A LIFTING THEOREM ON FORCING LCS SPACES

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Abstract. Denote by $\mathcal{THIN}(\alpha)$ the statement that there is an LCS space of height α and width ω. We prove, for each regular cardinal κ, that if there is a "natural" c.c.c poset P such that $\mathcal{THIN}(\kappa)$ holds in V^P then there is a "natural" c.c.c poset Q as well such that $\mathcal{THIN}(\delta)$ holds in V^Q for each $\delta < \kappa^+$.

1. Introduction

A topological space X is called *scattered* if its every non-empty subspace has an isolated point. Denoting by $\mathrm{I}(Y)$ the isolated points of a subspace $Y \subset X$ for each ordinal α define the α^{th} *Cantor-Bendixson level* of the space X, $\mathrm{I}_\alpha(X)$, as follows:

$$\mathrm{I}_\alpha(X) = \mathrm{I}(X \setminus \cup\{\mathrm{I}_\beta(X) : \beta < \alpha\}).$$

The minimal α with $\mathrm{I}_\alpha(X) = \emptyset$ is called the height of X and denoted by $\mathrm{ht}(X)$. Define the width of X, $wd(X)$, as follows: $wd(X) = \sup\{|I_\alpha(X)|\colon \alpha < ht(X)\}$. The *cardinal sequence* of X, $\mathrm{CS}(X)$, is the sequence of the cardinalities of its Candor-Bendixson levels, i.e.

$$\mathrm{CS}(X) = \left\langle\ |I_\alpha(X)| : \alpha < ht(X)\ \right\rangle.$$

The following problem was first posed by R. Telgarsky in 1968 (unpublished): *Does there exist a locally compact, scattered (in short: LCS) space with height ω_1 and width ω?* After some consistency results Rajagopalan, in [8], constructed such a space in ZFC.

To simplify the formulation of the forthcoming results denote by $\mathcal{THIN}(\alpha)$ the statement that there is an LCS space of height α and width ω. (A scattered space is called *thin* iff it has width ω.)

In [4] I. Juhász and W. Weiss showed $\mathcal{THIN}(\alpha)$ for each $\alpha < \omega_2$. W. Just proved, in [5], that this result is sharp in the following sense. Add ω_2 Cohen reals to a ZFC model satisfying CH. Then, in the

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generic extension, $2^\omega = \omega_2$ and $\mathcal{THIN}(\omega_2)$ fails. So you can not prove $\mathcal{THIN}(\alpha)$ for each $\alpha < (2^\omega)^+$ in ZFC.

Just's result was improved in [3] by I. Juhász, S. Shelah, L. Soukup and Z. Szentmiklóssy: if we add Cohen reals to a model of set theory satisfying CH, then, in the new model, every LCS space has at most ω_1 many countable levels.

The notion of Δ-function (see definition 1.1 below) was introduced in [2]. In that paper Baumgartner and Shelah proved that (a) the existence of a Δ-function is consistent with ZFC, (b) if there is a Δ-function then $\mathcal{THIN}(\omega_2)$ holds in a *natural* c.c.c forcing extension. We will explain later, in section 3, what we mean under "natural poset". Roughly speaking, "natural" means that the elements of the posets are just finite approximations of the locally compact right-separating neighbourhoods of the points of the desired space. Building on their method, but using much more involved combinatorics, Martinez [6] proved that if there is a strong Δ-function, then for each $\delta < \omega_3$ there is a c.c.c poset P_δ such that $\mathcal{THIN}(\delta)$ holds in V^{P_δ}. These results naturally raised the following problem.

Problem 1. *Does $\mathcal{THIN}(\omega_2)$ imply $\mathcal{THIN}(\delta)$ for each $\delta < \omega_3$?*

Although this question remains still open we prove a "lifting theorem" claiming that if there is a natural poset P_{ω_2} such that $\mathcal{THIN}(\omega_2)$ holds in $V^{P_{\omega_2}}$ then for each $\delta < \omega_3$ there is a natural poset P_δ such that $\mathcal{THIN}(\delta)$ holds in V^{P_δ}: the posets used by Martinez can be constructed directly from the poset applied by Baumgartner and Shelah without even mentioning the Δ-function. Moreover, our lifting theorem works for each cardinal κ, not only for ω_2! Since there is no Δ-function on ω_3 you can not expect to apply the method of Baumgartner and Shelah to prove $\mathcal{THIN}(\omega_3)$. However, if anybody can construct a "natural" c.c.c poset P such that $\mathcal{THIN}(\omega_3)$ holds in V^P then our theorem gives immediately the consistency of $\mathcal{THIN}(\alpha)$ for each $\alpha < \omega_4$.

To formulate this statement more precisely we introduce some notation, so we postpone the formulation of our main result till theorem 3.15.

First we recall some definition and results.

Definition 1.1. Let $f : \left[\omega_2\right]^2 \longrightarrow \left[\omega_2\right]^{\le\omega}$ be a function with $f\{\alpha,\beta\} \subset \alpha\cap\beta$ for $\{\alpha,\beta\} \in \left[\omega_2\right]^2$. (1) We say that two finite subsets x and y of ω_2 are *good for f* provided that for $\alpha \in x\cap y$, $\beta \in x\setminus y$ and $\gamma \in y\setminus x$ we always have

(a) $\alpha < \beta,\gamma \Longrightarrow \alpha \in f\{\beta,\gamma\}$,

(b) $\alpha < \beta \Longrightarrow f\{\alpha, \gamma\} \subset f\{\beta, \gamma\}$,
(c) $\beta < \gamma \Longrightarrow f\{\alpha, \beta\} \subset f\{\alpha, \gamma\}$.

(2) The function f is a *Δ-function* if every uncountable family of finite subsets of ω_2 contains two elements x and y which are are good for f.
(3) The function f is a *strong Δ-function* if every uncountable family $\mathcal{A}$ of finite subsets of ω_2 contains an uncountable subfamily $\mathcal{B}$ such that any two sets x and y from $\mathcal{B}$ are good for f.

Theorem (Velickovic)**.** *If $\square_{\omega_1}$ holds then there is a strong Δ-function.*

For the proof see [1].

2. A method to force thin LCS spaces with prescribed cardinal sequence

Recall that given a topological space $\langle X, \tau_X\rangle$ a function $f : X \longrightarrow \tau_X$ is called *neighbourhood assignment* iff $x \in f(x)$ for each $x \in X$.

Assume that X is an LCS space. Define the function $\mathrm{ht} : X \longrightarrow \mathrm{ht}(X)$ by the formula $x \in \mathrm{I}_{\mathrm{ht}(x)}(X)$. Since LCS spaces are 0-dimensional, we can fix a neighbourhood-assignment $\mathrm{U} : X \longrightarrow \tau_X$ such that $\mathrm{U}(x)$ is a compact-open neighbourhood of x with

$$\mathrm{U}(x) \setminus \{x\} \subset \mathrm{I}_{<\mathrm{ht}(x)}(X) = \bigcup\{\mathrm{I}_\alpha(X) : \alpha < \mathrm{ht}(x)\}.$$

The family $\{\mathrm{U}(x), X \setminus \mathrm{U}(x) : x \in X\}$ is a subbase of X.

The space is *coherent* iff we can choose U in such a way that $x \in \mathrm{U}(y)$ implies $\mathrm{U}(x) \subset \mathrm{U}(y)$. Such a U is a called *coherent neighbourhood assignment* .

If U is coherent then we can define a partial order $\lhd_U$ on X by taking $x \lhd_{\mathrm{U}} y$ iff $x \in \mathrm{U}(y)$. Since clearly $\mathrm{U}(x) = \{y \in X : y \lhd_{\mathrm{U}} x\}$ we have that $\lhd_{\mathrm{U}}$ determines the neighbourhood assignment U.

If $\lhd$ is an arbitrary partial order on X then define the topology $\tau_\lhd$ on X generated by the family $\{\mathrm{U}_\lhd(x), X \setminus \mathrm{U}_\lhd(x) : x \in X\}$ as a subbase, where $\mathrm{U}_\lhd(x) = \{y \in X : y \lhd x\}$. As we have seen if U witnesses that $\langle X, \tau\rangle$ is coherent then $\tau_{\lhd_{\mathrm{U}}} = \tau$.

So the topologies of coherent LCS spaces are determined by partial orderings. We would like to determine certain properties of a partial ordering in such a way that if some partial order $\langle X, \lhd\rangle$ has those properties then $\langle X, \tau_\lhd\rangle$ is an LCS-space with prescribed Cantor-Bendixson levels.

To formulate these properties we investigate some covering properties of the family $\{\mathrm{U}(x) : x \in X\}$, where U is a coherent neighbourhood assignment on some LCS-space X.

If $x \notin \mathrm{U}(y)$ and $y \notin \mathrm{U}(x)$ then

$$\mathrm{U}(x) \cap \mathrm{U}(y) \subset \bigcup \{\mathrm{U}(z) : z \in \mathrm{U}(x) \cap \mathrm{U}(y)\}.$$

Since $\mathrm{U}(x) \cap \mathrm{U}(y)$ is compact there is a finite set $i\{x, y\} \in \left[\mathrm{U}(x) \cap \mathrm{U}(y)\right]^{<\omega}$ such that

$$\mathrm{U}(x) \cap \mathrm{U}(y) \subset \bigcup \{\mathrm{U}(z) : z \in i\{x, y\}\}.$$

We will enumerate some properties of $\lhd$ and the function i. Let $\delta = \mathrm{ht}(X)$ and for $\alpha < \delta$ write $X_\alpha = \mathrm{I}_\alpha(X)$.

(I) if $x \in X_\alpha$, $y \in X_\beta$ and $x \lhd y$ then either $x = y$ or $\alpha < \beta$,
(II) $\forall \{x, y\} \in \left[X\right]^2$ $\big(\ \forall z \in X\ (z \lhd x \wedge z \lhd y)$ iff $\exists t \in i\{x, y\}\ z \lhd t\ \big)$.
(III) if $x \in X_\alpha$ and $\beta < \alpha$ then the set $\{y \in X_\beta : y \lhd x\}$ is infinite.

Proposition 2.1. *Assume that $\{X_\alpha : \alpha < \delta\}$ is a partition of a given set X, $\lhd$ is a partial order on X and $i : \left[X\right]^2 \longrightarrow \left[X\right]^{<\omega}$ is a function satisfying (I)–(III). Then $\mathcal{X} = \langle X, \tau_\lhd \rangle$ is a (coherent) LCS space with $\mathrm{I}_\alpha(\mathcal{X}) = X_\alpha$ for $\alpha < \delta$.*

Proof. X is right-separated, i.e. scattered, witnessed by any well-ordering extending the well-founded partial ordering $\lhd$ because of (I).

For each $x \in X$, the family

$$\mathbb{U}(x) = \Big\{\ \mathrm{U}_\lhd(x) \setminus \bigcup_{y \in F} \mathrm{U}_\lhd(y) : F \in \left[\mathrm{U}_\lhd(x) \setminus \{x\}\right]^{<\omega} \Big\}$$

is a neighbourhood base of x. Indeed, if $x \neq y$ then $\mathrm{U}_\lhd(x) \cap \mathrm{U}_\lhd(y) = \mathrm{U}_\lhd(x)$ provided $x \in \mathrm{U}_\lhd(y)$ and

$$\mathrm{U}_\lhd(x) \setminus \mathrm{U}_\lhd(y) = \mathrm{U}_\lhd(x) \setminus \bigcup_{z \in i\{x,y\}} \mathrm{U}_\lhd(z)$$

provided $x \notin \mathrm{U}_\lhd(y)$, where $i\{x, y\} \in \left[\mathrm{U}_\lhd(x) \setminus \{x\}\right]^{<\omega}$.

Lemma 2.2. $\mathrm{I}_\alpha(\mathcal{X}) = X_\alpha$.

Proof. First we show by induction on α that if $x \in X_\alpha$, $U \in \mathbb{U}(x)$ and $\beta \leq \alpha$ then $U \cap X_\beta \neq \emptyset$. For $\beta = \alpha$ we have $x \in U \cap X_\beta$ so we can assume $\beta < \alpha$. Assume that $U = \mathrm{U}_\lhd(x) \setminus \bigcup \{\mathrm{U}_\lhd(z) : z \in F\}$, where $F \in \left[\mathrm{U}_\lhd(x) \setminus \{x\}\right]^{<\omega}$. Let $\mu = \max\{\nu : F \cap X_\nu \neq \emptyset\}$ and $\gamma = \max\{\mu, \beta\}$. Since $\gamma < \alpha$ by (III) we can pick $t \in (X_\gamma \cap \mathrm{U}_\lhd(x)) \setminus F$. Then $\mathrm{U}_\lhd(t) \setminus \bigcup \{\mathrm{U}_\lhd(z) : z \in F\} \subset U$ is a neighbourhood of t which intersects X_β by the inductive hypothesis because $t \in X_\gamma$ and $\beta \leq \gamma < \alpha$.

Now prove the statement of the lemma by induction on α. Let $Y = X \setminus \bigcup_{\beta < \alpha} \mathrm{I}_\beta(X) = X \setminus \bigcup_{\beta < \alpha} X_\beta$. If $x \in X_\alpha$ then $U(x) \cap Y = \{x\}$, so $X_\alpha \subset I(Y)$. If $x \in X_\gamma$ for some $\gamma > \alpha$ then for any neighbourhood of U we have $U \cap X_\alpha \neq \emptyset$, i.e. $U \cap Y \neq \{x\}$, and so $x \notin I(Y)$. Thus $I(Y) = X_\alpha$ which was to be proved. $\square$

Lemma 2.3. $\mathrm{U}_\triangleleft(x)$ *is compact in* $\mathcal{X}$.

Proof. We prove this statement by induction on $\mathrm{ht}(x)$. By Alexander's subbase lemma it suffices to show that any cover $\mathcal{V}$ of $\mathrm{U}_\triangleleft(x)$ by members of $\{\mathrm{U}_\triangleleft(y) : y \in X\}$ and their complements has a finite subcover. Let $V \in \mathcal{V}$ be such that $x \in V$. If $V = \mathrm{U}_\triangleleft(y)$ then $\mathrm{U}_\triangleleft(x) \subset \mathrm{U}_\triangleleft(y)$ so we have a one element covering. So we can assume that $V = X \setminus \mathrm{U}_\triangleleft(y)$. Then

$$\mathrm{U}_\triangleleft(x) \setminus V = \mathrm{U}_\triangleleft(x) \cap \mathrm{U}_\triangleleft(y) = \bigcup \big\{ \mathrm{U}_\triangleleft(z) : z \in i\{x,y\} \big\}.$$

For each $z \in i\{x,y\}$ we have $\mathrm{ht}(z) < \mathrm{ht}(x)$ and so $\mathrm{U}_\triangleleft(z)$ is compact, and so $\mathrm{U}_\triangleleft(x) \setminus V$ is compact as well. Thus there is a finite $\mathcal{W} \subset \mathcal{V}$ with $\mathrm{U}_\triangleleft(x) \setminus V \subset \bigcup \mathcal{W}$. Hence $\mathcal{W} \cup \{V\}$ is a finite cover of $\mathrm{U}_\triangleleft(x)$ $\square$

This completes the proof of proposition 2.1. $\square_{2.1}$

We say that $\triangleleft$ is an *LCS-order on* X iff $\langle X, \tau_\triangleleft \rangle$ is an LCS-space.

So our strategy to force an LCS space with a prescribed cardinal sequence $\langle \kappa_\alpha : \alpha < \delta \rangle$ is the following. Let $X_\alpha = \{\alpha\} \times \kappa_\alpha$ for $\alpha < \delta$ and put $X = \bigcup \{X_\alpha : \alpha < \delta\}$. Now we try to add generically a partial ordering $\triangleleft$ on X and a function $i : \big[X\big]^2 \longrightarrow \big[X\big]^{<\omega}$ satisfying (I)–(III) using finite approximations. That is, a typical forcing condition is a triple $\langle a, \leq, i \rangle$, where a is a finite subset of X, $\leq$ is a partial order on a, and i is a function on $\big[a\big]^2$ such that $\langle a, \leq, i \rangle$ satisfies (I) and (II). (III) would be guaranteed by some density argument. This type of forcing was introduced by Judy Roitmann to get thin superatomic Boolean algebras (LCS spaces).

The main problem is that the poset of all the possible finite approximations may not satisfy c.c.c. That is the point where the Δ-function came into the picture. Baumgartner and Shelah, and later Martinez, applied this function to select a suitable subfamily of the conditions which satisfies c.c.c. Our strategy will be different: we show that if there is a suitable poset which introduces $\mathcal{THIN}(\kappa)$ then for each $\delta < \kappa^+$ there is a a suitable poset which introduces $\mathcal{THIN}(\delta)$.

This strategy will be carried out in the next section in a special situation.

3. Lifting theorem

Fix a cardinal $\kappa \geq \omega$ and let $\pi : \kappa^+ \times \omega \longrightarrow \kappa^+$ be the natural projection: $\pi(\langle \alpha, n \rangle) = \alpha$.

Define the poset $P^0 = \langle P^0, \prec\rangle$ as follows. The underlying set P^0 consists of triples $\langle a, \leq, i\rangle$ satisfying the following requirements:

(i) $a \in \left[\kappa^+ \times \omega\right]^{<\omega}$,
(ii) $\leq$ is a partial ordering on a,
(iii) $\forall \{x,y\} \in \left[a\right]^2$ if $x \leq y$ the $\pi(x) < \pi(y)$,
(iv) $i : \left[a\right]^2 \longrightarrow \mathcal{P}(a)$ is a function,
(v) $\forall \{x,y\} \in \left[a\right]^2$ if $\pi(x) = \pi(y)$ then $i\{x,y\} = \emptyset$,
(vi) $\forall \{x,y\} \in \left[a\right]^2$ if $x \leq y$ then $i\{x,y\} = \{x\}$.

Write $p = \langle a^p, \leq^p, i^p\rangle$ for $p \in P^0$. Define the function $h^p : a^p \longrightarrow \mathcal{P}(a^p)$ by the formula $h^p(x) = \{y \in a^p : y \leq^p x\}$. For $b \subset a^p$ write $h^p[b] = \bigcup\{h^p(x) : x \in b\}$.

Let $p \prec q$ iff $a^q \subset a^p$,
$\leq^q = \leq^p \cap (a^q \times a^q)$,
$i^q \subset i^p$.

Clearly $\prec$ is a partial ordering on P^0.

Let

$$P^* = \left\{\langle a, \leq, i\rangle \in P^0 : \forall \{x,y\} \in \left[a\right]^2\ \forall z \in a\right.$$
$$\left.(z \leq x \wedge z \leq y) \text{ iff } \exists t \in i\{x,y\}\ z \leq t.\right\}$$

Fact 3.1. *For $p \in P^0$,*

$$p \in P^* \textit{ iff } \forall \{x,y\} \in \left[a^p\right]^2\ h^p(x) \cap h^p(y) = h^p[i^p\{x,y\}].$$

The elements of P^* can be considered as the natural finite approximations of an LCS-order on $\kappa^+ \times \omega$ and the witnessing function i.

Definition 3.2. Two condition $p, q \in P^0$ are *twins* iff (i)–(ii) below hold, where $a = a^p \cap a^q$:

(i) $\leq^p \restriction a = \leq^q \restriction a$,
(ii) $i^p \restriction \left[a\right]^2 = i^q \restriction \left[a\right]^2$.

Definition 3.3. Let $p, q \in P^0$ be twins. A condition $r \in P^0$ is an *amalgamation of p and q* iff

(a) $a^r = a^p \cup a^q$
(b) $\leq^r$ is the partial ordering on a^r generated by $\leq^p \cup \leq^q$,
(c) $i^r \supset i^p \cup i^q$.

Let

$$\mathfrak{amalg}(p,q) = \{r : r \text{ is an amalgamation of } p \text{ and } q\}.$$

When we speak about amalgamations of two conditions we will always assume that these conditions are twins.

Fact 3.4. *If $r \in P^0$ is an amalgamation of p and q, then*
(1) $\le^r \restriction a^p = \le^p$,
(2) $r \prec p$ and $r \prec q$,
(3) If $x \in a^p$ and $y \in a^q$ then $x \le^r y$ iff there is $z \in a^p \cap a^q$ such that $x \le^p z \le^q y$.

Fact 3.5. *If $r \in P^0$ is an amalgamation of p and q, moreover $p, q \in P^*$ then*

$$\forall \{x, y\} \in \left[a^p\right]^2 \cup \left[a^q\right]^2 \ h^r(x) \cap h^r(y) = h^r[i\{x, y\}].$$

Proof. Assume that $\{x, y\} \in \left[a^p\right]^2$ and let $z \in (h^r(x) \cap h^r(y)) \cap a^q$. Then there are $u, v \in a^p \cap a^q$ with $z \le^q u \le^p x$ and $z \le^q v \le^p y$. Since $q \in P^*$ there is $w \in i^q\{u, v\}$ with $z \le^q w$. Since $i^p\{u, v\} = i^q\{u, v\}$ we have $w \in a^p \cap a^q$. Thus $w \in h^p(x) \cap h^p(y)$. Since $q \in P^*$, there is $t \in i^p\{x, y\}$ with $w \le^p t$. Thus $z \le^q w \le^p t \in i^p\{x, y\} = i^r\{x, y\}$ and hence $z \in h^r[i^r\{x, y\}]$. $\square_{3.5}$

For $A \subset \kappa^+$ let

$$P^*_A = \{p \in P^* : a^p \subset A \times \omega\}.$$

Next we introduce three properties, (K^+), D^A_1 and D^A_2, of posets $\langle P, \prec\rangle$, where $P \subset P^*_A$ for some $A \subset \kappa^+$. The first one is a strong version of property (K), the two others are density requirements.

Definition 3.6. Let $P \subset P^*$. The poset $\mathcal{P} = \langle P, \prec\rangle$ has *property (K^+)* iff

> $\forall S \in \left[P\right]^{\omega_1} \ \exists T \in \left[S\right]^{\omega_1} \ \forall \{p, q\} \in \left[T\right]^2$ p and q have an amalgamation in P.

Definition 3.7. For a condition $p \in P^0$ and $x \in (\kappa^+ \times \omega) \setminus a^p$ define $q = p \uplus \{x\} \in P^0$ as follows:

- $a^q = a^p \cup \{x\}$,
- $\le^q = \le^p \cup \{\langle x, x\rangle\}$,
- $i^p \subset i^q$,
- $i^q\{x, y\} = \emptyset$ for $y \in a^p$.

Fact 3.8. *$p \uplus \{x\} \in P^*_A$ for each $p \in P^*_A$ and $x \in (A \times \omega) \setminus a^p$.*

Definition 3.9. Let $P \subset P^*_A$. The poset $\mathcal{P} = \langle P, \prec\rangle$ has property D^A_1 iff

> $p \uplus \{x\} \in P$ for each $p \in P$ and $x \in (A \times \omega) \setminus a^p$.

Definition 3.10. For $p \in P_A$, $x \in a^p$, $y_0, y_1, \ldots, y_{n-1} \in (A \times \omega) \setminus a^p$ with $\pi(y_0) < \pi(y_1) < \ldots \pi(y_{n-1}) < \pi(x)$ define the condition $q = p \uplus_x \langle y_0, \ldots, y_{n-1}\rangle \in P^0$ as follows:

- $a^q = a^p \cup \{y_0, \dots, y_{n-1}\}$,
- $\leq^q = \leq^p \cup \{\langle y_i, y_j : i \leq j < n\rangle\} \cup \{\langle y_i, z\rangle : z \in a^p, x \leq^p z\}$,
- $i^p \subset i^q$,
- $i^q\{y_i, y_j\} = y_{\min(i,j)}$,
- $i^q\{y_i, z\} = \begin{cases} y_i & \text{if } x \leq^p z \\ \emptyset & \text{otherwise} \end{cases}$ for $z \in a^p$.

Fact 3.11. *If $p \in P^*_A$, $x \in a^p$, $y_0, y_1, \dots, y_{n-1} \in (A \times \omega) \setminus a^p$ with $\pi(y_0) < \pi(y_1) < \dots \pi(y_{n-1}) < \pi(x)$, then $q = p \uplus_x \langle x_0, \dots, x_{n-1}\rangle \in P^*_A$.*

Definition 3.12. Let $P \subset P^*_A$. The poset $\mathcal{P} = \langle P, \prec\rangle$ has property D^A_2 iff

> $\forall\{\alpha, \beta\} \in A$, $\alpha < \beta$, there is a finite set of ordinals $L^P(\alpha, \beta) = \{\alpha_0, \dots, \alpha_{n-1}\} \in [A]^{<\omega}$ such that $\alpha = \alpha_0 < \alpha_1 < \dots \alpha_{n-1} < \beta$ and if $p \in P$, $x \in a^p$ with $\pi(x) = \beta$ and $x_i \in (A \times \omega) \setminus a^p$ with $\pi(x_i) = \alpha_i$ for $i < n$, then $p \uplus_x \langle y_0, \dots y_{n-1}\rangle \in P$.

Definition 3.13. Let $A \subset \kappa^+$ and $P \subset P^*$. The poset $P = \langle P, \prec\rangle$ is *A-nice* iff $P \subset P^*_A$ and P has properties (K^+), (D^A_1) and (D^A_2). For $\delta < \kappa^+$ let $\mathcal{NAT}(\delta)$ be the statement that there is δ-nice poset P_δ.

Proposition 3.14. *If a poset P is δ-nice then P has property (K) and $\mathcal{THIN}(\delta)$ holds in V^P.*

Proof. By fact 3.4(2) property (K^+) implies property (K). Let $G \subset P$ be a generic filter. Put $A = \bigcup\{a^p : p \in G\}$, $i = \bigcup\{i^p : p \in G\}$ and $\prec = \bigcup\{\prec^p : p \in G\}$. Then $A = \delta \times \omega$ by (D^δ_1). The partial ordering $\prec$ satisfies (I) because every $p \in P$ satisfies (iii). The function $i : [\delta \times \omega]^2 \longrightarrow [\delta \times \omega]^{<\omega}$ satisfies (II) because every element of P is in P^*. Finally (III) holds because (D^2_δ) can be applied in a suitable density argument. Thus $\langle \delta \times \omega, \tau_\prec\rangle$ is an LCS space with levels $\{\alpha\} \times \omega$ for $\alpha < \delta$. $\square_{3.14}$

After this preparation we are able to formulate the main lifting theorem.

Theorem 3.15. *$\mathcal{NAT}(\kappa)$ implies $\mathcal{NAT}(\delta)$ for each cardinal κ and ordinal $\delta < \kappa^+$.*

First, in lemma 3.16 below, we show that our lifting theorem works downwards. Although $\mathcal{THIN}(\kappa)$ clearly implies $\mathcal{THIN}(\delta)$ for $\delta < \kappa$ we should prove $\mathcal{NAT}(\delta)$ for $\delta < \kappa$ as well, because we will use the posets witnessing this to prove $\mathcal{NAT}(\gamma)$ for $\gamma \geq \kappa$.

If $p \in P^0$ and $I \subset \kappa^+$ let

$$p \restriction I = \left\langle a^p \cap (I \times \omega), \leq^p \restriction (I \times \omega), i^p \restriction [I \times \omega]^2 \right\rangle.$$

Observe that

- $p \restriction I \in P^0$ iff $i^p\{x,y\} \subset I \times \omega$ for each $\{x,y\} \in \left[a^p \cap (I \times \omega)\right]^2$,
- if $p \in P^*$ and $p \restriction I \in P^0$ then $p \restriction I \in P^*$.

Lemma 3.16. *$\mathcal{NAT}(\kappa)$ implies $\mathcal{NAT}(\delta)$ for $\delta < \kappa$.*

Proof. Fix $P_\kappa \subset P^*_\kappa$ such that $\mathcal{P}_\kappa = \langle P_\kappa, \prec\rangle$ has properties (K^+), (D^κ_1) and (D^κ_2). Let $\mathcal{P}_\delta = \langle P_\delta, \prec\rangle$, where $P^\delta = \{p \restriction \delta : p \in P_\kappa\}$.

We should check that $\mathcal{P}_\delta$ also has has properties (K^+), (D^δ_1) and (D^δ_2).

(K^+): Let $\{p_\nu \restriction \delta : \nu < \omega_1\} \in \left[P_\delta\right]^{\omega_1}$. We can assume that for each $\{\nu,\mu\} \in \left[\omega_1\right]^2$ p_ν and μ have an amalgamation $r_{\nu,\mu} \in P_\kappa$. Hence $r_{\nu,\mu} \restriction \delta \in P_\delta$ is an amalgamation of $p_\nu \restriction \delta$ and $p_\mu \restriction \delta$.

(D^δ_1) is easy: $(p \restriction \delta) \uplus \{x\} = (p \uplus \{x\}) \restriction \delta$ for $\pi(x) < \delta$.

(D^δ_2) is also easy: $(p \restriction \delta) \uplus_x \langle y_0, \ldots, y_{n-1}\rangle = (p \uplus_x \langle y_0, \ldots, y_{n-1}\rangle) \restriction \delta$ for $\pi(x) < \delta$. $\square_{3.16}$

Proof of theorem 3.15. Since we know the statement for $\delta < \kappa$ we prove the theorem by induction on $\delta \geq \kappa$. When we constructed P_δ we will also have $P_A \subset P^*_A$ for each $A \subset \kappa^+$ with order type δ such that $\mathcal{P}_A = \langle P_A, \prec\rangle$ has properties (K^+), (D^A_1) and (D^A_2).

We will write $L^A(\alpha,\beta)$ for $L^{\mathcal{P}_A}(\alpha,\beta)$. Let $L^A(\alpha,\alpha) = \emptyset$.

Successor step:

Assume that $\mathcal{P}_\delta$ is constructed. Then we can get $\mathcal{P}_{\delta+1}$ as follows.

A p is in $P_{\delta+1}$ iff

(i) $p \in P^*_{\delta+1}$,
(ii) $p \restriction \delta \in P_\delta$,
(iii) $\forall \{x,y\} \in \left[a^p\right]^2$ if $\pi(x) < \delta$ and $\pi(y) = \delta$ then either $i\{x,y\} = x$ (i.e. $x \leq^p y$) or $i\{x,y\} = \emptyset$ (i.e. $h^p(x) \cap h^p(y) = \emptyset$).

We show that $\mathcal{P}_{\delta+1} = \langle P_\delta, \prec\rangle$ works, i.e. it satisfies properties (K^+), $(D^{\delta+1}_1)$ and $(D^{\delta+1}_2)$.

Lemma 3.17. *$\mathcal{P}_{\delta+1}$ satisfies (K^+).*

Proof. Let $\{p_\nu : \nu < \omega_1\} \in \left[P_{\delta+1}\right]^{\omega_1}$, $p_\nu = \langle a_\nu, \leq_\nu, i_\nu\rangle$, $h_\nu = h^{p_\nu}$. Without loss of generality

(a) $\forall \{\nu,\mu\} \in \left[\omega_1\right]^2$ $\exists r_{\nu,\mu} \in P_\delta$ $r_{\nu,\mu}$ is an amalgamation of p_ν and p_μ.
(b) $\exists t$ $a_\nu \cap (\{\delta\} \times \omega) = t$.
(c) $\{a_\nu : \nu < \omega_1\}$ forms a Δ-system with kernel a.
(d) $\leq_\nu \restriction a = \leq_\mu \restriction a$ for each $\{\nu,\mu\} \in \left[\omega_1\right]^2$

(iii) and (d) together imply that

(e) $\forall \{\nu,\mu\} \in \left[\omega_1\right]^2$ $\forall x \in t$ $\forall y \in (a \setminus t)$ $i_\nu\{x,y\} = i_\mu\{x,y\}$.

Now for each $\{\nu,\mu\}\in\left[\omega_1\right]^2$ the conditions p_ν and p_μ are twins and we can define $r\in P^0$ as follows:

- r is an amalgamation of p_ν and p_μ
- $r\restriction\delta=r_{\nu,\mu}$.

If $\{x,y\}\in\left[a^r\right]^2\setminus\left(\left[a^p\right]^2\cup\left[a^q\right]^2\right)$ then $\{x,y\}\in\left[a^{r_{\nu,\mu}}\right]^2$. Hence $r_{\nu,\mu}\in P^*_\delta$ and fact 3.5 imply that $r\in P^*_{\delta+1}$. Thus $r\in P_{\delta+1}$. $\square_{3.17}$

Lemma 3.18. *$\mathcal{P}_{\delta+1}$ satisfies ($D_1^{\delta+1}$).*

Straightforward.

Lemma 3.19. *$\mathcal{P}_{\delta+1}$ satisfies ($D_2^{\delta+1}$).*

Proof. For $\alpha<\beta<\delta$ let $L^{\delta+1}_{\alpha,\beta}=L^\delta_{\alpha,\beta}$. For $\alpha<\delta$ let $L^{\delta+1}_{\alpha,\delta}=\{\alpha\}$. $\square_{3.19}$

The successor step is done.

Limit step:

Assume that δ is limit ordinal, and $\mathcal{P}_A$ is constructed for each $A\subset\kappa^+$ with order type $<\delta$.

Fix a club $C\subset\delta$, $C=\{\gamma_\zeta:\zeta<\mathrm{cf}(\delta)\}$. Let $I_\zeta=[\gamma_\zeta,\gamma_{\zeta+1})$ for $\zeta<\mathrm{cf}(\delta)$.

Let $\rho:\delta\longrightarrow\mathrm{cf}(\delta)$ s.t. $\rho(\alpha)=\zeta$ iff $\alpha\in I_\zeta$.

Let $p\in P_\delta$ iff

(δ1) $p\in P^*_\delta$,

(δ2) $p\restriction C\in P_C$,

(δ3) $p\restriction I_\zeta\in P_{I_\zeta}$ for each $\zeta<\mathrm{cf}(\delta)$,

(δ4) $\forall x,y\in a^p$ if $x\le^p y$, $\gamma_\zeta<\pi(x)<\gamma_{\zeta+1}\le\pi(y)$ then $\exists u\in a^p$ $x\le^p u\le^p y$ and $\pi(u)=\gamma_{\zeta+1}$

(δ5) $\forall x,y\in a^p$ if $x\le^p y$, $\pi(x)<\gamma_\xi\le\pi(y)<\gamma_{\xi+1}$ then $\exists v\in a^p$ $x\le^p v\le^p y$ and $\pi(v)=\gamma_\xi$

(δ6) $\forall x,y\in a^p$, $\gamma_\zeta\le\pi(x)<\gamma_{\zeta+1}\le\gamma_\xi\le\pi(y)<\gamma_{\xi+1}$, $x\not\le^p y$ then

$$i^p\{x,y\}\subset\bigcup\{i^p\{u,v\}:u\le^p x,v\le^p y,\pi(u)=\gamma_\zeta,\pi(v)=\gamma_\xi\}.$$

We show that $\mathcal{P}_\delta=\langle P_\delta,\prec\rangle$ works, i.e. it satisfies properties (K^+), (D_1^δ) and (D_2^δ).

Lemma 3.20. *$\mathcal{P}_\delta$ satisfies (K^+).*

Proof. Let $\{p_\nu:\nu<\omega_1\}\in\left[P_\delta\right]^{\omega_1}$, $p_\nu=\langle a_\nu,\le_\nu,i_\nu\rangle$, $h_\nu=h^{p_\nu}$. Let $c_\nu=\{\eta<\mathrm{cf}(\delta):a_\nu\cap I_\eta\ne\emptyset\}$. By thinning out the sequence $\{p_\nu:\nu<\omega_1\}$ we can assume that

(a) $\{a_\nu:\nu\in\omega_1\}$ forms a Δ-system with kernel d,

(b) there is a partial ordering $\le^d$ on d such that $\le_\nu\restriction d = \le^d$ for each $\nu \in \omega_1$,
(c) $\{c_\nu : \nu < \omega_1\}$ forms a Δ-system with kernel c.
(d) $\forall \eta \in c\ \exists e_\eta\ \forall \nu \in \omega_1\ a_\nu \cap (\{\gamma_\eta, \gamma_\eta + 1\} \times \omega) = e_\eta$
(e) $\forall \eta \in c\ \forall \{\nu, \mu\} \in \left[\omega_1\right]^2$ the conditions $p_\nu \restriction I_\eta$ and $p_\mu \restriction I_\eta$ have an amalgamation $r^\eta_{\nu,\mu} = \left\langle a^\eta_{\nu,\mu}, \le^\eta_{\nu,\mu}, i^\eta_{\nu,\mu}\right\rangle$ in P_{I_η}.
(f) $\forall \{\nu, \mu\} \in \left[\omega_1\right]^2$ the conditions $p_\nu \restriction C$ and $p_\mu \restriction C$ have an amalgamation $r^C_{\nu,\mu} = \left\langle a^C_{\nu,\mu}, \le^C_{\nu,\mu}, i^C_{\nu,\mu}\right\rangle$ in P_C.
(g) $i_\nu\{x, y\} = i_\mu\{x, y\}$ for each $\{x, y\} \in \left[d\right]^2$ and $\{\nu, \mu\} \in \left[\omega_1\right]^2$.

To ensure (g) fix $\{x, y\} \in \left[d\right]^2$. If $\rho(x) = \rho(y) = \eta$ then (g) holds by (e): $i_\nu\{x, y\} = i_\mu\{x, y\}$. If $\{\pi(x), \pi(y)\} \in \left[C\right]^2$ then $i_\nu\{x, y\} = i_\mu\{x, y\} \overset{\text{def}}{=} i^C\{x, y\}$ by (f). If $\eta = \rho(x) \ne \rho(y) = \sigma$ then by (δ6) we have

$$i_\nu\{x, y\} \subset \bigcup\{i_\nu\{u, v\} : u \le_\nu x, v \le_\nu y, \pi(u) = \gamma_{\rho(x)}, \pi(v) = \gamma_{\rho(y)}\} \subset \bigcup\{i^C\{u, v\} : u \in e_{\rho(x)}, v \in e_{\rho(y)}\},$$

i.e. $i_\nu\{x, y\}$ is a subset of a fixed finite set for each $\nu \in \omega_1$. So, by thinning out our sequence we can guarantee that (g) holds.

Claim 3.20.1. *p_ν and p_μ are twins for each $\{\nu, \mu\} \in \left[\omega_1\right]^2$.*

Fix $\{\nu, \mu\} \in \left[\omega_1\right]^2$. Define $r = \langle a, \le, i\rangle \in P^0$ as follows:
(r1) $a = a_\nu \cup a_\mu$,
(r2) $\le$ is the partial ordering on a generated by $\le_\nu \cup \le_\mu$,
(r3)

$$i\{x, y\} = \begin{cases} i_\nu\{x, y\} & \text{if } \{x, y\} \in \left[a_\nu\right]^2, \\ i_\mu\{x, y\} & \text{if } \{x, y\} \in \left[a_\mu\right]^2, \\ i^C_{\nu,\mu}\{x, y\} & \text{if } \{x, y\} \in \left[C\right]^2, \\ i^\eta_{\nu,\mu}\{x, y\} & \text{if } \{x, y\} \in \left[I_\eta\right]^2, \\ M(x, y) & \text{otherwise,} \end{cases}$$

where

$$M(x, y) = \bigcup\{i\{u, v\} : \{u, v\} \in \left[a\right]^2, u \le x, v \le y, \pi(u) = \gamma_{\rho(x)}, \pi(v) = \gamma_{\rho(y)}\}.$$

Claim 3.20.2. *r is an amalgamation of p_ν and p_μ.*

Claim 3.20.3. $\le\restriction C\times\omega = \le^C_{\nu,\mu}$.

Proof. Let $x,y\in a\cap(C\times\omega)$, $x\le y$. We can assume that $x\in a_\nu$ and $y\in a_\mu$, and $\rho(x)<\rho(y)$. Then, by fact 3.4(3), there is $z\in d$ with $x\le_\nu z\le_\mu y$. Then, applying ($\delta 5$) for x and z in p_ν there is $v\in a_\nu$ such that $x\le_\nu v\le_\nu z$ and $\pi(\nu)=\gamma_{\rho(z)}$. Since $z\in a$ we have $\rho(z)\in c$ and so $v\in e_{\rho(z)}\subset d$. Thus $x\le^C_{\nu,\mu} y$ because $x\le_\nu v\le_\mu y$ and $v\in d\cap(C\times\omega)$. $\square_{3.20.3}$

Claim 3.20.4. *r satisfies ($\delta 2$) and ($\delta 3$).*

Proof. $r\restriction I_\eta = r^\eta_{\nu,\mu}\in P_{I_\eta}$ is clear for each $\eta<\mathrm{cf}(\delta)$ and $r\restriction C = r^C_{\nu,\mu}\in P_C$ follows from claim 3.20.3. $\square_{3.20.4}$

Claim 3.20.5. *r satisfies ($\delta 4$).*

Proof. Assume that $\{x,y\}\in\left[a\right]^2$, $x\le y$, $\gamma_\eta<\pi(x)<\gamma_{\eta+1}\le\pi(y)$. We can assume that $x\in a_\nu\setminus a_\mu$ and $y\in a_\mu\setminus a_\nu$. Pick $z\in d$ such that $x\le_\nu z\le_\mu y$.

If $\gamma_\eta<\pi(z)<\gamma_{\eta+1}$ then applying ($\delta 4$) for the pair $\{z,y\}$ in p_μ we obtain $u\in a_\mu$ such that $z\le_\mu u\le_\mu y$ and $\pi(u)=\gamma_{\eta+1}$. Then this u works for $\{x,y\}$.

If $\gamma_{\eta+1}\le\pi(z)$ then applying ($\delta 4$) for the pair $\{x,z\}$ in p_ν we obtain $v\in a_\nu$ such that $x\le_\nu v\le_\nu z$ and $\pi(v)=\gamma_{\eta+1}$. $\square_{3.20.5}$

Claim 3.20.6. *r satisfies ($\delta 5$).*

Proof. Assume that $\{x,y\}\in\left[a\right]^2$, $x\le y$, $\pi(x)<\gamma_\eta\le\pi(y)<\gamma_{\eta+1}$. We can assume that $x\in a_\nu\setminus a_\mu$ and $y\in a_\mu\setminus a_\nu$. Pick $z\in d$ such that $x\le_\nu z\le_\mu y$.

If $\gamma_\eta\le\pi(z)<\gamma_{\eta+1}$ then applying ($\delta 5$) for the pair $\{x,z\}$ in p_ν we obtain an $u\in a_\nu$ such that $x\le_\nu u\le_\nu z$ and $\pi(u)=\gamma_\eta$. Then this u works for $\{x,y\}$.

If $\gamma_{\eta+1}<\pi(z)$ then applying ($\delta 5$) for the pair $\{z,y\}$ in p_μ we obtain a $v\in a_\mu$ such that $z\le_\mu v\le_\mu y$ and $\pi(v)=\gamma_\eta$. $\square_{3.20.6}$

Claim 3.20.7. *r satisfies ($\delta 6$).*

Straightforward from the construction of i.

Claim 3.20.8. *r satisfies ($\delta 1$): $r\in P^*_\delta$.*

Proof. Write $h=h^r$. Let $\{x,y\}\in\left[a\right]^2$ be $\le$-incomparable elements.. By fact 3.5 we can assume that $x\in a_\nu\setminus a_\mu$ and $y\in a_\mu\setminus a_\nu$. Let $z\in h(x)\cap h(y)$.

Case 1. $\rho(x) = \rho(y)$.

Let $\eta = \rho(x)$. Since $r \restriction I_\eta = r^\eta_{\nu,\mu} \in P^*_{I_\eta}$ we can assume that $\pi(z) < \gamma_\eta$. Applying ($\delta 5$) for the pairs $\{z, x\}$ and $\{z, y\}$ we obtain u and v, respectively, such that $\pi(u) = \pi(v) = \gamma_\eta$, $z \le u \le x$, $z \le v \le y$. Since $\eta \in c_\nu \cap c_\mu = c$ we have $\{u, v\} \subset d$. Since $z \in h(u) \cap h(v)$ we have $u = v$ by fact 3.5. Hence there is $t \in i^\eta_{\nu,\mu}\{x, y\} = i\{x, y\}$ with $u \le^\eta_{\nu,\mu} t$. Thus $z \le t \in i\{x, y\}$.

Case 2. $\rho(z) = \rho(x) < \rho(y)$, $\pi(z) = \gamma_{\rho(x)}$.

Applying ($\delta 5$) for the pair $\{z, y\}$ there is $u \in a$ such that $z \le u \le y$ and $\pi(u) = \gamma_{\rho(y)}$. Then $i\{z, u\} = \{z\}$ and $i\{z, u\} \subset i\{x, y\}$.

Case 3. $\rho(z) = \rho(x) < \rho(y)$, $\pi(z) > \gamma_{\rho(x)}$.

Applying ($\delta 4$) for the pair $\{z, y\}$ there is $u \in a$ such that $z \le u \le y$ and $\pi(u) = \gamma_{\rho(x)+1}$. If $u \in a_\mu$ then there is $w \in d \cap (I_{\rho(X)} \times \omega)$ such that either $z \le_\nu w \le_\mu u$ or $z \le_\mu w \le_\nu x$ by fact 3.4(3). Hence $\rho(x) \in c$ and so $u \in e_{\rho(x)} \subset d \subset a_\nu$. Thus $u \in a_\nu$. Thus $z \in h_\nu(x) \cap h_\nu(u)$, hence by fact 3.5 and by ($\delta 6$) we have $x \le_\nu u$. Hence $x \le y$, contradiction, this case is not possible.

Case 4. $\rho(z) < \rho(x) < \rho(y)$.

Applying ($\delta 4$) for the pairs $\{z, x\}$ and $\{z, y\}$ we obtain u and v, respectively, such that $\pi(u) = \pi(v) = \sigma \in C$, $z \le u \le x$, $z \le v \le y$. Then $z \in h(u) \cap h(v)$ so, by case 1, we have $u = v$. Applying ($\delta 5$) for the pairs $\{u, x\}$ and $\{u, y\}$ we obtain t and w such that $u \le t \le x$, $u \le w \le y$, $\pi(v) = \gamma_{\rho(x)}$, $\pi(w) = \gamma_{\rho(y)}$. Since $r \restriction C = r^C_{\nu,\mu} \in P^*_C$ there is $s \in i\{t, w\}$ with $u \le s$. Then $z \le s$ and $s \in i\{x, y\}$.

$\square_{3.20.8}$

Hence P_δ satisfies (K^+). $\square_{3.20}$

Lemma 3.21. *P_δ satisfies (D^δ_1).*

Proof. Assume that $p \in P_\delta$ and $z \in (\delta \times \omega) \setminus a^p$. Let $q = p \uplus \{x\}$. We need to show that $q \in P_\delta$, i.e., q satisfies ($\delta 1$)–($\delta 6$).

($\delta 1$) follows from fact 3.8.

If $z \notin C \times \omega$ then $q \restriction C = p \restriction C \in P_C$ because $p \in P_\delta$. If $z \in C \times \omega$ then $q \restriction C = (p \restriction C) \uplus \{z\} \in P_C$ because $p \restriction C \in P_C$ and P_C satisfies (D^C_1). Hence ($\delta 2$) holds. Similar arguments work for ($\delta 3$).

As for ($\delta 4$), let $\{x, y\} \in \left[a^q\right]^2$ with $x \le^q y$. Then $\{x, y\} \in \left[a^p\right]^2$ because z and the elements of a^p are $\le^q$-incomparable. So we can apply property ($\delta 4$) for $\{x, y\}$ in p to get a suitable $u \in a^p \subset a^q$. Similar arguments work for ($\delta 5$).

As for ($\delta 6$), let $x, y \in \left[a^q\right]^2$. If $z \in \{x, y\}$ then $i^q\{x, y\} = \emptyset$ so the required inclusion holds trivially. Otherwise $\{x, y\} \in \left[a^p\right]^2$ so we can use property ($\delta 6$) for p to get the required inclusion. $\square_{3.21}$

Lemma 3.22. *P_δ satisfies (D_2^δ).*

Proof. If $\{\alpha, \beta\} \in \left[I_\eta\right]^2$ for some η then let $L^\delta(\alpha, \beta) = L^{I_\eta}(\alpha, \beta)$. Otherwise, if $\alpha \in I_\eta$, $\beta \in I_\sigma$, $\eta < \sigma$, then let $\alpha^+ = \min(C \setminus \alpha + 1)$ and put

$$L^\delta(\alpha, \beta) = \{\alpha, \alpha^+\} \cup L^C(\alpha^+, \gamma_\sigma) \cup L^{I_\sigma}(\gamma_\sigma, \beta).$$

Enumerate $L^P(\alpha, \beta)$ as $\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \beta$. Let $p \in P_\delta$, $z \in a^p$ with $\pi(z) = \beta$ and $z_i \in (\delta \times \omega) \setminus a^p$ with $\pi(z_i) = \alpha_i$ for $i < n$. Let $q = p \uplus_z \langle z_0, \ldots z_{n-1}\rangle$.

We should show that $q \in P_\delta$, i.e. q satisfies ($\delta 1$)–($\delta 6$).

We will consider only the harder case, i.e. when $\alpha \in I_\eta$, $\beta \in I_\sigma$, $\eta < \sigma$. Fix $1 \le m < n$ such that $L^C(\alpha^+, \gamma_\sigma) = \{\alpha_1, \ldots, \alpha_{k-1}\}$ and $L^{I_\sigma}(\gamma_\sigma, \beta) = \{\alpha_k, \ldots, \alpha_{m-1}\}$, i.e.

$$\alpha_0 = \alpha < \alpha_1 = \alpha^+ = \gamma_{\eta+1} < \cdots < \alpha_m = \gamma_\sigma < \cdots < \alpha_{n-1} < \beta.$$

($\delta 1$) follows from fact 3.11.

($\delta 2$): If $z \notin C \times \omega$ then

$$q \restriction C = ((p \restriction C) \uplus \{z_k\}) \uplus \{z_{k-1}\} \cdots \uplus \{z_\ell\} \in P_C,$$

where $\ell = 1$ if $\alpha_0 \notin C$ and $\ell = 0$ if $\alpha_0 \in C$, because P_C satisfies D_C^1.

If $z \in C \times \omega$ then

$$q \restriction C = (p \restriction p) \uplus_z \langle z_\ell, \ldots, z_k\rangle \in P_C,$$

where $\ell = 1$ if $\alpha_0 \notin C$ and $\ell = 0$ if $\alpha_0 \in C$, because P_C satisfies D_C^2.

($\delta 3$): Let $\zeta < \operatorname{cf} \delta$. If $\zeta = \sigma$ then

$$q \restriction I_\sigma = (p \restriction I_\sigma) \uplus_z \langle z_k, \ldots, z_{n-1}\rangle \in P_{I_\sigma}$$

because P_{I_σ} satisfies $D_{I_\sigma}^2$. If $\gamma_\zeta = \alpha_i$ for some $i \in \{0, \ldots k-1\}$ then

$$q \restriction I_\zeta = (p \restriction I_\zeta) \uplus \{z_i\} \in P_{I_\zeta}$$

because P_{I_ζ} satisfies $D_{I_\zeta}^1$.

Otherwise $q \restriction I_\zeta = p \restriction I_\zeta \in P_{I_\zeta}$.

($\delta 4$): Let $\{x, y\} \in \left[a^q\right]^2$ with $x \le^q y$ and $\gamma_\zeta < \pi(x) < \gamma_{\zeta+1} \le \pi(y)$. If $x \in a^p$ then $y \in a^p$ so we can apply ($\delta 4$) in p the get a suitable u. So we can assume that $x \in \{z_0, \ldots, z_{n-1}\}$. Since $\gamma_\zeta < \pi(x) < \gamma_{\zeta+1} \le \pi(y)$ we have $x = z_0$ or $z \in \{z_{k+1}, \ldots, z_{n-1}\}$. If $x = z_0$ then $u = z_1$ works. If $x = z_i$ for some $k < i < n$ then $\xi = \sigma$ so $\gamma_{\sigma+1} \le \pi(y)$ implies $y \in a^p$. Hence $\gamma_\sigma < \pi(z) < \gamma_{\sigma+1} \le \pi(y)$ and so applying ($\delta 4$) in p for the pair

$\{z, y\}$ we get $u \in a^p$ with $z \leq^p u \leq^y$ and $\pi(u) = \gamma_{\sigma+1}$. Thus this u works for $\{x, y\}$ in q.

($\delta 5$): Let $\{x, y\} \in \left[a^q\right]^2$ with $x \leq^q y$ and $\pi(x) < \gamma_\xi \leq \pi(y) < \gamma_{\xi+1}$. If $x \in a^p$ then $y \in a^q$ so we can apply $\delta 4$ in p the get a suitable v. So we can assume that $x \in \{z_0, \ldots, z_{n-1}\}$.

If $\xi = \sigma$ then $v = z_k$ works.

If $\xi > \sigma$ then $z \leq^p y$ and $\pi(z) = \gamma_\sigma < \gamma_\xi \leq \pi(y) < \gamma_{\zeta+1}$ so we can apply ($\delta 5$) in p for the pair $\{z, y\}$ to get a suitable v.

If $\xi < \sigma$ then $y \in \{z_1, \ldots z_{k-1}\}$ so $v = y$ works.

($\delta 6$): Let $\{x, y\} \in \left[a^p\right]^2$ If $\{x, y\} \in \left[a^p\right]^2$ then we can apply ($\delta 6$) for p to get the required inclusion. We can assume that $x \in \{z_0, \ldots, z_{n-1}\}$. Then $i^q\{z, y\} = \emptyset$ by the construction of $q = p \uplus_z \langle z_0, \ldots, z_{n-1}\rangle$ because x and y are incomparable and so $z \not\leq^p y$. $\Box_{3.22}$

Thus the limit step is done as well, which completes the inductive construction, so theorem 3.15 is proved. $\Box_{3.15}$

We conclude the paper with the result we quoted in the abstract.

Theorem 3.23. *If there is a κ-nice poset P for some regular cardinal κ then there is a c.c.c poset Q such that $\mathcal{THIN}(\delta)$ holds in V^Q for each $\delta < \kappa^+$.*

Proof. Using theorem 3.15 we fix, for each $\delta < \kappa^+$, a δ-nice poset P_δ. Let Q be the finite-support product of $\{P_\delta : \delta < \kappa^+\}$. Since every P_δ has property (K), so has Q.

Let $\mathcal{G}$ be a Q-generic filter and let $\delta < \kappa^+$ be arbitrary. Then $\mathcal{G}_\delta = \{p(\delta) : p \in \mathcal{G} \wedge \delta \in \operatorname{dom} p\}$ is a P_δ-generic filter, hence $\mathcal{THIN}(\delta)$ holds in $V[\mathcal{G}_\delta]$ winessed by some space X_δ by proposition 3.14. Since $V[\mathcal{G}_\delta] \subset V[\mathcal{G}]$ the space X_δ witnesses $\mathcal{THIN}(\delta)$ in $V[\mathcal{G}]$. $\Box$

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