

# LINDELÖF SPACES OF SINGULAR DENSITY

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ABSTRACT. A cardinal  $\lambda$  is called  $\omega$ -inaccessible if for all  $\mu < \lambda$  we have  $\mu^\omega < \lambda$ . We show that for every  $\omega$ -inaccessible cardinal  $\lambda$  there is a CCC (hence cardinality and cofinality preserving) forcing that adds a hereditarily Lindelöf regular space of density  $\lambda$ . This extends an analogous earlier result of ours that only worked for regular  $\lambda$ .

In [1] we have shown that for any cardinal  $\lambda$  a natural CCC forcing notion adds a hereditarily Lindelöf 0-dimensional Hausdorff topology on  $\lambda$  that makes the resulting space  $X_\lambda$  left-separated in its natural well-ordering. It was also shown there that the density  $d(X_\lambda) = \text{cf}(\lambda)$ , hence if  $\lambda$  is regular then  $d(X_\lambda) = \lambda$ . The aim of this paper is to show that a suitable extension of the construction given in [1] enables us to generalize this to many singular cardinals as well.

Note that the existence of an L-space, that we now know is provable in ZFC (see [3]), is equivalent to the existence of a hereditarily Lindelöf regular space of density  $\omega_1$ . Since the cardinality of a hereditarily Lindelöf  $T_2$  space is at most continuum, just in ZFC we cannot replace in this  $\omega_1$  with anything bigger. The following problem however, that is left open by our subsequent result, can be raised naturally.

**Problem 1.** *Assume that  $\omega_1 < \lambda \leq \mathfrak{c}$ . Does there exist then a hereditarily Lindelöf regular space of density  $\lambda$ ?*

We should emphasize that this problem is open for all cardinals  $\lambda$ , regular or singular, in particular for  $\lambda = \omega_2$ .

Before describing our new construction, let us recall that the one given in [1] is based on simultaneously and generically “splitting into two” the complements  $\lambda \setminus \alpha$  for all proper initial segments  $\alpha$  of  $\lambda$ . The novelty in the construction to be given is that we shall perform the same simultaneous splitting for the complements of the members of a

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family  $\mathcal{A}$  of subsets of  $\lambda$  that is, at least when  $\lambda$  is singular, much larger than the family of its proper initial segments (that is just  $\lambda$  if we are considering von Neumann ordinals). The following definition serves to describe the properties of such a family of subsets of  $\lambda$ .

**Definition 2.** Let  $\lambda$  be an infinite cardinal. A family  $\mathcal{A}$  of *proper* subsets of  $\lambda$  is said to be *good* over  $\lambda$  if it satisfies properties (i)-(iii) below:

- (i)  $\lambda \in \mathcal{A}$  that is, all proper initial segments of  $\lambda$  belong to  $\mathcal{A}$ ;
- (ii) for every subset  $S \subset \lambda$  with  $|S| < \lambda$  there is  $A \in \mathcal{A}$  with  $S \subset A$ ;
- (iii) for every subset  $S \subset \lambda$  with  $|S| = \omega_1$  there is  $T \in [S]^{\omega_1}$  such that if  $A \in \mathcal{A}$  then either  $|A \cap T| \leq \omega$  or  $T \subset A$ .

If  $\lambda$  is regular then  $\mathcal{A} = \lambda$ , the family of all proper initial segments of  $\lambda$ , is a good family over  $\lambda$ . Indeed, (i) and (ii) are obviously valid and if  $S \in [\lambda]^{\omega_1}$  then any subset  $T$  of  $S$  of order type  $\omega_1$  satisfies (iii). If, however,  $\lambda$  is singular then this  $\mathcal{A}$  definitely does not satisfy condition (ii). Actually, we do not know if it is provable in ZFC that for any (singular) cardinal  $\lambda$  there is a good family over  $\lambda$ . But we know that they do exist if  $\lambda$  is  $\omega$ -inaccessible, that is  $\mu^\omega < \lambda$  holds whenever  $\mu < \lambda$ .

**Theorem 3.** *If  $\lambda$  is an  $\omega$ -inaccessible cardinal then there exists a good family  $\mathcal{A} \subset [\lambda]^{<\lambda}$  over  $\lambda$ .*

*Proof.* It is well-known that there is a map  $G : [\omega]^\omega \rightarrow \omega$  with the property that for every  $a \in [\omega]^\omega$  we have  $G[[a]^\omega] = \omega$ . In other words: we may color the infinite subsets of  $\omega$  with countably many colors so that on the subsets of any infinite set all the colors are picked up. Such a coloring may be constructed by a simple transfinite recursion.

Next we fix a *maximal almost disjoint* family  $\mathcal{F}$  of subsets of order type  $\omega$  of our underlying set  $\lambda$  and then we “transfer” the coloring  $G$  to each member  $F$  of  $\mathcal{F}$ . More precisely, this means that for every  $F \in \mathcal{F}$  we fix a map  $G_F : [F]^\omega \rightarrow F$  such that  $G_F[[a]^\omega] = F$  whenever  $a \in [F]^\omega$ . Then we “fit together” these colorings  $G_F$  to obtain a coloring  $H : [\lambda]^\omega \rightarrow \lambda$  of all countable subsets of  $\lambda$  as follows: For any  $S \in [\lambda]^\omega$  we set  $H(S) = G_F(S)$  if there is an  $F \in \mathcal{F}$  with  $S \subset F$  and  $H(S) = 0$  otherwise. The coloring  $H$  is well-defined because, as  $\mathcal{F}$  is almost disjoint, for every  $S \in [\lambda]^\omega$  there is at most one  $F \in \mathcal{F}$  with  $S \subset F$ .

Now, a set  $C \subset \lambda$  is called *H-closed* if for every  $S \in [C]^\omega$  we have  $H(S) \subset C$ . Clearly, for every set  $A \subset \lambda$  there is a smallest *H-closed* set including  $A$  that will be denoted by  $cl_H(A)$  and is called the *H-closure* of  $A$ .

Let us set  $A^+ = A \cup H[[A]^\omega]$  for any  $A \subset \lambda$ . It is obvious that then we have

$$cl_H(A) = \bigcup_{\alpha < \omega_1} A^\alpha,$$

where the sets  $A^\alpha$  are defined by the following transfinite recursion:  $A^0 = A$ ,  $A^{\alpha+1} = (A^\alpha)^+$ , and  $A^\alpha = \bigcup_{\beta < \alpha} A^\beta$  for  $\alpha$  limit. Since

$$H[[A]^\omega] \subset \bigcup \{F \in \mathcal{F} : |F \cap A| = \omega\} \cup \{0\},$$

it is also obvious that we have  $|A^+| \leq |A|^\omega$  for all  $A \subset \lambda$  and consequently

$$|cl_H(A)| \leq |A|^\omega$$

as well. In particular,  $|A| < \lambda$  implies  $|cl_H(A)| < \lambda$  because  $\lambda$  is  $\omega$ -inaccessible.

Now we claim that the family  $\mathcal{A}$  of all  $H$ -closed sets of cardinality less than  $\lambda$  is good over  $\lambda$ . Indeed, first notice that because each  $F \in \mathcal{F}$  has order type  $\omega$ , for every set  $S \in [F]^\omega$  we have

$$H(S) = G_F(S) < \sup F = \sup S,$$

implying that every initial segment  $\alpha$  of  $\lambda$  is  $H$ -closed and so  $\mathcal{A}$  satisfies condition (i) of definition 2. Condition (ii) is satisfied trivially.

To see (iii) we first show that there is no infinite strictly descending sequence of  $H$ -closed subsets of  $\lambda$ , or in other words: the family of  $H$ -closed sets is well-founded with respect to inclusion. Assume, reasoning indirectly, that  $\{C_n : n < \omega\}$  is a strictly decreasing sequence of  $H$ -closed sets and for each  $n < \omega$  we have  $\alpha_n \in C_n \setminus C_{n+1}$ . By the maximality of  $\mathcal{F}$  then there is some  $F \in \mathcal{F}$  such that the set  $S = F \cap \{\alpha_n : n < \omega\}$  is infinite. Then, for any  $k < \omega$ , the set  $S \cap C_k$  is also infinite and consequently we have

$$H[[S \cap C_k]^\omega] = G_F[[S \cap C_k]^\omega] = F \subset C_k$$

because  $C_k$  is  $H$ -closed. But for any  $k < \omega$  such that  $\alpha_k \in S \subset F$  this would imply

$$\alpha_k \in F \subset C_{k+1},$$

which is clearly a contradiction.

Now let  $S \subset \lambda$  with  $|S| = \sigma$ . Our previous result clearly implies that there is a set  $T \in [S]^\sigma$  such that we have  $cl_H(U) = cl_H(T)$  whenever  $U \subset T$  with  $|U| = \sigma$ . In other words, this means that for every  $H$ -closed set  $C$  we have either  $|C \cap T| < \sigma$  or  $T \subset C$ . In particular, for  $\sigma = \omega_1$  this shows that our family  $\mathcal{A}$  satisfies condition (iii) of definition 2 as well, hence it is indeed good over  $\lambda$ .  $\square$

Next we present our main result that, in view of theorem 3, immediately implies the consistency of the existence of hereditarily Lindelöf regular spaces of density  $\lambda$  practically for any singular cardinal  $\lambda$ . (Of course, this has to be in a model in which  $\lambda \leq \mathfrak{c}$ .) We shall follow [2] in our notation and terminology concerning forcing.

**Theorem 4.** *Let  $\mathcal{A}$  be a good family over  $\lambda$ . Then there is a complete (hence CCC) subforcing  $\mathbb{Q}$  of the Cohen forcing  $F_n(\mathcal{A} \times \lambda, 2)$  such that in the generic extension  $V^{\mathbb{Q}}$  there is a hereditarily Lindelöf 0-dimensional Hausdorff topology  $\tau$  on  $\lambda$  that has density  $\lambda$ . If we also have  $\mathcal{A} \subset [\lambda]^{<\lambda}$  (as in theorem 3) then every subset of  $\lambda$  of size  $< \lambda$  is even  $\tau$ -nowhere dense.*

*Proof.* We start by defining the the subforcing  $\mathbb{Q}$  of  $F_n(\mathcal{A} \times \lambda, 2)$ :  $\mathbb{Q}$  consists of those  $p \in F_n(\mathcal{A} \times \lambda, 2)$  for which  $\langle A, \alpha \rangle \in \text{dom } p$  with  $\alpha \in A$  implies  $p(A, \alpha) = 0$  and  $\langle A, \gamma_A \rangle \in \text{dom } p$  implies  $p(A, \gamma_A) = 1$ , where  $\gamma_A = \min(\lambda \setminus A)$ . It is straight-forward to check that  $\mathbb{Q}$  is a complete suborder of  $F_n(\mathcal{A} \times \lambda, 2)$ .

For any condition  $p \in \mathbb{Q}$  and any set  $A \in \mathcal{A}$  we define

$$U_A^p = \{\alpha : p(A, \alpha) = 1\},$$

and if  $G \subset \mathbb{Q}$  is generic then, in  $V[G]$ , we set

$$U_A = \bigcup \{U_A^p : p \in G\}.$$

Next, let  $U_A^1 = U_A$  and  $U_A^0 = \lambda \setminus U_A$  and  $\tau$  be the topology on  $\lambda$  generated by the sets  $\{U_A^i : i < 2, A \in \mathcal{A}\}$ . Note that then the family  $\mathcal{B} = \{B_\varepsilon : \varepsilon \in F_n(\mathcal{A}, 2)\}$  is a base for  $\tau$ , where  $B_\varepsilon = \bigcap_{A \in \text{dom } \varepsilon} U_A^{\varepsilon(A)}$ . It is clear from the definition that each  $B_\varepsilon$  is clopen, hence  $\tau$  is 0-dimensional. Now, if  $\beta < \alpha < \lambda$  then we have  $\alpha \in \mathcal{A}$  by (i) and hence  $\beta \in \alpha \subset U_\alpha^0$  while  $\alpha = \gamma_\alpha \in U_\alpha^1$ , which shows that  $\tau$  is also Hausdorff. It is also immediate from (ii) that no set  $S \in [\lambda]^{<\lambda}$  is  $\tau$ -dense, hence the space  $\langle \lambda, \tau \rangle$  has density  $\lambda$ . Indeed, if  $S \subset A \in \mathcal{A}$  then we have  $S \cap U_A^1 = \emptyset$ , while  $U_A^1 \neq \emptyset$ . Thus it only remains for us to prove that the topology  $\tau$  is hereditarily Lindelöf.

Assume, reasoning indirectly, that some condition  $p \in \mathbb{Q}$  forces that  $\tau$  is not hereditarily Lindelöf, i. e. there is a right separated  $\omega_1$ -sequence in  $\lambda$ . More precisely, this means that there are  $\mathbb{Q}$ -names  $\dot{s}$  and  $\dot{e}$  such that  $p$  forces “ $\dot{s} : \omega_1 \rightarrow \lambda$ ,  $\dot{e} : \omega_1 \rightarrow F_n(\mathcal{A}, 2)$ ,  $\dot{s}(\alpha) \in B_{\dot{e}(\alpha)}$ , and  $\dot{s}(\beta) \notin B_{\dot{e}(\alpha)}$  whenever  $\alpha < \beta < \lambda$ .” Then, in the ground model  $V$ , for each  $\alpha < \omega_1$  we may pick a condition  $p_\alpha \leq p$ , an ordinal  $\nu_\alpha < \lambda$ , and a finite function  $\varepsilon_\alpha \in F_n(\mathcal{A}, 2)$  such that

$$p_\alpha \Vdash \dot{s}(\alpha) = \nu_\alpha \wedge \dot{e}(\alpha) = \varepsilon_\alpha.$$

Since  $\mathbb{Q}$  is a complete suborder of  $F_n(\mathcal{A} \times \lambda, 2)$  it has property K, hence we may assume without any loss of generality that the conditions  $p_\alpha$  are pairwise compatible. By extending the conditions  $p_\alpha$ , if necessary, we may assume that  $\text{dom } p_\alpha = I_\alpha \times a_\alpha$  with  $I_\alpha \in [\mathcal{A}]^{<\omega}$  and  $a_\alpha \in [\lambda]^{<\omega}$ , moreover  $\text{dom } \varepsilon_\alpha \subset I_\alpha$  and  $\nu_\alpha \in a_\alpha$  whenever  $\alpha < \omega_1$ . With an appropriate thinning out (and re-indexing) we can achieve that if  $\alpha < \beta < \omega_1$  then

$$\nu_\beta \notin a_\alpha \cup \{\gamma_A : A \in I_\alpha\}.$$

Using standard counting and delta-system arguments, we may assume that each  $\varepsilon_\alpha$  has the same size  $n < \omega$ , moreover the sets

$$\text{dom } \varepsilon_\alpha = \{A_{i,\alpha} : i < n\} \in [\mathcal{A}]^n$$

form a delta-system, so that for some  $m < n$  we have  $A_{i,\alpha} = A_i$  if  $i < m$  for all  $\alpha < \omega_1$ , and the families  $\{A_{m,\alpha}, \dots, A_{n-1,\alpha}\}$  are pairwise disjoint. We may also assume that for every  $i < n$  there is a fixed value  $l_i < 2$  such that  $\varepsilon_\alpha(A_i) = l_i$  for all  $\alpha < \omega_1$ . With a further thinning out we may achieve to have

$$\text{dom } \varepsilon_\alpha \cap I_\beta = \{A_i : i < m\}$$

whenever  $\alpha < \beta < \omega_1$ .

Finally, by property (iii) of the good family  $\mathcal{A}$ , we may also assume that the set  $T = \{\nu_\alpha : \alpha < \omega_1\} \in [\lambda]^{\omega_1}$  satisfies either  $|A \cap T| \leq \omega$  or  $T \subset A$  whenever  $A \in \mathcal{A}$ .

Now, after all this thinning out, we claim that there is a countable ordinal  $\alpha > 0$  such that, for every  $i < n$ , if  $\nu_\alpha \in A_{i,0}$  then  $l_i = 0$ . Indeed, arguing indirectly, assume that for every  $0 < \alpha < \omega_1$  there is an  $i_\alpha < n$  with  $\nu_\alpha \in A_{i_\alpha,0}$  and  $l_{i_\alpha} = 1$ . Then there is a fixed  $j < n$  such that the set  $\{\alpha : i_\alpha = j\}$  is uncountable and  $l_j = 1$ . But the first part implies  $|A_{j,0} \cap T| = \omega_1$ , hence  $\nu_0 \in T \subset A_{j,0} \subset U_{A_{j,0}}^0$  that would imply  $\varepsilon_0(A_{j,0}) = l_j = 1$ , a contradiction.

So, let us choose  $\alpha > 0$  as in our above claim. We then define a finite function  $q \in F_n(\mathcal{A} \times \lambda, 2)$  by setting  $q \supset p_0 \cup p_\alpha$ ,

$$\text{dom } q = \text{dom } p_0 \cup \text{dom } p_\alpha \cup \{\langle A_{i,0}, \nu_\alpha \rangle : m \leq i < n\},$$

and finally

$$q(A_{i,0}, \nu_\alpha) = l_i$$

for all  $m \leq i < n$ . We have  $\nu_\alpha \notin a_0$ , and also  $A_{i,0} \notin I_\alpha$  for  $m \leq i < n$  by our construction, hence this definition of  $q$  is correct. Moreover, by the above claim if  $\nu_\alpha \in A_{i,0}$  then  $l_i = 0$  and if  $\nu_\alpha \notin A_{i,0}$  then  $\nu_\alpha \neq \gamma_{A_{i,0}}$ , consequently we actually have  $q \in \mathbb{Q}$ .

Let us observe, however, that we have  $q(A_{i,0}, \nu_\alpha) = l_i$  for all  $i < n$ . Indeed, if  $i < m$  then this holds because  $p_\alpha(A_{i,0}, \nu_\alpha) = p_\alpha(A_i, \nu_\alpha) = l_i$ .

But this implies that  $q \Vdash \nu_\alpha \in B_{\varepsilon_0}$  and hence  $q \Vdash \dot{s}(\alpha) \in B_{\varepsilon(0)}$  that is clearly a contradiction because  $q$  extends  $p$ .

Now assume that we also have  $\mathcal{A} \subset [\lambda]^{<\lambda}$  (in  $V$ ). Since  $\mathbb{Q}$  is CCC, every subset of  $\lambda$  in  $V^{\mathbb{Q}}$  is covered by a ground model set of the same size, hence it suffices to show that any ground model member  $Y$  of  $[\lambda]^{<\lambda}$  is  $\tau$ -nowhere dense. To see this, we first note that it follows from a straight-forward density argument that for every  $\varepsilon \in Fn(\mathcal{A}, 2)$  we have  $|B_\varepsilon| = \lambda$ . (Actually, this only uses the assumption that  $|\lambda \setminus \cup \mathcal{A}_0| = \lambda$  for every  $\mathcal{A}_0 \in [\mathcal{A}]^{<\omega}$  which is weaker than  $\mathcal{A} \subset [\lambda]^{<\lambda}$ .)

Next, consider any set  $Y \in [\lambda]^{<\lambda} \cap V$  and a fixed  $\varepsilon \in Fn(\mathcal{A}, 2)$ . Since  $\mathcal{A}$  satisfies condition (ii) of definition 2, we may clearly find an  $A \in \mathcal{A}$  such that  $Y \subset A$  and  $A \notin \text{dom } \varepsilon$ . Let  $\varepsilon' = \varepsilon \cup \{\langle A, 1 \rangle\}$ , then  $B_{\varepsilon'} = B_\varepsilon \cap U_A^1$  is a non-empty open subset of  $B_\varepsilon$  that is clearly disjoint from  $A$  and hence from  $Y$  as well. This shows that  $Y$  is indeed  $\tau$ -nowhere dense.  $\square$

For a singular cardinal  $\lambda$  of cofinality  $\omega$  the results of [1] did imply the existence of hereditarily Lindelöf regular spaces of density  $\lambda$ , by taking the topological sum of those of density  $\lambda_n$  with  $\lambda_n$  regular and  $\lambda = \sum_{n < \omega} \lambda_n$ . It should be emphasized, however, that the hereditarily Lindelöf regular spaces obtained in this way clearly do not have the stronger property we obtained in theorem 4 that all subsets of size less than  $\lambda$  are nowhere dense. So, we do have here something new even in the case of singular cardinals of cofinality  $\omega$ .

Finally, we would like to point out that the forcing construction given in [1] may be considered as a particular case of that in theorem 4, where the good family  $\mathcal{A}$  over  $\lambda$  happens to be equal to the family of all proper initial segments of  $\lambda$ .

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