Mathematicae, 166 (2000), no 1-2, pp 1-82.

Alfréd Rényi Institute of Mathematics
E-mail address: juhasz@renyi.hu

The Hebrew University of Jerusalem
E-mail address: shelah@math.huji.ac.il

Alfréd Rényi Institute of Mathematics
E-mail address: soukup@renyi.hu

Eötvös University of Budapest
E-mail address: zoli@renyi.hu