CARDINAL SEQUENCES OF LCS SPACES UNDER GCH

JUAN CARLOS MARTINEZ AND LAJOS SOUKUP

Abstract. Let $\mathcal{C}(\alpha)$ denote the class of all cardinal sequences of length α associated with compact scattered spaces. Also put

$$\mathcal{C}_\lambda(\alpha) = \{f \in \mathcal{C}(\alpha) : f(0) = \lambda = \min[f(\beta) : \beta < \alpha]\}.$$

If λ is a cardinal and $\alpha < \lambda^{++}$ is an ordinal, we define $\mathcal{D}_\lambda(\alpha)$ as follows: if $\lambda = \omega$,

$$\mathcal{D}_\omega(\alpha) = \{f \in {}^\alpha\{\omega, \omega_1\} : f(0) = \omega\},$$

and if λ is uncountable,

$$\mathcal{D}_\lambda(\alpha) = \{f \in {}^\alpha\{\lambda, \lambda^+\} : f(0) = \lambda, \\ f^{-1}\{\lambda\} \text{ is } < \lambda\text{-closed and successor-closed in } \alpha\}.$$

We show that for each uncountable regular cardinal λ and ordinal $\alpha < \lambda^{++}$ it is consistent with GCH that $\mathcal{C}_\lambda(\alpha)$ is as large as possible, i.e.

$$\mathcal{C}_\lambda(\alpha) = \mathcal{D}_\lambda(\alpha).$$

This yields that under GCH for any sequence f of regular cardinals of length α the following statements are equivalent:

(1) $f \in \mathcal{C}(\alpha)$ in some cardinal preserving and GCH-preserving generic-extension of the ground model.

(2) for some natural number n there are infinite regular cardinals $\lambda_0 > \lambda_1 > \cdots > \lambda_{n-1}$ and ordinals $\alpha_0, \ldots, \alpha_{n-1}$ such that $\alpha = \alpha_0 + \cdots + \alpha_{n-1}$ and $f = f_0 {}^\frown f_1 {}^\frown \cdots {}^\frown f_{n-1}$ where each $f_i \in \mathcal{D}_{\lambda_i}(\alpha_i)$.

The proofs are based on constructions of *universal* locally compact scattered spaces.

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1. Introduction

Given a locally compact scattered T_2 (in short : LCS) space X the α^{th} Cantor-Bendixson level will be denoted by $\mathrm{I}_\alpha(X)$. The *height of* X, $\mathrm{ht}(X)$, is the least ordinal α with $\mathrm{I}_\alpha(X) = \emptyset$. The *reduced height* $\mathrm{ht}^-(X)$ is the smallest ordinal α such that $\mathrm{I}_\alpha(X)$ is finite. Clearly, one has $\mathrm{ht}^-(X) \le \mathrm{ht}(X) \le \mathrm{ht}^-(X) + 1$. The *cardinal sequence* of X, denoted by $\mathrm{SEQ}(X)$, is the sequence of cardinalities of the infinite Cantor-Bendixson levels of X, i.e.

$$\mathrm{SEQ}(X) = \Big\langle\ |I_\alpha(X)| : \alpha < \mathrm{ht}(X)^-\ \Big\rangle.$$

A characterization in ZFC of the sequences of cardinals of length $\le \omega_1$ that arise as cardinal sequences of LCS spaces is proved in [4]. However, no characterization in ZFC is known for cardinal sequences of length $< \omega_2$.

For an ordinal α we let $\mathcal{C}(\alpha)$ denote the class of all cardinal sequences of length α of LCS spaces. We also put, for any fixed infinite cardinal λ,

$$\mathcal{C}_\lambda(\alpha) = \{s \in \mathcal{C}(\alpha) : s(0) = \lambda \wedge \forall\beta < \alpha\ [s(\beta) \ge \lambda]\}.$$

In [2], the authors show that a class $\mathcal{C}(\alpha)$ is characterized if the classes $\mathcal{C}_\lambda(\beta)$ are characterized for every infinite cardinal λ and every ordinal $\beta \le \alpha$. Then, they obtain under GCH a characterization of the classes $\mathcal{C}(\alpha)$ for any ordinal $\alpha < \omega_2$ by means of a a full description under GCH of the classes $\mathcal{C}_\lambda(\alpha)$ for any ordinal $\alpha < \omega_2$ and any infinite cardinal λ. The situation becomes, however, more complicated when we consider the class $\mathcal{C}(\omega_2)$. We can characterize under GCH the classes $\mathcal{C}_\lambda(\omega_2)$ for $\lambda > \omega_1$, by using the description given in [2] and the following simple observation.

Observation 1.1. *If $\lambda \ge \omega_2$, then $f \in \mathcal{C}_\lambda(\omega_2)$ iff $f \restriction \alpha \in \mathcal{C}_\lambda(\alpha)$ for each $\alpha < \omega_2$.*

Proof. If $\mathrm{SEQ}(X_\alpha) = f \restriction \alpha$ for $\alpha < \omega_2$ then take X as the disjoint union of $\{X_\alpha : \alpha < \omega_2\}$. Then $\mathrm{SEQ}(X) = f$ because for any $\beta < \omega_2$ we have $\mathrm{I}_\beta(X) = \bigcup\{\mathrm{I}_\beta(X_\alpha) : \beta < \alpha < \omega_2\}$ and so

$$|\,\mathrm{I}_\beta(X)| = \sum_{\beta<\alpha<\omega_2} |\,\mathrm{I}_\beta(X_\alpha)| = \omega_2 \cdot f(\beta) = f(\beta).$$

□

If α is any ordinal, a subset $L \subset \alpha$ is called *κ-closed in α*, where κ is an infinite cardinal, iff $\sup\langle\alpha_i : i < \kappa\rangle \in L \cup \{\alpha\}$ for each increasing sequence $\langle\alpha_i : i < \kappa\rangle \in {}^\kappa L$. The set L is *$< \lambda$-closed in α* provided it is κ-closed in α for each cardinal $\kappa < \lambda$. We say that L is *successor closed in* α if $\beta + 1 \in L \cup \{\alpha\}$ for all $\beta \in L$.

For a cardinal λ and ordinal $\delta < \lambda^{++}$ we define $\mathcal{D}_\lambda(\delta)$ as follows: if $\lambda = \omega$,

$$\mathcal{D}_\omega(\delta) = \{f \in {}^{\delta}\{\omega, \omega_1\} : f(0) = \omega\},$$

and if λ is uncountable,

$$\mathcal{D}_\lambda(\delta) = \{s \in {}^{\delta}\{\lambda, \lambda^+\} : s(0) = \lambda, \\ s^{-1}\{\lambda\} \text{ is } < \lambda\text{-closed and successor-closed in } \delta\}.$$

The observation 1.1 above left open the characterization of $\mathcal{C}_{\omega_1}(\omega_2)$ under GCH. In [2, Theorem 4.1] it was proved that if GCH holds then

$$\mathcal{C}_{\omega_1}(\delta) \subseteq \mathcal{D}_{\omega_1}(\delta),$$

and we have equality for $\delta < \omega_2$. In Theorem 1.3 we show that it is consistent with GCH that we have equality not only for $\delta = \omega_2$ but even for each $\delta < \omega_3$.

To formulate our results we need to introduce some more notation.

We shall use the notation $\langle\kappa\rangle_\alpha$ to denote the constant κ-valued sequence of length α. Let us denote the concatenation of a sequence f of length α and a sequence g of length β by $f^\frown g$ so that the domain of $f^\frown g$ is $\alpha + \beta$ and $f^\frown g(\xi) = f(\xi)$ for $\xi < \alpha$ and $f^\frown g(\alpha + \xi) = g(\xi)$ for $\xi < \beta$.

Definition 1.2. An LCS space X is called $\mathcal{C}_\lambda(\alpha)$-*universal* iff $\mathrm{SEQ}(X) \in \mathcal{C}_\lambda(\alpha)$ and for each sequence $s \in \mathcal{C}_\lambda(\alpha)$ there is an open subspace Y of X with $\mathrm{SEQ}(Y) = s$.

In this paper we prove the following result:

Theorem 1.3. *If κ is an uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$ then for each $\delta < \kappa^{++}$ there is a κ-complete κ^+-c.c poset P of cardinality κ^+ such that in V^P*

$$\mathcal{C}_\kappa(\delta) = \mathcal{D}_\kappa(\delta)$$

and there is a $\mathcal{C}_\kappa(\delta)$-universal LCS space.

How do the universal spaces come into the picture? The first idea to prove the consistency of $\mathcal{C}_\lambda(\alpha) = \mathcal{D}_\lambda(\alpha)$ is to try to carry out an iterated forcing. For each $f \in \mathcal{D}_\lambda(\alpha)$ we can try to find a poset P_f such that

$$1_{P_f} \Vdash \text{There is an LCS space } X_f \text{ with cardinal sequence } f.$$

Since typically $|X_f| = \lambda^+$, if we want to preserve the cardinals and CGH we should try to find a λ-complete, λ^+-c.c. poset P_f of cardinality λ^+. In this case forcing with P_f introduces λ^+ new subsets of λ because P_f has cardinality λ^+. However $|\mathcal{D}_\lambda(\alpha)| = \lambda^{++}$! So the length of the iteration is at least λ^{++}, hence in the final model the cardinal λ will have $\lambda^+ \cdot \lambda^{++} = \lambda^{++}$ many new subsets, i.e. $2^\lambda > \lambda^+$.

A $\mathcal{C}_\lambda(\delta)$-universal space has cardinality λ^+ so we may hope that there is a λ-complete, λ^+-c.c. poset P of cardinality λ^+ such that V^P contains a $\mathcal{C}_\lambda(\delta)$-universal space. In this case $(2^\lambda)^{V^P} \le ((|P|^\lambda)^\lambda)^V = \lambda^+$. So in the generic extension we might have GCH.

In this paper, we shall use the notion of a universal LCS space in order to prove Theorem 1.3. Further constructions of universal LCS spaces will be carried out in [6].

Problem 1.4. *Assume that s is a sequence of cardinals of length α, $s \notin \mathcal{C}(\alpha)$. Is it possible that there is a $|\alpha|^+$-Baire ($|\alpha|^+$-complete) poset P such that $s \in \mathcal{C}(\alpha)$ in V^P?*

For an ordinal $\delta < \kappa^{++}$ let $\mathcal{L}_\kappa^\delta = \{\alpha < \delta : \mathrm{cf}(\alpha) \in \{\kappa, \kappa^+\}\}$.

Definition 1.5. An LCS space X is called $\mathcal{L}_\kappa^\delta$-good iff X has a partition $X = Y \cup^* \bigcup^* \{Y_\zeta : \zeta \in \mathcal{L}_\kappa^\delta\}$ such that

(1) Y is an open subspace of X, $\mathrm{SEQ}(Y) = \langle\kappa\rangle_\delta$,
(2) $Y \cup Y_\zeta$ is an open subspace of X with $\mathrm{SEQ}(Y \cup Y_\zeta) = \langle\kappa\rangle_\zeta {}^\frown \langle\kappa^+\rangle_{\delta-\zeta}$.

Theorem 1.3 follows immediately from Theorem 1.6 and Proposition 1.7 below.

Theorem 1.6. *If κ is an uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$ then for each $\delta < \kappa^{++}$ there is a κ-complete κ^+-c.c poset $\mathcal{P}$ of cardinality κ^+ such that in $V^{\mathcal{P}}$ there is an $\mathcal{L}_\kappa^\delta$-good space.*

Proposition 1.7. *Let κ be an uncountable regular cardinal, $\delta < \kappa^{++}$ and X be an $\mathcal{L}_\kappa^\delta$-good space. Then for each $s \in \mathcal{D}_\kappa(\delta)$ there is an open subspace Z of X with $\mathrm{SEQ}(Z) = s$. Especially, under GCH an $\mathcal{L}_\kappa^\delta$-good space is $\mathcal{C}_\kappa(\delta)$-universal.*

Proof. Let $J = s^{-1}\{\kappa^+\} \cap \mathcal{L}_\kappa^\delta$. For each $\zeta \in J$ let

$$f(\zeta) = \min((\delta + 1) \setminus (s^{-1}\{\kappa^+\} \cup \zeta)).$$

Let

$$Z = Y \cup \bigcup \{\mathrm{I}_{<f(\zeta)}(Y \cup Y_\zeta) : \zeta \in J\}.$$

Since $Y \cup Y_\zeta$ is an open subspace of X it follows that $\mathrm{I}_{<f(\zeta)}(Y \cup Y_\zeta)$ is an open subspace of Z. Hence for every $\alpha < \delta$

$$
\begin{aligned}
(1)\quad \mathrm{I}_\alpha(Z) &= \mathrm{I}_\alpha(Y) \cup \bigcup\{\mathrm{I}_\alpha(\mathrm{I}_{<f(\zeta)}(Y \cup Y_\zeta)) : \zeta \in J\} \\
&= \mathrm{I}_\alpha(Y) \cup \bigcup\{\mathrm{I}_\alpha(Y \cup Y_\zeta) : \zeta \in J, \zeta \le \alpha < f(\zeta)\}.
\end{aligned}
$$

Since $[\zeta, f(\zeta)) \subset s^{-1}\{\kappa^+\}$ for $\zeta \in J$ it follows that if $s(\alpha) = \kappa$ then $\mathrm{I}_\alpha(Z) = \mathrm{I}_\alpha(Y)$, and so

$$(2)\qquad |\,\mathrm{I}_\alpha(Z)| = |\,\mathrm{I}_\alpha(Y)| = \kappa.$$

If $s(\alpha) = \kappa^+$, let $\zeta_\alpha = \min\{\zeta \le \alpha : [\zeta, \alpha] \subset s^{-1}\{\kappa^+\}\}$. Then $\zeta_\alpha \in J$ because $s(0) = \kappa$ and $s^{-1}\{\kappa\}$ is $< \kappa$-closed and successor-closed in δ. Thus $\zeta_\alpha \le \alpha < f(\zeta_\alpha)$ and so

$$(3)\qquad |\,\mathrm{I}_\alpha(Z)| \ge |\,\mathrm{I}_\alpha(Y \cup Y_{\zeta_\alpha})| = \kappa^+.$$

Since $|Z| \le |X| = \kappa^+$ we have $|\,\mathrm{I}_\alpha(Z)| = \kappa^+$. Thus $\mathrm{SEQ}(Z) = s$. □

Theorem 1.3 yields the following characterization:

Theorem 1.8. *Under GCH for any sequence f of regular cardinals of length α the following statements are equivalent:*

(A) $f \in \mathcal{C}(\alpha)$ in some cardinal preserving and GCH-preserving generic-extension of the ground model.

(B) for some natural number n there are infinite regular cardinals $\lambda_0 > \lambda_1 > \cdots > \lambda_{n-1}$ and ordinals $\alpha_0, \ldots, \alpha_{n-1}$ such that $\alpha = \alpha_0 + \cdots + \alpha_{n-1}$ and $f = f_0{}^\frown f_1{}^\frown \cdots {}^\frown f_{n-1}$ where each $f_i \in \mathcal{D}_{\lambda_i}(\alpha_i)$.

Proof. (A) clearly implies (B) by [2].

Assume now that (B) holds. Without loss of generality, we may suppose that $\lambda_{n-1} = \omega$. Since the notion of forcing defined in Theorem 1.3 preserves GCH, we can carry out a cardinal-preserving and GCH-preserving iterated forcing of length $n-1$, $\langle P_m : m < n-1\rangle$, such that for $m < n-1$

$$V^{P_m} \models \mathcal{C}_{\lambda_m}(\alpha_m) = \mathcal{D}_{\lambda_m}(\alpha_m).$$

Put $k = n-2$, $\beta = \alpha_0 + \cdots + \alpha_k$ and $g = f_0{}^\frown f_1{}^\frown \cdots {}^\frown f_k$. Since $f_m \in \mathcal{D}_{\lambda_m}(\alpha_m) \cap V$, in V^{P_k} we have $f_m \in \mathcal{C}_{\lambda_m}(\alpha_m)$ for each $m < n-1$. Hence in V^{P_k} we have $g \in \mathcal{C}(\beta)$ by [2, Lemma 2.2]. Also, by using [4, Theorem 9], we infer that $f_{n-1} \in \mathcal{C}(\alpha_{n-1})$ in ZFC. Then as $f = g^\frown f_{n-1}$, in V^{P_k} we have $f \in \mathcal{C}(\alpha)$ again by [2, Lemma 2.2]. □

Problem 1.9. **(1)** *Are (A) and (B) below equivalent under GCH for every sequence f of regular cardinals?*

(A) $f \in \mathcal{C}(\alpha)$.

(B) for some natural number n there are infinite regular cardinals $\lambda_0 > \lambda_1 > \cdots > \lambda_{n-1}$ and ordinals $\alpha_0, \ldots, \alpha_{n-1}$ such that $\alpha = \alpha_0 + \cdots + \alpha_{n-1}$ and $f = f_0 {}^\frown f_1 {}^\frown \cdots {}^\frown f_{n-1}$ where each $f_i \in \mathcal{D}_{\lambda_i}(\alpha_i)$.

(2) *Is it consistent with GCH that (A) and (B) above are equivalent for every sequence of regular cardinals?*

Juhász and Weiss proved in [3] that $\langle\omega\rangle_\delta \in \mathcal{C}(\delta)$ for each $\delta < \omega_2$.

Also, it was shown in [5] that for every specific regular cardinal κ it is consistent that $\langle\kappa\rangle_\delta \in \mathcal{C}(\delta)$ for each $\delta < \kappa^{++}$. However, the following problem is open:

Problem 1.10. *Is it consistent with GCH that $\langle\omega_1\rangle_\delta \in \mathcal{C}(\delta)$ for each $\delta < \omega_3$?*

2. Proof of theorem 1.6

This section is devoted to the proof of Theorem 1.6, so κ is an uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$, and $\delta < \kappa^{++}$ is an ordinal.

If $\alpha \le \beta$ are ordinals let

$$[\alpha, \beta) = \{\gamma : \alpha \le \gamma < \beta\}. \tag{4}$$

We say that I is an *ordinal interval* iff there are ordinals α and β with $I = [\alpha, \beta)$. Write $I^- = \alpha$ and $I^+ = \beta$.

If $I = [\alpha, \beta)$ is an ordinal interval let $\mathrm{E}(I) = \{\varepsilon^I_\nu : \nu < \mathrm{cf}(\beta)\}$ be a cofinal closed subset of I having order type $\mathrm{cf}\,\beta$ with $\alpha = \varepsilon^I_0$ and put

$$\mathcal{E}(I) = \{[\varepsilon^I_\nu, \varepsilon^I_{\nu+1}) : \nu < \mathrm{cf}\,\beta\} \tag{5}$$

provided β is a limit ordinal, and let $\mathrm{E}(I) = \{\alpha, \beta'\}$ and put

$$\mathcal{E}(I) = \{[\alpha, \beta'), \{\beta'\}\} \tag{6}$$

provided $\beta = \beta' + 1$.

Define $\{\mathcal{I}_n : n < \omega\}$ as follows:

$$\mathcal{I}_0 = \{[0, \delta)\} \text{ and } \mathcal{I}_{n+1} = \bigcup\{\mathcal{E}(I) : I \in \mathcal{I}_n\}. \tag{7}$$

Put $\mathbb{I} = \bigcup\{\mathcal{I}_n : n < \omega\}$. Note that $\mathbb{I}$ is a *cofinal tree of intervals* in the sense defined in [5]. Then, for each $\alpha < \delta$ we define

$$\mathrm{n}(\alpha) = \min\{n : \exists I \in \mathcal{I}_n \text{ with } I^- = \alpha\}, \tag{8}$$

and for each $\alpha < \delta$ and $n < \omega$ we define

$$\mathrm{I}(\alpha, n) \in \mathcal{I}_n \text{ such that } \alpha \in \mathrm{I}(\alpha, n). \tag{9}$$

Proposition 2.1. *Assume that $\zeta < \delta$ is a limit ordinal. Then, there is a $j(\zeta) \in \omega$ and an interval $J(\zeta) \in \mathcal{I}_{j(\zeta)}$ such that ζ is a limit point of $E(J(\zeta))$. Also, we have $\mathrm{n}(\zeta) - 1 \le j(\zeta) \le \mathrm{n}(\zeta)$, and $j(\zeta) = n(\zeta)$ if $cf(\zeta) = \kappa^+$.*

Proof. Clearly $j(\zeta)$ and $\mathrm{J}(\zeta)$ are unique if defined.

If there is an $I \in \mathcal{I}_{\mathrm{n}(\zeta)}$ with $I^+ = \zeta$ then $J(\zeta) = I$, and so $j(\zeta) = \mathrm{n}(\zeta)$. If there is no such I, then ζ is a limit point of $\mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1))$, so $J(\zeta) = \mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1)$ and $j(\zeta) = \mathrm{n}(\zeta) - 1$.

Assume now that $\mathrm{cf}(\zeta) = \kappa^+$. Then $\zeta \in \mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1))$, but $|\,\mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1)) \cap \zeta| \leq \kappa$, so ζ can not be a limit point of $\mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1))$. Therefore, it has a predecessor ξ in $\mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1))$, i.e $[\xi, \zeta) \in \mathcal{I}_{\mathrm{n}(\zeta)}$, and so $J(\zeta) = [\xi, \zeta)$ and $j(\zeta) = \mathrm{n}(\zeta)$. □

Example 2.2. Put $\delta = \omega_2 \cdot \omega_2 + 1$. We define

$E([0, \delta)) = \{0, \omega_2 \cdot \omega_2\}$,

$E([0, \omega_2 \cdot \omega_2)) = \{\omega_2 \cdot \xi : 0 \leq \xi < \omega_2\}$,

$E([\omega_2 \cdot \xi, \omega_2 \cdot (\xi + 1))) = \{\zeta : \omega_2 \cdot \xi \leq \zeta < \omega_2 \cdot (\xi + 1)\}$,

$E(\{\zeta\}) = \{\zeta\}$ for each $\zeta \leq \omega_2 \cdot \omega_2$.

Then, we have $\mathrm{n}(\omega_2 \cdot \omega_2) = 1$, $\mathrm{n}(\omega_2 \cdot \omega_1) = 2$, $\mathrm{n}(\omega_2 \cdot \omega_1 + \omega) = 3$. Also, we have $j(\omega_2 \cdot \omega_2) = \mathrm{j}(\omega_2 \cdot \omega_1) = 1$ and $J(\omega_2 \cdot \omega_2) = J(\omega_2 \cdot \omega_1) = [0, \omega_2 \cdot \omega_2)$.

If $\mathrm{cf}(J(\zeta)^+) \in \{\kappa, \kappa^+\}$, we denote by $\{\epsilon^\zeta_\nu : \nu < \mathrm{cf}(J(\zeta)^+)\}$ the increasing enumeration of $\mathrm{E}(J(\zeta))$, i.e. $\epsilon^\zeta_\nu = \varepsilon^{J(\zeta)}_\nu$ for $\nu < \mathrm{cf}(J(\zeta)^+)$.

Now if $\zeta < \delta$, we define the *basic orbit* of ζ (with respect to $\mathbb{I}$) as

$$\mathrm{o}(\zeta) = \bigcup \{(\mathrm{E}(\mathrm{I}(\zeta, m)) \cap \zeta) : m < \mathrm{n}(\zeta)\}. \tag{10}$$

Note that this is the notion of orbit used in [5] in order to construct by forcing an LCS space X such that $\mathrm{SEQ}(X) = \langle \kappa \rangle_\eta$ for any specific regular cardinal κ and any ordinal $\eta < \kappa^{++}$. However, this notion of orbit can not be used to construct an LCS space X such that $\mathrm{SEQ}(X) = \langle \kappa \rangle_{\kappa^+} {}^\frown \langle \kappa^+ \rangle$. To check this point, assume on the contrary that such a space X can be constructed by forcing from the notion of a basic orbit. Then, since the basic orbit of κ^+ is $\{0\}$, we have that if x, y are any two different elements of $I_{\kappa^+}(X)$ and U, V are basic neighbourhoods of x, y respectively, then $U \cap V \subset I_0(X)$. But then, we deduce that $|I_1(X)| = \kappa^+$.

However, we will show that a refinement of the notion of basic orbit can be used to proof Theorem 1.6.

If $\zeta < \delta$ with $\mathrm{cf}\,\zeta \geq \kappa$, we define the *extended orbit* of ζ by

$$\overline{\mathrm{o}}(\zeta) = \mathrm{o}(\zeta) \cup (\mathrm{E}(J(\zeta)) \cap \zeta). \tag{11}$$

Consider the tree of intervals defined in Example-2.2. Then, we have $\mathrm{o}(\omega_2 \cdot \omega_1) = \overline{o}(\omega_2 \cdot \omega_1) = \{\omega_2 \cdot \xi : 0 \leq \xi < \omega_1\}$, $o(\omega_2 \cdot \omega_2) = \{0\}$, $\overline{o}(\omega_2 \cdot \omega_2) = \{\omega_2 \cdot \xi : 0 \leq \xi < \omega_2\}$.

Note that if $\zeta < \delta$, the basic orbit of ζ is a set of cardinality at most κ (see [5, Proposition 1.3]). Then, it is easy to see that for any $\zeta < \delta$ with $\operatorname{cf}\zeta \geq \kappa$, the extended orbit of ζ is a cofinal subset of ζ of cardinality $\operatorname{cf}\zeta$.

In order to define the desired notion of forcing, we need some preparations. The underlying set of the desired space will be the union of a collection of blocks.

Let

$$\mathbb{B} = \{S\} \cup \{\langle\zeta, \eta\rangle : \zeta < \delta, \operatorname{cf}\zeta \in \{\kappa, \kappa^+\}, \eta < \kappa^+\}. \tag{12}$$

Let

$$B_S = \delta \times \kappa \tag{13}$$

and

$$B_{\zeta,\eta} = \{\langle\zeta, \eta\rangle\} \times [\zeta, \delta) \times \kappa \tag{14}$$

for $\langle\zeta, \eta\rangle \in \mathbb{B} \setminus \{S\}$.

Let

$$X = \bigcup\{B_T : T \in \mathbb{B}\}. \tag{15}$$

The underlying set of our space will be X. We should produce a partition $X = Y \cup^* \bigcup^*\{Y_\zeta : \zeta \in \mathcal{L}^\delta_\kappa\}$ such that

(1) Y is an open subspace of X with $\mathrm{SEQ}(Y) = \langle\kappa\rangle_\delta$,

(2) $Y \cup Y_\zeta$ is an open subspace of X with $\mathrm{SEQ}(Y \cup Y_\zeta) = \langle\kappa\rangle_\zeta {}^\frown \langle\kappa^+\rangle_{\delta-\zeta}$.

We will have $Y = B_S$, $Y_\zeta = \bigcup\{B_{\zeta,\eta} : \eta < \kappa^+\}$ for $\zeta \in \mathcal{L}^\delta_\kappa$.

Let

$$\pi : X \longrightarrow \delta \text{ such that } \begin{array}{l} \pi(\langle\alpha, \nu\rangle) = \alpha, \\ \pi(\langle\zeta, \eta, \alpha, \nu\rangle) = \alpha. \end{array} \tag{16}$$

Let

$$\pi_- : X \longrightarrow \delta \text{ such that } \begin{array}{l} \pi_-(\langle\alpha, \nu\rangle) = \alpha, \\ \pi_-(\langle\zeta, \eta, \alpha, \nu\rangle) = \zeta. \end{array} \tag{17}$$

Define

$$\pi_B : X \longrightarrow \mathbb{B} \text{ by the formula } x \in B_{\pi_B(x)}. \tag{18}$$

Define the *block orbit* function $\mathrm{o_B} : \mathbb{B} \setminus \{S\} \longrightarrow [\delta]^{\leq\kappa}$ as follows:

$$\mathrm{o_B}(\langle\zeta, \eta\rangle) = \begin{cases} \overline{\mathrm{o}}(\zeta) & \text{if } \operatorname{cf}\zeta = \kappa, \\ \mathrm{o}(\zeta) \cup \{\epsilon^\zeta_\nu : \nu < \eta\} & \text{if } \operatorname{cf}\zeta = \kappa^+. \end{cases} \tag{19}$$

That is, if $\operatorname{cf}\zeta = \kappa^+$ then

$$\mathrm{o}_{\mathrm{B}}(\langle\zeta,\eta\rangle) = \overline{\mathrm{o}}(\zeta) \cap \epsilon^{\zeta}_{\eta}.$$

Finally we define the *orbits* of the elements of X as follows:

(20)

$$\mathrm{o}^* : X \longrightarrow \big[\delta\big]^{\le\kappa} \text{ such that } \begin{array}{l} \mathrm{o}^*(\langle\alpha,\nu\rangle) = \mathrm{o}(\alpha), \\ \mathrm{o}^*(\langle\zeta,\eta,\alpha,\nu\rangle) = \mathrm{o}_{\mathrm{B}}(\langle\zeta,\eta\rangle) \cup (\mathrm{o}(\alpha) \setminus \zeta). \end{array}$$

Let $\Lambda \in \mathbb{I}$ and $\{x, y\} \in \big[X\big]^2$. We say that Λ *isolates x from y* if

(i) $\Lambda^- < \pi(x) < \Lambda^+$,
(ii) $\Lambda^+ \le \pi(y)$ provided $\pi_B(x) = \pi_B(y)$,
(iii) $\Lambda^+ \le \pi_-(y)$ provided $\pi_B(x) \ne \pi_B(y)$.

Now, we define the poset $\mathcal{P} = \langle P, \le\rangle$ as follows: $\langle A, \preceq, i\rangle \in P$ iff

(P1) $A \in \big[X\big]^{<\kappa}$.
(P2) $\preceq$ is a partial order on A such that $x \preceq y$ implies $x = y$ or $\pi(x) < \pi(y)$.
(P3) Let $x \preceq y$.
  (a) If $\pi_B(y) = \langle\zeta,\eta\rangle$ and $\zeta \le \pi(x)$ then $\pi_B(x) = \pi_B(y)$.
  (b) If $\pi_B(y) = \langle\zeta,\eta\rangle$ and $\zeta > \pi(x)$ then $\pi_B(x) = S$.
  (c) If $\pi_B(y) = S$ then $\pi_B(x) = S$.
(P4) $\mathrm{i} : \big[A\big]^2 \longrightarrow A \cup \{\mathrm{undef}\}$ such that for each $\{x, y\} \in \big[A\big]^2$ we have

$$\forall a \in A([a \preceq x \wedge a \preceq y] \text{ iff } a \preceq \mathrm{i}\{x, y\}).$$

(P5) $\forall\{x, y\} \in \big[A\big]^2$ if x and y are $\preceq$-incomparable but $\preceq$-compatible, then $\pi(\mathrm{i}\{x, y\}) \in \mathrm{o}^*(x) \cap \mathrm{o}^*(y)$.
(P6) Let $\{x, y\} \in [A]^2$ with $x \preceq y$. Then:
  (a) If $\pi_B(x) = S$ and $\Lambda \in \mathbb{I}$ isolates x from y, then there is $z \in A$ such that $x \preceq z \preceq y$ and $\pi(z) = \Lambda^+$.
  (b) If $\pi_B(x) \ne S$, $\pi(x) \ne \pi_-(x)$ and $\Lambda \in \mathbb{I}$ isolates x from y, then there is $z \in A$ such that $x \preceq z \preceq y$ and $\pi(z) = \Lambda^+$.

The ordering on P is the extension: $\langle A, \preceq, \mathrm{i}\rangle \le \langle A', \preceq', \mathrm{i}'\rangle$ iff $A' \subset A$, $\preceq' = \preceq \cap (A' \times A')$, and $\mathrm{i}' \subset \mathrm{i}$.

By using (P3), we obtain:

Claim 2.3. *Assume that x, y, z and Λ are as in (P6). Then we have:*
(a) If $\pi_B(x) = \pi_B(y)$, then $\pi_B(z) = \pi_B(x) = \pi_B(y)$.
(b) If $\pi_B(x) \ne \pi_B(y)$ and $\Lambda^+ < \pi_-(y)$, then $\pi_B(z) = \pi_B(x)$.
(c) If $\pi_B(x) \ne \pi_B(y)$ and $\Lambda^+ = \pi_-(y)$, then $\pi_B(z) = \pi_B(y)$.

Since $\kappa^{<\kappa} = \kappa$ implies $(\kappa^+)^{<\kappa} = \kappa^+$, we have that the cardinality of P is κ^+. Then, using the arguments of [5] it is enough to prove that Lemmas 2.4, 2.5 and 2.6 below hold.

Lemma 2.4. *$\mathcal{P}$ is κ-complete.*

Lemma 2.5. *$\mathcal{P}$ satisfies the κ^+-c.c.*

Lemma 2.6. *Assume that $p = \langle A, \preceq, \mathrm{i}\rangle \in P$, $x \in A$, and $\alpha < \pi(x)$. Then there is $p' = \langle A', \preceq', \mathrm{i}'\rangle \in P$ with $p' \leq p$ and there is $b \in A' \setminus A$ with $\pi(b) = \alpha$ such that $b \preceq' y$ iff $x \preceq y$ for $y \in A$.*

Since κ is regular, Lemma 2.4 clearly holds.

PROOF of Lemma 2.6. Let $\beta = \pi(x)$. Let K be a countable subset of $[\alpha, \beta)$ such that $\alpha \in K$ and $\mathrm{I}(\gamma, n)^+ \in K \cup [\beta, \delta)$ for $\gamma \in K$ and $n < \omega$. For each $\gamma \in K$ pick $b_\gamma \in X \setminus A$ such that $\pi(b_\gamma) = \gamma$ and
(1) if $\pi_B(x) = S$ then $\pi_B(b_\gamma) = S$.
(2) if $\pi_B(x) \neq S$ and $\gamma \geq \pi_-(x)$ then $\pi_B(b_\gamma) = \pi_B(x)$.
(3) if $\pi_B(x) \neq S$ and $\gamma < \pi_-(x)$ then $\pi_B(b_\gamma) = S$.

Let $A' = A \cup \{b_\gamma : \gamma \in K\}$,

$$\preceq' = \preceq \cup \{\langle b_\gamma, b_{\gamma'}\rangle : \gamma, \gamma' \in K, \gamma \leq \gamma'\} \cup \{\langle b_\gamma, z\rangle : \gamma \in K, z \in A, x \preceq z\}.$$

The definition of i' is straightforward because if $y \in A'$ and $\gamma \in K$ then either y and b_γ are $\preceq'$-comparable or they are $\preceq'$-incompatible.

Then $p' = \langle A', \preceq', i'\rangle$ and $b = b_\alpha$ satisfy the requirements. □

Finally we should prove Lemma 2.5.

Proof of Lemma 2.5. Assume that $\langle r_\nu : \nu < \kappa^+\rangle \subset P$ with $r_\nu \neq r_\mu$ for $\nu < \mu < \kappa^+$.

Write $r_\nu = \langle A_\nu, \preceq_\nu, \mathrm{i}_\nu\rangle$ and $A_\nu = \{x_{\nu,i} : i < \sigma_\nu\}$.

Since we are assuming that $\kappa^{<\kappa} = \kappa$, by thinning out $\langle r_\nu : \nu < \kappa^+\rangle$ by means of standard combinatorial arguments, we can assume the following:

(A) $\sigma_\nu = \sigma$ for each $\nu < \kappa^+$.
(B) $\{A_\nu : \nu < \kappa^+\}$ forms a Δ-system with kernel A.
(C) For each $\nu < \mu < \kappa^+$ there is an isomorphism $h = h_{\nu,\mu} : \langle A_\nu, \preceq_\nu, \mathrm{i}_\nu\rangle \longrightarrow \langle A_\mu, \preceq_\mu, \mathrm{i}_\mu\rangle$ such that for every $i < \sigma$ and $x, y \in A_\nu$ the following holds:
   (a) $h \restriction A = \mathrm{id}$.
   (b) $h(x_{\nu,i}) = x_{\mu,i}$.
   (c) $\pi_B(x) = \pi_B(y)$ iff $\pi_B(h(x)) = \pi_B(h(y))$.
   (d) $\pi_B(x) = S$ iff $\pi_B(h(x)) = S$.
   (e) $\pi(x) = \pi_-(x)$ iff $\pi(h(x)) = \pi_-(h(x))$.
   (f) if $\{x, y\} \in \left[A\right]^2$ then $\mathrm{i}_\nu\{x, y\} = \mathrm{i}_\mu\{x, y\}$.

Note that in order to obtain (C)(f) we use condition (P5) and the fact that $|o^*(x)| \leq \kappa$ for every $x \in A$. Also, we may assume the following:

(D) There is a partition $\sigma = K \cup^* F \cup^* L \cup^* D \cup^* M$ such that for each $\nu < \mu < \kappa^+$:
  (a) $\forall i \in K\ x_{\nu,i} \in A$ and so $x_{\nu,i} = x_{\mu,i}$. $A = \{x_{\nu,i} : i \in K\}$.
  (b) $\forall i \in F\ x_{\nu,i} \neq x_{\mu,i}$ but $\pi_B(x_{\nu,i}) = \pi_B(x_{\mu,i}) \neq S$.
  (c) $\forall i \in L\ \pi_B(x_{\nu,i}) \neq \pi_B(x_{\mu,i})$ but $\pi_-(x_{\nu,i}) = \pi_-(x_{\mu,i})$.
  (d) $\forall i \in D\ \pi_B(x_{\nu,i}) = S$ and $\pi(x_{\nu,i}) \neq \pi(x_{\mu,i})$.
  (e) $\forall i \in M\ \pi_B(x_{\nu,i}) \neq S$ and $\pi_-(x_{\nu,i}) \neq \pi_-(x_{\mu,i})$.

(E) If $\pi_B(x_{\nu,i}) = \pi_B(x_{\nu,j})$ then $\{i,j\} \in \big[K \cup D\big]^2 \cup \big[K \cup F\big]^2 \cup \big[L\big]^2 \cup \big[M\big]^2$.

It is well-known that if $\gamma < \kappa = \kappa^{<\kappa}$ then the following partition relation holds:

$$\kappa^+ \longrightarrow (\kappa^+, (\omega)_\gamma)^2.$$

Hence we can assume:

(F) If $\nu < \mu < \kappa^+$ then for each $i \in \sigma$ we have
  (a) $\pi(x_{\nu,i}) \leq \pi(x_{\mu,i})$,
  (b) $\pi_-(x_{\nu,i}) \leq \pi_-(x_{\mu,i})$.

By (F)(a) and (F)(b) the sequences $\{\pi(x_{\nu,i}) : \nu < \kappa^+\}$ and $\{\pi_-(x_{\nu,i}) : \nu < \kappa^+\}$ are increasing for each $i \in \sigma$, hence the following definition is meaningful:

For $i \in \sigma$ let

$$\delta_i = \begin{cases} \pi(x_{\nu,i}) & \text{if } i \in K, \\ \sup\{\pi(x_{\nu,i}) : \nu < \kappa^+\} & \text{if } i \in F \cup D, \\ \pi_-(x_{\nu,i}) & \text{if } i \in L, \\ \sup\{\pi_-(x_{\nu,i}) : \nu < \kappa^+\} & \text{if } i \in M. \end{cases}$$

By using Proposition 2.1, (C)(c) and condition (P3), we obtain:

Claim 2.7. *(a) If $i \in F \cup D \cup M$, then $cf(\delta_i) = \kappa^+$ and $sup(J(\delta_i)) = \delta_i$. Moreover for every $\nu < \kappa^+$ we have $\pi(x_{\nu,i}) < \delta_i$ if $i \in F \cup D$, and $\pi_-(x_{\nu,i}) < \delta_i$ if $i \in M$.*
(b) If $\{i,j\} \in [L]^2 \cup [M]^2$ and $x_{\nu,i} \prec_\nu x_{\nu,j}$ for $\nu < \kappa^+$, then $\delta_i = \delta_j$.

Indeed, (b) holds for large enough ν, and so (C)(c) implies that it holds for each ν.

We put

$$Z_0 = \{\pi_-(x_{\nu,i}) : i \in F \cup K, \pi_B(x_{\nu,i}) \neq S\} \cup \{\delta_i : i \in \sigma\}. \tag{21}$$

Since $\pi''A = \{\delta_i : i \in K\}$ we have $\pi''A \subset Z_0$. Then, we define Z as the closure of Z_0 with respect to $\mathbb{I}$:

$$Z = Z_0 \cup \{I^+ : I \in \mathbb{I}, I \cap Z_0 \neq \emptyset\}. \tag{22}$$

Since $|Z| < \kappa$, we can assume:

(G) $A = \{x_{\nu,i} : i \in K \cup F \cup D, \pi(x_{\nu,i}) \in Z\}$.

Equivalently,

$$\text{if } i \in F \cup D \text{ then } \pi(x_{\nu,i}) \notin Z. \tag{23}$$

Let us remark that for $i \in L \cup M$ we may have that $\pi(x_{\nu,i}) \in Z$.

Our aim is to show that there are $\nu < \mu < \kappa^+$ such that r_ν and r_μ are compatible. Note that if $x, y \in A$ with $x \neq y$ then, by (C)(f), we may assure that $i_\nu\{x,y\} = i_\mu\{x,y\}$. However, if $x \in A_\nu \setminus A$ and $y \in A_\mu \setminus A$ it may happen that for infinitely many $v \in A$ we have $v \preceq_\nu x$ and $v \preceq_\mu y$. Then, in order to amalgamate r_ν and r_μ in such a way that any pair of such elements has an infimum in the amalgamation, we will need to add new elements to $A_\nu \cup A_\mu$. Then, the next definitions will permit us to find suitable room for adding new elements to the domains of the conditions.

Let

$$\sigma_1 = \{i \in \sigma \setminus K : \mathrm{cf}(\delta_i) = \kappa\}$$

and

$$\sigma_2 = \{i \in \sigma \setminus K : \mathrm{cf}(\delta_i) = \kappa^+\}.$$

Assume that $i \in \sigma \setminus K$. Put $I_i = J(\delta_i)$. Let

$$\xi_i = \min\{\nu \in \mathrm{cf}\,\delta_i : \epsilon^{I_i}_\nu > \sup(\delta_i \cap Z)\}.$$

Then, if $i \in \sigma_1$ we put

$$\underline{\gamma}(\delta_i) = \epsilon^{I_i}_{\xi_i} \text{ and } \gamma(\delta_i) = \delta_i,$$

and if $i \in \sigma_2$ we put

$$\underline{\gamma}(\delta_i) = \epsilon^{I_i}_{\xi_i} \text{ and } \gamma(\delta_i) = \epsilon^{I_i}_{\xi_i+\kappa}.$$

Claim 2.8. *For each $i \in F \cup D \cup M$ there is $\nu_i < \kappa^+$ such that for all $\nu_i \leq \nu < \kappa^+$ we have:*

$$\textit{if } i \in F \cup D \textit{ then } \pi(x_{\nu,i}) \in J(\delta_i) \setminus \gamma(\delta_i) \tag{24}$$

and

$$\text{if } i \in M \text{ then } \pi_-(x_{\nu,i}) \in J(\delta_i) \setminus \gamma(\delta_i). \tag{25}$$

Proof. For $i \in F \cup D \cup M$ we have

$$\delta_i = \begin{cases} \sup\{\pi(x_{\nu,i}) : \nu < \kappa^+\} & \text{if } i \in F \cup D, \\ \sup\{\pi_-(x_{\nu,i}) : \nu < \kappa^+\} & \text{if } i \in M, \end{cases} \tag{26}$$

and $\gamma(\delta_i) < \sup(\mathrm{J}(\delta_i)) = \delta_i$. □

Claim 2.9. *For each $i \in L$ with $cf(\delta_i) = \kappa^+$ there is $\nu_i < \kappa^+$ such that for all $\nu_i \leq \nu < \kappa^+$, $o^*(x_{\nu,i}) \supset \overline{o}(\delta_i) \cap \gamma(\delta_i)$.*

Definition 2.10. r_ν is *good* iff

(i) $\forall i \in F \cup D$ $\pi(x_{\nu,i}) \in \mathrm{J}(\delta_i) \setminus \gamma(\delta_i)$.
(ii) $\forall i \in M$ $\pi_-(x_{\nu,i}) \in J(\delta_i) \setminus \gamma(\delta_i)$.
(iii) $\forall i \in L$ if $\mathrm{cf}\,\delta_i = \kappa^+$ then $\mathrm{o}^*(x_{\nu,i}) \supset \overline{o}(\delta_i) \cap \gamma(\delta_i)$.

Using Claims 2.8 and 2.9 we can assume:

(H) r_ν is good for $\nu < \kappa^+$.

By using (H), we will prove that r_ν and r_μ are compatible for $\{\nu, \mu\} \in [\kappa^+]^2$. First, we need to prove some fundamental facts.

By using (P3), (E) and (C)(c) we obtain:

Claim 2.11. *If $x_{\nu,i} \preceq_\nu x_{\nu,j}$ then either $\pi_B(x_{\nu,i}) = S$ or $\pi_B(x_{\nu,i}) = \pi_B(x_{\nu,j})$ and $\{i,j\} \in [K \cup F]^2 \cup [L]^2 \cup [M]^2$.*

Indeed, (P3) and (E) imply that Claim 2.11 holds for large enough ν, and then (C)(c) yields that it holds for each ν.

Claim 2.12. *If $x_{\nu,i} \preceq_\nu x_{\nu,j}$ then $\delta_i \leq \delta_j$.*

Proof. If $x_{\nu,i} \preceq_\nu x_{\nu,j}$ then $x_{\mu,i} \preceq_\mu x_{\mu,j}$ for each $\mu < \kappa^+$, and so we have:

(a) $\pi(x_{\mu,i}) \leq \pi(x_{\mu,j})$,
(b) $\pi_-(x_{\mu,i}) \leq \pi_-(x_{\mu,j})$,
(c) if $\pi_B(x_{\mu,i}) \neq \pi_B(x_{\mu,j})$ then $\pi(x_{\mu,i}) \leq \pi_-(x_{\mu,j})$.

Hence if $\pi_B(x_{\nu,i}) \neq \pi_B(x_{\nu,j})$ then

$$\delta_i = \sup\{\pi(x_{\mu,i}) : \mu < \kappa^+\} \leq \sup\{\pi_-(x_{\mu,j}) : \mu < \kappa^+\} \leq \delta_j. \tag{27}$$

If $\pi_B(x_{\nu,i}) = \pi_B(x_{\nu,j})$ then either $\{i,j\} \in \left[K \cup F\right]^2 \cup \left[K \cup D\right]^2$ and so

$$\delta_i = \sup\{\pi(x_{\mu,i}) : \mu < \kappa^+\} \leq \sup\{\pi(x_{\mu,j}) : \mu < \kappa^+\} = \delta_j, \tag{28}$$

or $\{i,j\} \in \left[L\right]^2 \cup \left[M\right]^2$ and so

$$\delta_i = \sup\{\pi_-(x_{\mu,i}) : \mu < \kappa^+\} \leq \sup\{\pi_-(x_{\mu,j}) : \mu < \kappa^+\} = \delta_j. \tag{29}$$

□

Claim 2.13. *Assume $i, j \in \sigma$. If $x_{\nu,i} \preceq_\nu x_{\nu,j}$ then either $\delta_i = \delta_j$ or there is $a \in A$ with $x_{\nu,i} \preceq_\nu a \preceq_\nu x_{\nu,j}$.*

Proof. Put $x_i = x_{\nu,i}, x_j = x_{\nu,j}$. Assume that $i, j \notin K$ and $\delta_i \neq \delta_j$. By Claim 2.12, we have $\delta_i < \delta_j$. Since $i \in L \cup M$ implies $\delta_i = \delta_j$, we have that $i \in F \cup D$, and so $\pi(x_i) < \delta_i$, $\operatorname{cf}(\delta_i) = \kappa^+$ and $J(\delta_i)^+ = \delta_i$. We distinguish the following cases:

Case 1. $i \in D$ and $j \in D \cup L \cup M$.

Since $\delta_i < \delta_j$, we have that $J(\delta_i)$ isolates x_i from x_j. Also, note that if $j \in L \cup M$, then $J(\delta_i)^+ = \delta_i < \pi_-(x_j)$. By (P6)(a), we infer that there is an $x = x_{\nu,k} \in A_\nu$ such that $\pi(x) = \delta_i$ and $x_i \prec_\nu x \prec_\nu x_j$. Now, by Claim 2.3(a)-(b), we deduce that $k \in K \cup D$. But as $\delta_i \in Z$, by (G), we have that $x \in A$, and so we are done.

Case 2. $i \in D$ and $j \in F$.

We have that $\pi_B(x_i) \neq \pi_B(x_j)$. By using (P3), we infer that $\delta_i \leq \pi_-(x_j)$, and so $J(\delta_i)$ isolates x_i from x_j. If $\delta_i < \pi_-(x_j)$, we proceed as in Case 1. So, assume that $\delta_i = \pi_-(x_j)$. By (P6)(a), we deduce that there is an $x = x_{\nu,k} \in A_\nu$ such that $\pi(x) = \delta_i$ and $x_i \prec_\nu x \prec_\nu x_j$. By Claim 2.3(c), we infer that $k \in K \cup F$. Then as $\delta_i \in Z$, we have that $x \in A$ by (G).

Case 3. $i, j \in F$.

We have that $\pi_B(x_i) = \pi_B(x_j) \neq S$ and $J(\delta_i)$ isolates x_i from x_j. Since $\pi_-(x_i) \in Z$ and we are assuming that $i \notin K$, we infer that $\pi(x_i) \neq \pi_-(x_i)$. Now, applying (P6)(b), we deduce that there is an $x = x_{\nu,k} \in A_\nu$ such that $\pi(x) = \delta_i$ and $x_i \prec_\nu x \prec_\nu x_j$. Now we deduce from Claim 2.3(a) that $k \in K \cup F$. Then as $\delta_i \in Z$, we have that $x \in A$ by (G). □

Claim 2.14. *If $x \in A$ and $y \in A_\nu$, and x and y are compatible but incomparable in r_ν, then* $\mathrm{i}_\nu\{x, y\} \in A$.

Proof. Indeed, $\pi(\mathrm{i}_\nu\{x, y\}) \in \mathrm{o}^*(x)$ by (P5) and $|\mathrm{o}^*(x)| \leq \kappa$. □

Claim 2.15. *Assume that $x_{\nu,i}$ and $x_{\nu,j}$ are compatible but incomparable in r_ν. Let $x_{\nu,k} = \mathrm{i}_\nu\{x_{\nu,i}, x_{\nu,j}\}$. Then either $x_{\nu,k} \in A$ or $\delta_i = \delta_j = \delta_k$.*

Proof. Assume $x_{\nu,k} \notin A$. Then $k \notin K$. If $\delta_k \neq \delta_i$, we infer that there is $b \in A$ with $x_{\nu,k} \preceq_\nu b \preceq_\nu x_{\nu,i}$ by Claim 2.13.. So $x_{\nu,k} = \mathrm{i}_\nu\{b, x_{\nu,j}\}$ and thus $x_{\nu,k} \in A$ by Claim 2.14, contradiction.

Thus $\delta_i = \delta_k$, and similarly $\delta_j = \delta_k$. □

After this preparation fix $\{\nu, \mu\} \in \left[\kappa^+\right]^2$. We do not assume that $\nu < \mu$! Let $p = r_\nu$ and $q = r_\mu$. Our purpose is to show that p and q

are compatible. Write $p = \langle A_p, \preceq_p, \mathrm{i}_p \rangle$ and $q = \langle A_q, \preceq_q, \mathrm{i}_q \rangle$, $x_i^p = x_{\nu,i}$ and $x_i^q = x_{\mu,i}$, $\delta_{x_i^p} = \delta_{x_i^q} = \delta_i$.

If $s = x_i^p$ write $s \in K$ iff $i \in K$. Define $s \in L$, $s \in F$, $s \in M$, $s \in D$ similarly.

In order to amalgamate conditions p and q, we will use a refinement of the notion of amalgamation given in [5, Definition 2.4].

Let $A' = \{x_i^p : i \in F \cup D \cup M \cup L\}$.

Let $\mathrm{rk} : \langle A', \preceq_p \restriction A' \rangle \longrightarrow \theta$ be an order-preserving injective function for some ordinal $\theta < \kappa$.

For $x \in A'$, by induction on $\mathrm{rk}(x) < \theta$ choose $\beta_x \in \delta$ as follows:

Assume that $\mathrm{rk}(x) = \tau$ and β_z is defined provided $\mathrm{rk}(z) < \tau$.

Let

$$\beta_x = \min\big((\overline{\mathrm{o}}(\delta_x) \cap [\underline{\gamma}(\delta_x), \gamma(\delta_x))) \setminus \sup\{\beta_z : z \prec_p x\}\big). \tag{30}$$

Since $z \preceq_p x$ implies $\delta_z \leq \delta_x$ by Claim 2.12, we have $\beta_z < \gamma(\delta_x)$ for $z \prec_p x$. Since $\mathrm{cf}(\gamma(\delta_x)) = \kappa$ and $|A'| < \kappa$ we have $\sup\{\beta_z : z \prec_p x\} < \gamma(\delta_x)$, so β_x is always defined.

For $x \in A'$ let

$$y_x = \begin{cases} \langle \beta_x, \mathrm{rk}(x) \rangle & \text{if } x \in L \cup D \cup M, \\ \langle \zeta, \eta, \beta_x, \mathrm{rk}(x) \rangle & \text{if } x \in F,\ \pi_B(x) = \langle \zeta, \eta \rangle. \end{cases} \tag{31}$$

Put

$$Y = \{y_x : x \in A'\}. \tag{32}$$

For $x \in A'$ put

$$g(y_x) = x \text{ and } \bar{g}(y_x) = x', \tag{33}$$

where x' is the "twin" of x in A_q (i.e. $h_{\nu,\mu}(x) = x'$).

We will include the elements of Y in the domain of the amalgamation r of p and q. In this way, we will be able to define the infimum in r of elements s, t where $s \in A_p \setminus A_q$ and $t \in A_q \setminus A_p$.

We need to prove some basic facts.

Claim 2.16. *If $x \in A'$ then*

$$\overline{o}(\delta_x) \cap [\underline{\gamma}(\delta_x), \gamma(\delta_x)) \subset o^*(x) \cap o^*(x').$$

Proof. Let $\alpha \in \overline{o}(\delta_x) \cap [\underline{\gamma}(\delta_x), \gamma(\delta_x))$. It is enough to show that $\alpha \in o^*(x)$. Note that if $x \in D$, then $\alpha \in o(\pi(x)) = o^*(x)$. If $x \in M$, we have that $\alpha \in o(\pi_-(x)) \subset o_B(\pi_B(x)) \subset o^*(x)$. Also, if $x \in L$ then as p is good we have that $\alpha \in o_B(\pi_B(x)) \subset o^*(x)$. Now, assume that $x \in F$. Since $\pi_-(x) \in Z$, we have that $\pi_-(x) < \underline{\gamma}(\delta_x)$, hence $\alpha \in o(\pi(x)) \setminus \pi_-(x)$, and so $\alpha \in o^*(x)$. □

Note that we obtain as an immediate consequence of Claim 2.16 that $\beta_x \in o^*(x) \cap o^*(x')$ for every $x \in A'$.

Claim 2.17. *If $x \in A'$ then*

$$\mathrm{o}^*(y_x) \supset (\mathrm{o}^*(x) \cap \pi(y_x)) \cup \{\beta_z : \delta_z = \delta_x \wedge z \prec_p x\}. \tag{34}$$

Proof. Note that if $I \in \mathbb{I}$ and $\alpha, \beta \in E(I)$ with $\alpha < \beta$, we have that $\alpha \in o(\beta)$. By using this fact, it is easy to verify that $\{\beta_z : \delta_z = \delta_x$ and $z \prec_p x\} \subset o^*(y_x)$.

Now we prove that $o^*(y_x) \supset o^*(x) \cap \pi(y_x)$. Suppose that $\zeta \in o^*(x) \cap \pi(y_x)$. We distinguish the following three cases:

Case 1. $x \in D$.

Then $x, y_x \in B_S$, and so we have $o^*(x) = o(\pi(x))$ and $o^*(y_x) = o(\pi(y_x)) = o(\beta_x)$. Let $k = j(\delta_x)$, i.e. $J(\delta_x) \in \mathcal{I}_k$. Since $\zeta \in o(\pi(x)) \cap \pi(y_x)$, we infer that $\zeta \in E(I(\pi(x), m)) \cap \pi(y_x)$ for some $m \leq k$. Note that for $m \leq k$ we have $I(\pi(x), m) = I(\pi(y_x), m)$. So, $\zeta \in o(\pi(y_x)) = o^*(y_x)$.

Case 2. $x \in L \cup M$.

Since $\zeta \in o^*(x) \cap \pi(y_x)$, we infer that $\zeta \in o_B(\pi_B(x))$. Then as $y_x \in B_S$, we can show that $\zeta \in o(\pi(y_x)) = o^*(y_x)$ by using an argument similar to the one given in Case 1.

Case 3. $x \in F$.

We have $\pi_B(x) = \pi_B(y_x) \neq S$. Put $(\xi, \eta) = \pi_B(x) = \pi_B(y_x)$. So,
$o^*(x) = o_B((\xi, \eta)) \cup (o(\pi(x)) \setminus \pi_-(x))$,
$o^*(y_x) = o_B((\xi, \eta)) \cup (o(\pi(y_x)) \setminus \pi_-(x))$.

So we may assume that $\zeta \in o(\pi(x)) \setminus \pi_-(x)$, and then we can proceed as in Case 1. □

Claim 2.18. *There are no $y \in Y$ and $a \in A$ such that $a \preceq_p g(y), \overline{g}(y)$ and $\pi(y) \leq \pi(a)$.*

Proof. Assume that $y \in Y$. Put $x = g(y)$ and $I = J(\delta_x)$. Note that if $x \in F \cup D \cup M$, then since $\sup(I \cap Z) < \underline{\gamma}(\delta_x)$ we infer that there is no $a \in A$ such that $a \preceq_p x$ and $\pi(a) \geq \pi(y)$.

Now, suppose that $x \in L$. Note that there is no $a \in A$ such that $a \prec_p x$ and $\pi_B(a) = \pi_B(x)$. Also, as $\sup(\delta_x \cap Z) < \underline{\gamma}(\delta_x)$, we infer that there is no $a \in A \cap B_S$ such that $a \preceq_p x$ and $\pi(a) \geq \pi(y)$. □

Claim 2.19. *If $x \in F \cup D \cup M$, then there is no interval that isolates y_x from x.*

Proof. By Claim 2.7(a), we have $\mathrm{cf}(\delta_x) = \kappa^+$ and $\pi(x) < \delta_x$. By Proposition-2.1, we have $j(\delta_x) = n(\delta_x)$ and $\delta_x = J(\delta_x)^+$. Then, assume on the contrary that there is an interval $\Lambda \in \mathbb{I}$ that isolates y_x from x. Let $m < \omega$ such that $\Lambda = I(\pi(y_x), m)$. As Λ isolates y_x from x and $x, y_x \in J(\delta_x)$, we deduce that $m > j(\delta_x)$. But from $m > j(\delta_x)$ and $\pi(y_x) \in E(J(\delta_x))$ we infer that $\pi(y_x) = \Lambda^-$. Hence, Λ does not isolate y_x from x. □

However, if $x \in L$ it may happen that there is a $\Lambda \in \mathbb{I}$ that isolates y_x from x.

Now, we are ready to start to define the common extension $r = (A_r, \prec_r, i_r)$ of p and q. First, we define the universe A_r. Put $L^+ = \{x \in L : \pi(x) \neq \pi_-(x)\}$. Then, if $x \in L^+$ and x' is the twin element of x, we consider new elements $u_x, u_{x'} \in X \setminus (A_p \cup A_q \cup Y)$ such that $\pi_B(u_x) = \pi_B(x)$, $\pi(u_x) = \pi_-(x)$, $\pi_B(u_{x'}) = \pi_B(x')$ and $\pi(u_{x'}) = \pi_-(x')$. We suppose that $u_x, u_z, u_{x'}, u_{z'}$ are different if x, z are different elements of L^+. We put $U = \{u_x : x \in L^+\}$ and $U' = \{u_{x'} : x \in L^+\}$. Then, we define

$$A_r = A_p \cup A_q \cup Y \cup U \cup U'.$$

Clearly, A_r satisfies (P1). Now, our purpose is to define $\preceq_r$. First, for $x, y \in \left[A_p \cup A_q\right]^2$ let

(35) $x \preceq_{p,q} y$ iff $\exists z \in A_p \cup A_q \; [x \preceq_p z \vee x \preceq_q z] \;\wedge\; [z \preceq_p y \vee z \preceq_q y]$.

The following claim is straightforward.

Claim 2.20. *$\preceq_{p,q}$ is the partial order on $A_p \cup A_q$ generated by $\preceq_p \cup \preceq_q$.*

Next, we define the relation $\preceq^*$ on $A_p \cup A_q \cup Y$ as follows. Let us recall that $A = A_p \cap A_q$. Informally, $\preceq^*$ will be the ordering on $A_p \cup A_q \cup Y$ generated by

$$\preceq_{p,q} \cup \{\langle y, g(y)\rangle, \langle y, \bar{g}(y)\rangle : y \in Y\} \cup \{\langle y, y'\rangle : y, y' \in Y, g(y) \preceq_p g(y')\} \cup \{\langle a, y\rangle : a \in A, y \in Y, a \preceq_p g(y)\}.$$

The formal definition is a bit different, but its formulation simplifies the separation of different cases later. So we introduce five relations on $A_p \cup A_q \cup Y$ as follows:

$$\begin{aligned}
\prec^{R1_p} &= \{\langle y, a\rangle : y \in Y, a \in A_p, g(y) \preceq_p a\},\\
\prec^{R1_q} &= \{\langle y, a\rangle : y \in Y, a \in A_q, \bar{g}(y) \preceq_q a\},\\
\preceq^{R2} &= \{\langle y, y'\rangle : y, y' \in Y, g(y) \preceq_p g(y')\},\\
\prec^{R3_p} &= \{\langle x, y\rangle : x \in A_p, y \in Y, \exists a \in A \; x \preceq_p a \preceq_p g(y)\},\\
\prec^{R3_q} &= \{\langle x, y\rangle : x \in A_q, y \in Y, \exists a \in A \; x \preceq_q a \preceq_q \bar{g}(y)\}.
\end{aligned}$$

Then, we put

$$\preceq^* = \preceq_{p,q} \cup \prec^{R1_p} \cup \prec^{R1_q} \cup \preceq^{R2} \cup \prec^{R3_p} \cup \prec^{R3_q} . \tag{36}$$

The partial order $\preceq_r$ will be an extension of $\preceq^*$. So, we need to prove the following lemma:

Lemma 2.21. *$\preceq^*$ is a partial order on $A_p \cup A_q \cup Y$.*

Proof. Let $s \preceq_r t \preceq_r u$. We should show that $s \preceq_r u$.

We can assume that $t \notin A_q \setminus A_p$.

Case I. *$s \in A_p \cup A_q$, $t \in A_p$ and $s \preceq_{p,q} t$.*

Without loss of generality, we may assume that $u \in Y$ and $t \prec^{R3p} u$, i.e. there is $a \in A$ such that $t \preceq_p a \preceq_p g(u)$.

Case I.1. *$s \in A_p$.*

Then $s \preceq_p a \preceq_p g(u)$ and so $s \prec^{R3p} u$.

Case I.2. *$s \in A_q \setminus A_p$.*

Then there is $b \in A$ such that $s \preceq_q b \preceq_p t \preceq_p a \preceq_p g(u)$. Then $s \preceq_q a \preceq_q \bar{g}(u)$ so $s \prec^{R3q} u$.

Case II. *$s \in Y$, $t \in A_p$ and $s \prec^{R1p} t$.*

Case II.1. *$u \in A_p \cup A_q$ and $s \prec^{R1p} t \preceq_{p,q} u$.*

Case II.1.i. *$u \in A_p$.*

Then $g(s) \preceq_p t \preceq_p u$ hence $s \prec^{R1p} u$.

Case II.1.ii. *$u \in A_q \setminus A_p$.*

Then there is $a \in A$ such that $g(s) \preceq_p t \preceq_p a \preceq_q u$. Hence $\bar{g}(s) \preceq_q a \preceq_q u$ and so $\bar{g}(s) \preceq_q u$. Thus $s \prec^{R1q} u$.

Case II.2. *$u \in Y$ and $s \prec^{R1p} t \prec^{R3p} u$.*

Then there is $a \in A$ such that $g(s) \preceq_p t \preceq_p a \preceq_p g(u)$ and so $s \preceq^{R2} u$.

Case III. *$s, t \in Y$ and $s \preceq^{R2} t$.*

Case III.1. *$u \in A_p$ and $s \preceq^{R2} t \prec^{R1p} u$.*

Then $g(s) \preceq_p g(t) \preceq_p u$ so $s \prec^{R1p} u$.

Case III.2. *$u \in A_q$ and $s \preceq^{R2} t \prec^{R1q} u$.*

Then $g(s) \preceq_p g(t)$ and $\bar{g}(t) \preceq_q u$. Thus $\bar{g}(s) \preceq_q \bar{g}(t) \preceq_q u$ so $s \prec^{R1q} u$.

Case III.3. *$u \in Y$ and $s \preceq^{R2} t \preceq^{R2} u$.*

Then $g(s) \preceq_p g(t) \preceq_p g(u)$ so $s \preceq^{R2} u$.

Case IV. *$s \in A_p$, $t \in Y$ and $s \prec^{R3p} t$.*

Case IV.1. *$u \in A_p$ and $s \prec^{R3p} t \prec^{R1p} u$.*

Then there is $a \in A$ such that $s \preceq_p a \preceq_p g(t) \preceq_p u$ so $s \preceq_p u$.

Case IV.2. *$u \in A_q$ and $s \prec^{R3p} t \prec^{R1q} u$.*

Then there is $a \in A$ such that $s \preceq_p a \preceq_p g(t)$ and $\bar{g}(t) \preceq_q u$. So $a \preceq_q \bar{g}(t)$ and hence $s \preceq_p a \preceq_q u$. Thus $s \preceq_{p,q} u$.

Case IV.3. *$u \in Y$ and $s \prec^{R3p} t \preceq^{R2} u$.*

Then there is $a \in A$ such that $s \preceq_p a \preceq_p g(t) \preceq_p g(u)$ and so $s \prec^{R3p} u$.

Case V. *$s \in A_q$, $t \in Y$ and $s \prec^{R3q} t$.*

Only case (3) is different from (IV):

Case V.3. *$u \in Y$ and $s \prec^{R3q} t \preceq^{R2} u$.*

Then there is $a \in A$ such that $s \preceq_q a \preceq_q \bar{g}(t)$ and $g(t) \preceq_p g(u)$. Then $\bar{g}(t) \preceq_q \bar{g}(u)$, so $s \preceq_q a \preceq_q \bar{g}(u)$, thus $s \prec^{R3q} u$. □

Informally, $\preceq_r$ will be the ordering on $A_p \cup A_q \cup Y \cup U \cup U'$ generated by

$$\preceq^* \cup \{\langle y_s, u_s\rangle : s \in A_p \cup A_q\} \cup \{\langle u_s, s\rangle : s \in A_p \cup A_q\}.$$

Now, in order to define $\preceq_r$ we need to make the following definitions:

$$\begin{aligned}
\prec^{R4_p} &= \{\langle s, u_x\rangle : s \in A_p \cup A_q \cup Y, x \in L^+ \text{ and } s \preceq^* y_x\},\\
\prec^{R4_q} &= \{\langle s, u_{x'}\rangle : s \in A_p \cup A_q \cup Y, x \in L^+ \text{ and } s \preceq^* y_x\},\\
\prec^{R5_p} &= \{\langle u_x, t\rangle : x \in L^+, t \in A_p \text{ and } x \preceq_p t\},\\
\prec^{R5_q} &= \{\langle u_{x'}, t\rangle : x \in L^+, t \in A_q \text{ and } x' \preceq_q t\},\\
=^U &= \{\langle u_x, u_x\rangle : x \in L^+\},\\
=^{U'} &= \{\langle u_{x'}, u_{x'}\rangle : x \in L^+\}.
\end{aligned}$$

Then, we define:

$$\preceq_r = \preceq^* \cup \prec^{R4_p} \cup \prec^{R4_q} \cup \prec^{R5_p} \cup \prec^{R5_q} \cup =^U \cup =^{U'} . \tag{37}$$

Write $x \prec_r y$ iff $x \preceq_r y$ and $x \neq y$.

Lemma 2.22. *$\preceq_r$ is a partial order on A_r.*

Proof. Assume that $s \prec_r t \prec_r v$. We have to show that $s \prec_r v$. Note that if $s, t, v \in A_p \cup A_q \cup Y$, then $s \prec^* t \prec^* v$, and so we are done by Lemma 2.21. Also, it is impossible that two elements of $\{s, t, v\}$ are in $U \cup U'$. To check this point, assume that $s, v \in U$. Put $s = u_x$, $v = u_z$ for $x, z \in L^+$. As $u_x \prec_r t$, we have $u_x \prec^{R5p} t$ and so $x \preceq_p t$. As $t \prec_r u_z$, we have $t \prec^{R4p} u_z$ and so $t \prec^* y_z$. Hence, $x \preceq_p t \prec^* y_z \prec^* z$. Since $x \preceq_p t$ and $x \in L$, we infer that $t \in L$. Also, from $t \prec^* y_z$ we deduce that $t \prec^{R3p} y_z$ and so there is an $a \in A$ such that $t \preceq_p a \preceq_p z$. But since $t \in L$, it is impossible that there is an $a \in A$ with $t \preceq_p a$. Proceeding in an analogous way, we arrive to a contradiction if we assume that $s \in U$ and $v \in U'$. So, at most one element of $\{s, t, v\}$ is in $U \cup U'$. Then, we consider the following cases:

Case 1. $s \in U$.

We have that $t, v \in A_p \cup A_q \cup Y$. Put $s = u_x$ for some $x \in L^+$. Since $u_x \prec_r t$, we have $u_x \prec^{R5p} t$ and so $x \preceq_p t$. As $t \prec_r v$, we have $t \prec^* v$. So, $x \preceq_p t \prec^* v$. But as $x \in L$ and $x \preceq_p t$, we infer that $t \in L$. Hence, $t \prec_p v$. Thus $x \prec_p v$, therefore $u_x \prec^{R5p} v$, and so $u_x \prec_r v$.

Case 2. $t \in U$.

We have that $s, v \in A_p \cup A_q \cup Y$. Put $t = u_x$ for $x \in L^+$. From $s \prec_r u_x$, we infer that $s \prec^{R4p} u_x$ and so $s \preceq^* y_x$. From $u_x \prec_r v$, we deduce that $u_x \prec^{R5p} v$ and hence $x \preceq_p v$. So we have $s \preceq^* y_x \prec^* x \preceq_p v$, and therefore $s \prec_r v$.

Case 3. $v \in U$.

We have that $s, t \in A_p \cup A_q \cup Y$. Put $v = u_x$ for $x \in L^+$. Since $t \prec_r u_x$, we have that $t \prec^{R4p} u_x$ and so $t \preceq^* y_x$. And from $s \prec_r t$ we deduce that $s \prec^* t$. So $s \prec^* y_x$, hence $s \prec^{R4p} u_x$, and thus $s \prec_r u_x$. □

Now note that $s \prec^{R3_p} t$ implies $\pi(s) < \pi(t)$ by Claim 2.18, and so it is clear that $s \prec_r t$ implies $\pi(s) < \pi(t)$. Thus, condition (P2) holds. Also, it is easy to verify that $\preceq_r$ satisfies (P3).

If $x \in A_p$ denote its "twin" in A_q by x', and vice versa, if $x \in A_q$ denote its "twin" in A_p by x'.

Extend the definition of g as follows: $g : A_r \longrightarrow A_p$ is a function,

$$g(x) = \begin{cases} x & \text{if } x \in A_p, \\ x' & \text{if } x \in A_q, \\ s & \text{if } x = y_s \text{ for some } s \in A_p, \\ t & \text{if } x = u_t \text{ for some } t \in A_p, \\ t' & \text{if } x = u_t \text{ for some } t \in A_q. \end{cases}$$

For $\{s,t\} \in \left[A_r\right]^2$ we will be able to define the infimum of s,t in $(A_r, \preceq_r)$ from the infimum of $g(s), g(t)$ in p. Now, we need to prove some facts concerning the behavior of the function g on A_r.

Claim 2.23. *Let $a \in A$ and $x \in A_r$. Then*
(1) $x \preceq_r a$ iff $g(x) \preceq_p a$,
(2) $a \preceq_r x$ iff $a \preceq_p g(x)$.

Proof. (1) $x \preceq_r a$ iff $x \preceq_{p,q} a$ or $x \prec^{R1p} a$ and (1) holds in both cases. (2) $a \preceq_r x$ iff $a \preceq_{p,q} x$ or $a \prec^{R3p} x$ or $a \prec^{R4p} x$ or $a \prec^{R4q} x$, and (2) holds in every case. □

Claim 2.24. *If $x \preceq_r y$ then $g(x) \preceq_p g(y)$ for $x, y \in A_r$.*

Proof. $x \prec_r y$ iff $x \prec_{p,q} y$ or $x \prec^{R1p} y$ or $x \prec^{R1q} y$ or $x \prec^{R2} y$ or $x \prec^{R3p} y$ or $x \prec^{R3q} y$ or $x \prec^{R4p} y$ or $x \prec^{R4q} y$ or $x \prec^{R5p} y$ or $x \prec^{R5q} y$, and the implication holds in every case. □

Claim 2.25. *If $v \preceq_p g(s)$ then $y_v \preceq_r s$ for $v \in A_p \setminus A$ and $s \in A_r$.*

Proof. If $s \in A_p$ ($s \in A_q$) then $g(s) = s$ ($g(s) = s'$) and so $y_v \prec^{R1p} s$ ($y_v \prec^{R1q} s$).

If $s = y_x$ for some $x \in A_p$ then $g(s) = x$ and so $y_v \preceq^{R2} y_x$.

If $s = u_x$ for some $x \in L^+$ then $y_v \preceq_r y_x$, and so $y_v \prec^{R4p} u_x$. □

Claim 2.26. *If $x \preceq_r y$ and $\delta_{g(x)} < \delta_{g(y)}$ then there is $a \in A$ such that $x \preceq_r a \preceq_r y$.*

Proof. By Claim 2.24 we have $g(x) \preceq_p g(y)$. Hence, by Claim 2.13, there is $a \in A$ such that $g(x) \preceq_p a \preceq_p g(y)$. Then, by Claim 2.23, we have $x \preceq_r a \preceq_r y$. □

Claim 2.27. *If $a \in A$ and $x \in A_r$, $a \preceq_r x$, then $\pi(a) \in \mathrm{o}^*(x)$ iff $\pi(a) \in \mathrm{o}^*(g(x))$.*

Proof. We can assume that $x \notin A_p \cup A_q$. If $x \in Y$ then Claim 2.17 implies the statement. If $x = u_z$ for some $z \in L^+$ then $g(x) = z$, $\pi(a) < \delta_z$ and $\mathrm{o}^*(z) \cap \delta_z = \mathrm{o}^*(u_z) \cap \delta_z = o_B(\pi_B(z))$, and so we are done. □

Claim 2.28. *If $x \in A_r \setminus A$, $v \in A_p \setminus A$, $v \prec_p g(x)$ and $\delta_v = \delta_{g(x)}$ then $\pi(y_v) \in \mathrm{o}^*(x)$.*

Proof. We have $\pi(y_v) = \beta_v \in \overline{\mathrm{o}}(\delta_v) \cap [\underline{\gamma}(\delta_v), \gamma(\delta_v))$. If $x \in (A_p \cup A_q) \setminus A$, then $\beta_v \in \mathrm{o}^*(x)$ by Claim 2.16.

If $x = y_z$ for some $z \in A_p$, we have $z = g(x)$ and then $\beta_v \in \mathrm{o}^*(y_z)$ by Claim 2.17.

If $x = u_z$ for some $z \in L^+$ then $\beta_v \in \mathrm{o}^*(z)$ because p is good. Now as $\beta_v < \delta_z$ and $\mathrm{o}^*(z) \cap \delta_z = \mathrm{o}^*(u_z) \cap \delta_z$, the statement holds. □

Claim 2.29. *If $s \in A_r \setminus (A \cup Y)$ and $v = g(s)$ then $\pi(y_v) \in \mathrm{o}^*(s)$.*

Proof. We have $\pi(y_v) = \beta_v \in \overline{\mathrm{o}}(\delta_v) \cap \gamma(\delta_v)$. If $s \in A_p \cup A_q$ then $\overline{\mathrm{o}}(\delta_v) \cap \gamma(\delta_v) \subset \mathrm{o}^*(s)$ because p and q are good. If $s = u_{g(s)}$ then the block orbit of s and the block orbit of $g(s)$ are the same and the block orbit of $g(s)$ contains $\overline{\mathrm{o}}(\delta_v) \cap \gamma(\delta_v)$ because p is good. □

Claim 2.30. *If $w \in A_p$, $s \in A_r$, $w \preceq_r s$ and $\delta_w = \delta_{g(s)}$ then $s \in A_p$.*

Proof. If $s \in A_q \setminus A_p$ then $w \preceq_{p,q} s$ and so there is $a \in A$ such that $w \preceq_p a \preceq_q s$ which contradicts $\delta_w = \delta_{g(s)}$.

If $s = y_{g(s)}$ then $w \prec^{R3p} s$, i.e. there is $a \in A$ with $w \preceq_p a \preceq_p g(s)$ which contradicts $\delta_w = \delta_{g(s)}$.

If $s = u_{g(s)}$ then $w \prec^{R4p} u_{g(s)}$, i.e. $w \preceq_r y_{g(s)}$, but this was excluded in the previous paragraph. □

Lemma 2.31. *There is a function $\mathrm{i}_r \supset \mathrm{i}_p \cup \mathrm{i}_q$ such that $\langle A_r, \preceq_r, \mathrm{i}_r \rangle$ satisfies (P4) and (P5).*

Proof. If $\{s,t\} \in \left[A_p\right]^2$ ($\{s,t\} \in \left[A_q\right]^2$) we will have $\mathrm{i}_r\{s,t\} = \mathrm{i}_p\{s,t\}$ ($\mathrm{i}_r\{s,t\} = \mathrm{i}_q\{s,t\}$), and so (P5) holds because p and q satisfy (P5).

To check (P4) we should prove that $\mathrm{i}_p\{s,t\}$ is the greatest common lower bound of s and t in $(A_r, \preceq_r)$.

Indeed, let $x \preceq_r s, t$. We can assume that $x \notin A_p$. Then, we distinguish the following three cases.

Case i. $x \in A_q \setminus A_p$.

Then there are $a, b \in A$ such that $x \preceq_q a \preceq_p s$ and $x \preceq_q b \preceq_p t$. Thus $x \preceq_q \mathrm{i}_q\{a,b\} = \mathrm{i}_p\{a,b\} \preceq_p \mathrm{i}_p\{s,t\}$ and so $x \preceq_{p,q} \mathrm{i}_p\{s,t\}$.

Case ii. $x \in Y$.

Then $x \prec^{R1p} s$ and $x \prec^{R1p} t$, i.e. $g(x) \preceq_p s$ and $g(x) \preceq_p t$. So $g(x) \preceq_p \mathrm{i}_p\{s,t\}$ and hence $x \prec^{R1p} \mathrm{i}_p\{s,t\}$.

Case iii. $x \in U$.

Put $x = u_z$ for some $z \in L^+$. Since $x \preceq_r s, t$, we have that $u_z \prec^{R5p} s, t$, and thus $z \preceq_p s, t$. So $z \preceq_p i_p\{s,t\}$, and hence $x \preceq_r i_p\{s,t\}$.

Assume now that $s, t \in A_r$ are $\preceq_r$-compatible, but $\preceq_r$-incomparable elements, $\{s,t\} \notin \left[A_p\right]^2 \cup \left[A_q\right]^2$. Write $v = \mathrm{i}_p\{g(s), g(t)\}$. Note that, by Claim 2.24, $g(s)$ and $g(t)$ are compatible in p and hence $v \in A_p$. Let

$$\mathrm{i}_r\{s,t\} = \begin{cases} v & \text{if } v \in A, \\ y_v & \text{otherwise.} \end{cases}$$

Case I. $v \in A$.

Then $g(s)$ and $g(t)$ are incomparable in A_p. Indeed, $g(s) \preceq_p g(t)$ implies $v = g(s)$ and so $s = g(s) \preceq_r t$ by Claim 2.23.

Thus $\pi(v) \in \mathrm{o}^*(g(s)) \cap \mathrm{o}^*(g(t))$ by applying (P5) in p. Note that $v \preceq_r s, t$ by Claim 2.23. So, $\pi(v) \in \mathrm{o}^*(s) \cap \mathrm{o}^*(t)$ by Claim 2.27. Hence (P5) holds.

We have to check that v is the greatest lower bound of s, t in $(A_r, \preceq_r)$. We have $v \preceq_r s, t$ by Claim 2.23.

Let $w \in A_r$, $w \preceq_r s, t$. Then $g(w) \preceq_p g(s), g(t)$ by Claim 2.24. So $g(w) \preceq_p v$. Then $w \preceq_r v$ by Claim 2.23.

Case II. $v \notin A$.

Then $\delta_{g(s)} = \delta_{g(t)} = \delta_v$ by Claim 2.23 and Claim 2.13 if $g(s)$ and $g(t)$ are comparable in A_p, and by Claim 2.15 if $g(s)$ and $g(t)$ are incomparable in A_p.

If $g(s)$ and $g(t)$ are incomparable in A_p then $v \prec_p g(s), g(t)$ and $s, t \notin A$ by Claim 2.14. So, $\pi(y_v) \in \mathrm{o}^*(s) \cap \mathrm{o}^*(t)$ by Claim 2.28.

If $g(s) \prec_p g(t)$ then $s \notin Y$ by Claim 2.25 and $s \notin A$ because $v = g(s) \notin A$. Then $\pi(y_v) \in \mathrm{o}^*(s)$ by Claim 2.29. Also, since $v = g(s) \prec_p g(t)$ we infer from Claim 2.23 that $t \notin A$ and so we have that $\pi(y_v) \in \mathrm{o}^*(t)$ by Claim 2.28. Hence (P5) holds.

We have to check that y_v is the greatest common lower bound of s, t in $(A_r, \preceq_r)$. First observe that $y_v \preceq_r s, t$ by Claim 2.25.

Let $w \preceq_r s, t$.

Assume first that $\delta_{g(w)} < \delta_v$. Then there are $a, b \in A$ with $w \preceq_r a \preceq_r s$ and $w \preceq_r b \preceq_r t$ by Claim 2.26 and so $g(w) \preceq_p \mathrm{i}_p\{a, b\} \preceq_p v$ by using Claim 2.23. Now since $g(y_v) = v$, we obtain $w \preceq_r \mathrm{i}_p\{a, b\} \preceq_r y_v$ again by Claim 2.23.

Assume now that $\delta_{g(w)} = \delta_v$. Since $\{s, t\} \notin [A_p]^2 \cup [A_q]^2$, we have that $w \notin U \cup U'$. Then, by Claim 2.30, $w = y_z$ for some $z \in A_p$. Then $z \preceq_p g(s)$ and $z \preceq_p g(t)$ by Claim 2.24, and so $z \preceq_p v$. Thus $y_z \preceq_r y_v$. □

Now our aim is to verify condition (P6). First, we need some preparations.

For every $x, y \in A_r$ with $x \preceq_r y$ let

$$\pi_x(y) = \begin{cases} \pi(y) & \text{if } \pi_B(x) = \pi_B(y), \\ \pi_-(y) & \text{if } \pi_B(x) \neq \pi_B(y). \end{cases}$$

Note that for every $x, y \in A_r$ with $x \preceq_r y$, an interval $\Lambda \in \mathbb{I}$ isolates x from y iff $\Lambda^- < \pi(x) < \Lambda^+ \leq \pi_x(y)$.

Claim 2.32. *Let $a \in A$ and $t \in A_r$, $a \preceq_r t$. If Λ isolates a from t then Λ isolates a from $g(t)$.*

Proof. The statement is obvious if $t \in A_p$. Assume that $t \in A_q \setminus A_p$. Note that since Λ contains an element of A, we have that $\Lambda^+ \in Z$. Now if $t \in D \cup F \cup M$ we have that $Z \cap \pi(t) = Z \cap \pi(g(t)) = Z \cap \gamma(\delta_t)$, and so we are done. If $t \in L$ then as $a \preceq_r t$ we infer that $\pi_B(a) \neq \pi_B(t)$ and $\pi(a) < \delta_t = \pi_-(t)$, hence we have $\pi(a) < \Lambda^+ \leq \pi_a(t) = \pi_a(g(t)) = \pi_-(t)$, and so the statement holds.

If $t = y_v$ for some $v \in A_p$, then $a \prec_p v = g(t)$ and $\pi_a(y_v) \leq \pi_a(v)$, and so we are done.

If $t = u_v$ for some $v \in L^+$, we have $a \prec_p v = g(t)$ and $\pi_a(u_v) = \pi_a(v) = \pi_-(v)$. □

Claim 2.33. *Let $a \in A$ and $x \in A_r \setminus (A_p \cup A_q)$, $x \preceq_r a$. If Λ isolates x from a then $x = y_{g(x)}$ and Λ isolates $g(x)$ from a.*

Proof. We have $g(x) \preceq_p a$ by Claim 2.23, so as $a \in A$ we infer that $g(x) \notin L \cup M$, and thus $x \notin U \cup U'$. Hence $x \in Y$ and $g(x) \in D \cup F$, and so $x = y_{g(x)}$ and $\pi(g(x)) < \delta_{g(x)}$.

Let $J(\delta_{g(x)}) = \mathrm{I}(\pi(g(x)), j)$ and $\Lambda = \mathrm{I}(\pi(x), \ell)$. If $\ell > j$ then $\Lambda^- = \pi(y_{g(x)}) = \pi(x)$, which is impossible. If $\ell \leq j$ then $J(\delta_{g(x)}) \subset \Lambda$ and so $\Lambda^- < \pi(g(x)) < \Lambda^+$, i.e. Λ isolates $g(x)$ from a. □

Lemma 2.34. *$(A_r, \preceq_r, i_r)$ satisfies $(P6)$.*

Proof. Assume that $\{s,t\} \in \left[A_r\right]^2$, $s \preceq_r t$ and Λ isolates s from t. Suppose that $\pi(s) \neq \pi_-(s)$ if $s \notin B_S$. So, $s \notin U \cup U'$. We should find $v \in A_r$ such that $s \preceq_r v \preceq_r t$ and $\pi(v) = \Lambda^+$. Note that since $s \preceq_r t$, we have $\delta_{g(s)} \leq \delta_{g(t)}$ by Claims 2.24 and 2.12.

We can assume that $\{s,t\} \notin \left[A_p\right]^2 \cup \left[A_q\right]^2$ because p and q satisfy (P6).

Case 1. $\delta_{g(s)} < \delta_{g(t)}$.

By Claim 2.26 there is $a \in A$ with $s \preceq_r a \preceq_r t$. Moreover, $g(s) \preceq_p a \preceq_p g(t)$ by Claim 2.23.

Case 1.1. $\pi(a) \in \Lambda$.

Then $\pi_B(s) = \pi_B(a)$ and so $\pi_s(t) = \pi_a(t)$. Thus Λ isolates a from t.

If $t \in A_p$ ($t \in A_q$) then applying (P6) in p (in q) for a, t and Λ we obtain $b \in A_p$ ($b \in A_q$) such that $a \preceq_p b \preceq_p t$ ($a \preceq_q b \preceq_q t$) and $\pi(b) = \Lambda^+$. Then $s \preceq_r a \preceq_{p,q} b \preceq_{p,q} t$, so we are done.

Assume now that $t \notin A_p \cup A_q$.

By Claim 2.32, the interval Λ isolates a from $g(t)$. Since $\pi_-(a) \neq \pi(a)$ if $a \notin B_S$, we can apply (P6) in p to get a $b \in A_p$ with $\pi(b) = \Lambda^+$ and $a \preceq_p b \preceq_p g(t)$.

Note that as $\pi(a) \in \Lambda, a \in A$ and $\pi(b) = \Lambda^+$, we have that $\pi(b) \in Z$.

If $\pi_B(a) = \pi_B(b)$, we have $b \notin M \cup L$ because $a \in A$.

If $\pi_B(a) \neq \pi_B(b)$, then $\pi_-(b) = \pi(b) = \Lambda^+ \leq \pi(t)$. Note that if $t \in U \cup U'$, then $\pi(t) = \Lambda^+$, and so we are done. Thus, we may assume that $t \in Y$. Then, we have $\pi_B(b) = \pi_B(t) = \pi_B(g(t))$ and $g(t) \in F$. Hence $b \in K \cup F$.

In both cases we have $b \notin M \cup L$, so $\pi(b) \in Z$ implies $b \in A$. Thus $b \preceq_r t$ by Claim 2.23, and so b witnesses (P6).

Case 1.2. $\pi(a) \notin \Lambda$.

Since p and q satisfy $(P6)$ and Λ isolates s from a, we can assume that $s \notin A_p \cup A_q$.

Hence $s = y_{g(s)}$ and Λ isolates $g(s)$ from a by Claim 2.33. Since $\pi(g(s)) \neq \pi_-(g(s))$ if $g(s) \notin B_S$, there is $v \in A_p$ with $g(s) \preceq_p v \preceq_p a$ and $\pi(v) = \Lambda^+$. Since $y_{g(s)} \preceq_r g(s)$ by the definition of $\preceq_r$, we have that v witnesses (P6).

Case 2. $\delta_{g(s)} = \delta_{g(t)}$.

Case 2.1. $s \in A_p$.

Since $s \in A_p$, $s \preceq_r t$ and $\delta_s = \delta_{g(t)}$ we infer from Claim 2.30 that $t \in A_p$, which was excluded.

By means of a similar argument, we can show that $s \in A_q$ is also impossible.

Case 2.2. $s = y_v$ *for some* $v \in A_p$.

We have that $\delta_v = \delta_{g(t)}$. Note that since $\Lambda^- < \pi(s) < \Lambda^+$, we have $\delta_v \leq \Lambda^+$.

Thus $\pi(t) \geq \Lambda^+ \geq \delta_v = \delta_{g(t)}$. Since we can assume that $\pi(t) > \Lambda^+$, we have $\pi(t) > \delta_{g(t)}$. If $t \in A_p \cup A_q$ and $g(t) \in F \cup D \cup M$, or $t \in Y$, or $t \in U \cup U'$ then $\pi(t) \leq \delta_{g(t)}$. Thus we have $t \in A_p \cup A_q$ and $g(t) \in L$.

Note that as $\pi_B(t) \neq S$, if $\pi_B(y_v) = \pi_B(t)$ we would infer that $v \in F$ and hence $\delta_t = \delta_{g(t)} < \delta_v$. So $\pi_B(s) \neq \pi_B(t)$. Now since Λ isolates s from t, we deduce that $\delta_v = \delta_t = \Lambda^+$, and hence $\Lambda = \mathrm{J}(\delta_t)$.

Assume that $t \in A_q$ (the case $t \in A_p$ is simpler). Then $g(t) = t' \in L$. Since $\pi(t) > \delta_t = \pi_-(t)$ we have $\pi(t') > \pi_-(t')$ and so $t' \in L^+$.

Since $y_v \preceq_r t$ we have $y_v \prec^{R1q} t$, i.e. $v \preceq_p t'$ and so $y_v \preceq^{R2} y_{t'}$. Thus $y_v \prec^{R4q} u_t$. Hence $y_v \preceq_r u_t \preceq_r t$ and $\pi(u_t) = \delta_t = \Lambda^+$, i.e. u_t witnesses that (P6) holds. □

This completes the proof of Lemma 2.5, i.e. $\mathcal{P}$ satisfies κ^+-c.c. □

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Facultat de Matemàtiques, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain
E-mail address: jcmartinez@ub.edu

Alfréd Rényi Institute of Mathematics
E-mail address: soukup@renyi.hu