INDESTRUCTIBLE PROPERTIES OF S- AND L-SPACES

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Abstract. Building on a method of U. Abraham and S. Todorčevič we prove a preservation theorem on certain properties under c.c.c forcings. Applying this result we show that (1) an uncountable, first countable, 0-dimensional space containing only countable and co-countable open subspaces, and (2) S- and L-groups can exist under Martin's Axiom.

1. Introduction

An uncountable topological space is called *O-space* if its every open subspace is either countable or co-countable.

In [8], a locally compact (and so first countable) O-space on ω_1 was constructed using $\diamondsuit$. Such a space can not exist under Martin's Axiom because O-spaces are S-spaces and, by [10] or [11], there are no (locally) compact S-spaces under $MA_{\aleph_1}$.

On the other hand, by [1] a first countable S-space can exist under $MA_{\aleph_1}$. This result will be strengthened here: in theorem 3.7 we show that a first countable O-space can exist under $MA_{\aleph_1}$. Answering a question of J. Roitman [9] we show that S- and L-groups can also exist under $MA_{\aleph_1}$. These proofs are based on the preservation theorem 2.2 proved in section 2.

We use the standard notation, see e.g. [5]. Given a structure X and a property φ we say that *the property φ of X is c.c.c-indestructible* if for each c.c.c poset Q we have $1_Q \Vdash_Q$ "X has property φ".

2. The preservation theorem

Given a set K and $m \in \omega$ denote by $\mathrm{Fn}_m(\omega_1, K)$ the family of functions mapping an m-element subset of ω_1 into K. A function s with $\mathrm{ran}(s) \subset \mathrm{Fn}_m(\omega_1, K)$ is called *dom-disjoint* iff $\mathrm{dom}\left(s(t)\right) \cap \mathrm{dom}\left(s(t')\right) = \emptyset$ for each $\{t, t'\} \in \left[\mathrm{dom}(s)\right]^2$. Especially, a sequence $\langle s_\alpha : \alpha < \omega_1\rangle \subset \mathrm{Fn}_m(\omega_1, K)$ is *dom-disjoint* iff $\mathrm{dom}(s_\alpha) \cap \mathrm{dom}(s_\beta) = \emptyset$ for all $\alpha < \beta < \omega_1$. Given two disjoint sets s and t write $\left[s, t\right] = \left\{\{\alpha, \beta\} : \alpha \in s \wedge \beta \in t\right\}$.

Definition 2.1. Let G be a graph on $\omega_1 \times K$, $m \in \omega$. We say that G is *m-solid* if given any dom-disjoint sequence $\langle s_\alpha : \alpha < \omega_1\rangle \subset \mathrm{Fn}_m(\omega_1, K)$ there are $\alpha < \beta < \omega_1$ such that

$$[s_\alpha, s_\beta] \subset G.$$

G is called *strongly solid* iff it is m-solid for each $m \in \omega$.

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Theorem 2.2. *Assume $2^{\omega_1} = \omega_2$. If G is a strongly solid graph on $\omega_1 \times K$, where $|K| \le 2^{\omega_1}$, then for each $m \in \omega$ there is a c.c.c poset P of size ω_2 such that*

$$V^P \models \text{“}G \text{ is c.c.c-indestructibly } m\text{-solid.”}$$

Proof. We will argue in the following way. In definition 2.3 we introduce property $(*)_m$ and in lemma 2.4 we show that in some c.c.c extension of the ground model G has property $(*)_m$. Then, in lemma 2.7 we prove that property $(*)_m$ is c.c.c-indesctuctible. This concludes the proof of theorem 2.2 because it will be clear from definition 2.3 that property $(*)_m$ implies that G is m-solid.

To start with let T_{m+1} be the everywhere ω_1-branching tree of height $m+1$ with ω_1 minimal points whose underlying set is $\bigcup\{{}^{\ell}\omega_1 : 1 \le \ell \le m+1\}$ ordered by the inclusion. The $m+1$-branches of T_{m+1} will be denoted by $\mathbf{b}_{m+1}$. Let

$$\mathfrak{B}_{m+1} = \{\mathcal{B} \subset \mathbf{b}_{m+1} : b \cap b' = \emptyset \text{ for each } \{b, b'\} \in \big[\mathcal{B}\big]^2 \}$$

and

$$\mathfrak{B}^{\omega_1}_{m+1} = \{\mathcal{B} \in \mathfrak{B}_{m+1} : |\mathcal{B}| = \omega_1\}.$$

The following definition is modeled after [1, Definition 4.2]: their property $(*)$ corresponds to our property $(*)_1$ for a certain graph G .

Definition 2.3. We say that G has *property* $(*)_m$ iff for each dom-disjoint function $s : T_{m+1} \longrightarrow \mathrm{Fn}_m(\omega_1, K)$ there is $\mathcal{B} \in \mathfrak{B}^{\omega_1}_{m+1}$ such that

$$(\nabla_{s,\mathcal{B}}) \qquad \forall \{b, b'\} \in \big[\mathcal{B}\big]^2 \ \exists\, t \in b\ \exists t' \in b' \ [s(t), s(t')] \subset G.$$

Lemma 2.4. *There is a c.c.c poset P of size ω_2 such that $V^P \models$ “G has property $(*)_m$”.*

Proof of lemma 2.4. We start with some definitions. Let $C = \{\gamma_\eta : \eta < \omega_1\} \subset \omega_1$ be a club set. We say that two subsets A and B of ω_1 are *C-separated* iff $A \cap [\gamma_\eta, \gamma_{\eta+1}) = \emptyset$ or $B \cap [\gamma_\eta, \gamma_{\eta+1}) = \emptyset$ for each $\eta < \omega_1$. A function $s : I \longrightarrow \mathrm{Fn}_m(\omega_1, K)$ is called *dom-separated by C* iff $\mathrm{dom}(s(i))$ and $\mathrm{dom}(s(i'))$ are C-separated for each $\{i, i'\} \in \big[I\big]^2$, i.e. for each $\eta < \omega_1$ there is at most one $i \in I$ such that $\big(\mathrm{dom}\, s(i)\big) \cap [\gamma_\eta, \gamma_{\eta+1}) \ne \emptyset$. Let us remark that s is dom-disjoint iff it is dom-separated by ω_1.

Definition 2.5. If $\mathcal{B} \in \mathfrak{B}_{m+1}$ and $s : \bigcup \mathcal{B} \longrightarrow \mathrm{Fn}_m(\omega_1, K)$ is dom-disjoint put

$$E(s) = \big\{\gamma < \omega_1 : \forall b \in \mathcal{B} \ \big[\big(\cup_{t \in b} \mathrm{dom}\, s(t)\big) \subset \gamma \vee \big(\cup_{t \in b} \mathrm{dom}\, s(t)\big) \cap \gamma = \emptyset\big]\big\}.$$

Observe that $E(s)$ is club in ω_1 because the sets $\big(\cup_{t \in b} \mathrm{dom}\, s(t)\big)$ for $b \in \mathcal{B}$ are pairwise disjoint.

We will define a c.c.c. iterated forcing $\Big\langle P_\nu, \dot{Q}_\nu : \nu < \omega_2 \Big\rangle$ with finite support such that $1_{P_\nu} \Vdash$“$|\dot{Q}_\nu| = \omega_1$”. In this case $(2^{\omega_1})^{V^{P_{\omega_2}}} = \omega_2$ and so the family

$$\mathfrak{S} = \{s \in V^{P_{\omega_2}} : s : T_{m+1} \longrightarrow \mathrm{Fn}_m(\omega_1, K) \textit{ is a dom-disjoint function } \}$$

will be of size ω_2 because $|\mathrm{Fn}_m(\omega_1, K)| = \omega_1 + |K| \le \omega_2$. Thus, using a bookkeeping function, we can pick $s_\nu \in \mathfrak{S} \cap V^{P_\nu}$ for $\nu < \omega_2$ such that $\{s_\nu : \nu < \omega_2\}$ enumerates $\mathfrak{S}$.

We will also define a mod countable decreasing sequence of club sets $\langle C_\nu : \nu < \omega_2 \rangle$ and sequences $\mathcal{B}_\nu \in \mathfrak{B}^{\omega_1}_{m+1}$ such that $s_\nu \restriction \cup\mathcal{B}_\nu$ will be dom-separated by C_ν and $C_{\nu+1} \subset C_\nu \cap E(s_\nu \restriction \cup\mathcal{B}_\nu)$.

Assume that P_ν and C_ν are constructed.

We will work in V^{P_ν}. Using our book-keeping function pick the next dom-disjoint function $s_\nu : T_{m+1} \longrightarrow \mathrm{Fn}_m(\omega_1, K)$.

Since s_ν is dom-disjoint, we can find a $\mathcal{B}_\nu \in \mathfrak{B}^{\omega_1}_{m+1}$ such that $s_\nu \restriction \cup\mathcal{B}_\nu$ is dom-separated by C_ν. Let $C_{\nu+1}$ be a club subset of $C_\nu \cap E(s_\nu \restriction \cup\mathcal{B}_\nu)$ from the ground model.

Let $\mathbf{Q}'_\nu = \langle Q'_\nu, \supset\rangle$, where

$$Q'_\nu = \{B \in [\mathcal{B}_\nu]^{<\omega} : \forall\{b, b'\} \in [B]^2\ \exists t \in b\ \exists t' \in b'\ \ [s_\nu(t), s_\nu(t')] \subset G\}.$$

Thus, if Γ_ν is the Q'_ν-generic filter then $\mathcal{B} = \cup\Gamma$ satisfies $(\triangledown_{s,\mathcal{B}})$. In lemma 2.6 we will show that $\mathbf{Q}'_\nu$ satisfies c.c.c thus we can pick $B \in Q'_\nu$ such that $B \Vdash$ "*the Q'_ν-generic filter is uncountable*". Put $\mathbf{Q}_\nu = \{B' \in Q'_\nu : B' \supset B\}$ and $P_{\nu+1} = P_\nu * \mathbf{Q}_\nu$. Thus $\cup\Gamma_\nu$ witnesses property $(*)_m$ for s_ν in $V^{P_{\nu+1}}$.

If $\mu < \omega_2$ is limit and $\langle C_\nu : \nu < \mu\rangle$ is constructed, then let C_μ be a club subset of ω_1 from the ground model such that $|C_\mu \setminus C_\nu| \le \omega$ for each $\nu < \mu$.

Lemma 2.6. *P_ν satisfies c.c.c. for $\nu \le \omega_2$.*

Proof of lemma 2.6. By induction on $\nu \le \omega_2$ we prove statement $(\bullet_\nu)$ below which clearly yields that P_ν is c.c.c:

$(\bullet_\nu)$ If $\{p_\xi : \xi < \omega_1\} \subset P_\nu$, $\{I_\xi : \xi < \omega_1\}$ are pairwise disjoint finite sets, $I = \bigcup_{\xi<\omega_1} I_\xi$, $s : I \longrightarrow \mathrm{Fn}_m(\omega_1, K)$ is dom-separated by C_ν, then there is a pair $\{\xi_0, \xi_1\} \in [\omega_1]^2$ such that p_{ξ_0} and p_{ξ_1} are compatible in P_ν and $[s(\rho_0), s(\rho_1)] \subset G$ for each $\rho_0 \in I_{\xi_0}$ and $\rho_1 \in I_{\xi_1}$.

Case 1. $\nu = 0$.

Let $t_\xi = \bigcup\{s(\rho) : \rho \in I_\xi\}$ for $\xi < \omega_1$. Since s is dom-disjoint it follows that $t_\xi \in \mathrm{Fn}_{|I_\xi|\cdot m}(\omega_1, K)$ and that the sequence $\langle t_\xi : \xi \in \omega_1\rangle$ is dom-disjoint. So there is a pair $\{\xi_0, \xi_1\} \in [\omega_1]^2$ such that $[t_{\xi_0}, t_{\xi_1}] \subset G$ because G is strongly solid. Thus $[s(\rho_0), s(\rho_1)] \subset G$ for each $\rho_0 \in I_{\xi_0}$ and $\rho_1 \in I_{\xi_1}$. Since $P_0 = \{1\}$ and so $p_{\xi_0} = p_{\xi_1} = 1_{P_0}$, we are done.

Case 2. *ν is limit.*

Let $J_\xi = \operatorname{supp} p_\xi$. By thinning out our sequences we can assume that $\{J_\xi : \xi < \omega_1\}$ forms a Δ-system with kernel J. Let $\mu = (\max J) + 1 < \nu$. Since $|C_\nu \setminus C_\mu| \le \omega$, there is $\zeta < \omega_1$ such that $s' = s \restriction \cup_{\xi\ge\zeta} I_\xi$ is dom-separated by C_μ. Applying $(\bullet_\mu)$ for $\{p_\xi \restriction \mu : \xi \ge \zeta\}$, $\{I_\xi : \xi \ge \zeta\}$ and s' we can find $\{\xi_0, \xi_1\} \in [\omega_1 \setminus \zeta]^2$ such that $p_{\xi_0} \restriction \mu$ and $p_{\xi_1} \restriction \mu$ are compatible in P_μ and $[s(\rho_0), s(\rho_1)] \subset G$ for each $\rho_0 \in I_{\xi_0}$ and $\rho_1 \in I_{\xi_1}$. But $\operatorname{supp} p_{\xi_0} \cap \operatorname{supp} p_{\xi_1} \subset \mu$, so p_{ξ_0} and p_{ξ_1} are compatible in P_ν as well.

Case 3. $\nu = \mu + 1$.

We can assume that for each $\xi < \omega_1$ we have a finite set $B_\xi \subset \mathbf{b}_{m+1}$ and a finite function $r_\xi : \cup B_\xi \longrightarrow \mathrm{Fn}_m(\omega_1, K)$ such that

$$p_\xi \restriction \mu \Vdash \text{“}\dot{p_\xi(\mu)} = B_\xi \wedge \dot{s}_\mu \restriction \cup B_\xi = r_\xi\text{”}.$$

By thinning out our sequences we can assume that (i)–(v) below hold:

(i) $\{B_\xi : \xi < \omega_1\}$ forms a Δ-system with kernel B.

(ii) $r_\xi(t)$ is independent of ξ for each $t \in \cup B$.

(iii) Writing $B'_\xi = B_\xi \setminus B$ the sets $\{\cup B'_\xi : \xi < \omega\}$ are pairwise disjoint, i.e. $\left\langle B'_\xi : \xi < \omega_1\right\rangle \in \mathfrak{B}^{\omega_1}_{m+1}$.

(iv) the sets $D_\xi = \cup\{\operatorname{dom} r_\xi(t) : t \in \cup B'_\xi\}$ are pairwise disjoint.

(v) for each $\{\xi, \zeta\} \in [\omega_1]^2$ the sets D_ξ and D_ζ are separated by C_ν.

Indeed, (i) and (ii) are straightforward. As for (iii), since $1_{P_\mu} \Vdash$ "*the elements of* $\mathcal{B}_\mu$ *are pairwise disjoint*" and P_μ satisfies c.c.c by the induction hypothesis, for each $t \in T_{m+1}$ the set $\{b \in \mathbf{b}_{m+1} : t \in b \wedge \exists p \in P_\mu\ p \Vdash b \in \mathcal{B}_\mu\}$ is countable and so $|\{\xi : t \in \cup B'_\xi\}| \le \omega$ as well. Thus (iii) can be guaranteed. Similarly, since $1_{P_\mu} \Vdash$ "$\dot{s}_\mu \restriction \cup\mathcal{B}_\mu$ *is dom-disjoint*" and P_μ satisfies c.c.c, for each $\alpha \in \omega_1$ the set $\{t \in T_{m+1} : \exists p \in P_\mu\ p \Vdash \alpha \in \operatorname{dom} \dot{s}_\mu(t)\}$ is countable. Thus $\big|\{\xi : \alpha \in \cup\{\operatorname{dom} r_\xi(t) : t \in \cup B'_\xi\}\}\big| \le \omega$, too. So we can ensure (iv). Finally, (v) is straightforward by (iv).

Since $p_\xi \Vdash$ "$r_\xi \subset \dot{s}_\mu$" it follows that $p_\xi \Vdash$ "$E(r_\xi) \supset E(\dot{s}_\mu) \supset C_\nu$", and so $E(r_\xi) \supset C_\nu$. Thus for each $b \in B_\xi$ there is $\gamma \in C_\nu$ such that $\cup\{\operatorname{dom} r_\xi(t) : t \in b\} \subset [\gamma, \gamma')$, where $\gamma' = \min(C \setminus \gamma + 1)$. Since s is dom-separated by C_ν there is at most one $i_b \in I$ such that $\operatorname{dom} s(i_b)$ intersects $[\gamma, \gamma')$ and so

$$(\dagger) \qquad \operatorname{dom} s(i) \text{ and } \cup\{\operatorname{dom} r_\xi(t) : t \in b\} \text{ are } C_\nu\text{-separated for each } i \in I \setminus \{i_b\}.$$

Since $|\operatorname{dom} s(i_b)| = m < m+1 = |b|$ and the sets $\{\operatorname{dom}(r_\xi(t)) : t \in b\}$ are pairwise C_μ-separated, there is $t_b \in b$ such that $\operatorname{dom}(s(i_b))$ and $\operatorname{dom}(r_\xi(t))$ are C_μ-separated. Thus

$$(\ddagger) \qquad \operatorname{dom} s(i) \text{ and } \operatorname{dom} r_\xi(t_b) \text{ are } C_\mu\text{-separated for each } i \in I.$$

Let $I^*_\xi = I_\xi \cup B_\xi$ for $\xi < \omega_1$, $I^* = \cup\{I^*_\xi : \xi < \omega_1\}$ and define the function $s^* : I^* \longrightarrow \operatorname{Fn}_m(\omega_1, K)$ by stipulations $s^* \restriction I = s$ and $s^*(b) = r_\xi(t_b)$ for $b \in B_\xi$. By ($\ddagger$) and by (v) the function s^* is dom-separated by C_μ. So we can apply induction hypothesis ($\bullet_\mu$) for $\{p_\xi \restriction \mu : \xi < \omega_1\}$, $\{I^*_\xi : \xi < \omega_1\}$ and for s^* to find $\{\xi_0, \xi_1\} \in \big[\omega_1\big]^2$ such that $p_{\xi_0} \restriction \mu$ and $p_{\xi_1} \restriction \mu$ are compatible in P_μ and $[s^*(\rho_0), s^*(\rho_1)] \subset G$ for each $\rho_0 \in I^*_{\xi_0}$ and $\rho_1 \in I^*_{\xi_1}$. Then $(p_{\xi_0} \restriction \mu) \wedge (p_{\xi_1} \restriction \mu) \Vdash_{P_\mu}$ "$B_{\xi_0} \cup B_{\xi_1} \in Q_\mu$" because for each $b_0 \in B_{\xi_0}$ and $b_1 \in B_{\xi_1}$ we have $[r_{\xi_0}(t_{b_0}), r_{\xi_1}(t_{b_1})] = [s^*(b_0), s^*(b_1)] \subset G$. Thus $(p_{\xi_0} \restriction \mu) \wedge (p_{\xi_1} \restriction \mu) \Vdash_{P_\mu}$ "$p_{\xi_0}(\mu)$ *and* $p_{\xi_1}(\mu)$ *are compatible in* Q_μ", i.e. p_{ξ_0} and p_{ξ_1} are compatible in P_ν. Thus the pair $\{\xi_0, \xi_1\}$ witnesses ($\bullet_\nu$). Lemma 2.6 is proved. □

To verify property $(*)_m$ in $V^{P_{\omega_2}}$ let $s : T_{m+1} \longrightarrow \operatorname{Fn}_m(\omega_1, K)$ from $V^{P_{\omega_2}}$. Since P_{ω_2} is c.c.c and we used a suitable book-keeping function there is $\nu < \omega_2$ such that $s_\nu = s$. Let $\mathcal{G}$ be the Q_ν-generic filter over V^{P_ν}. Then $\cup\mathcal{G} \in \mathfrak{B}^{\omega_1}_{m+1}$ witnesses $(*)_m$ for s. This completes the proof of 2.4. □

Lemma 2.7. *Property* $(*)_m$ *is c.c.c-indestructible.*

Proof of lemma 2.7. Let Q be a c.c.c poset and assume that $1_Q \Vdash$ "$\dot{s} : T_{m+1} \longrightarrow \operatorname{Fn}_m(\omega_1, K)$ *is dom-disjoint*". Let $q \in Q$ be arbitrary. By induction on $\ell \le m$ for each $t \in {}^{\ell}\omega_1$ choose a condition $q(t) \le q$ from Q and an element $r(t) \in \operatorname{Fn}_m(\omega_1, K)$ such that

(a) $(q(\emptyset) = q)$ and $q(t) \le q(t \restriction \ell - 1)$ for $\ell > 0$,
(b) $q(t) \Vdash$ "$\dot{s}(t) = r(t)$".

Since Q is c.c.c and $1_Q \Vdash$ "$\dot{s}$ *is dom-disjoint*", for each $\eta \in \omega_1$

$$|\{t \in T_{m+1} : \eta \in \operatorname{dom} r(t)\}| \le \omega.$$

Thus there is an everywhere ω_1-branching subtree $T \subset T_{m+1}$ of height $m+1$ with ω_1-many minimal points such that $r \restriction T$ is dom-disjoint. Since T and T_{m+1} are isomorphic, we can apply $(*)_m$ for r in the ground model to find $\mathcal{B} \in \mathfrak{B}^{\omega_1}_{m+1}$ such that

$$(\circ) \qquad \forall \{b, b'\} \in \big[\mathcal{B}\big]^2\ \exists\, t \in b\ \exists t' \in b'\ [r(t), r(t')] \subset G.$$

For $b \in \mathcal{B}$ let t_b be the maximal element of the branch b in T_{m+1} and put $q'(b) = q(t_b)$.

Since Q is c.c.c. there is a condition $q' \le q$ such that

$$(\circ\circ) \qquad q' \Vdash \text{"the set } \dot{\mathcal{C}} = \{b \in \mathcal{B} : q'(b) \in \mathcal{G}\} \text{ is uncountable,"}$$

where $\mathcal{G}$ is the canonical name of the Q-generic filter.

Since $q'(b) \le_Q q(t)$ for each $t \in b$ it follows that $q'(b) \Vdash$ "$\dot{s}(t) = r(t)$ *for each* $t \in b$". Thus by ($\circ$) and ($\circ\circ$)

$$q' \Vdash \text{“property } (*)_m \text{ for } \dot{s} \text{ is witnessed by } \dot{\mathcal{C}}\text{”},$$

which completes the proof of lemma 2.7. □

It is straightforward from the definitions that if G has property $(*)_m$ then G is m-solid. Thus theorem 2.2 is proved. □

3. Applications

We start this section with a simple application: we show that if $2^{\omega_1} = \omega_2$ then every strong HFD_w (strong HFC_w) gives a c.c.c-indestructible S-space (L-space)in a suitable generic extension Let us recall the definition:

Definition 3.1. A subset $X = \{x_\nu : \nu < \omega_1\} \subset 2^{\omega_1}$ is called HFD^n_w (HFC^n_w) iff

$$\forall f : \omega_1 \times n \xrightarrow{1-1} \omega_1 \ \forall m < \omega \ \forall g : \omega_1 \times m \xrightarrow{1-1} \omega_1 \ \forall H : n \times m \longrightarrow 2$$
$$\exists \alpha < \beta < \omega_1 \ (\exists \beta < \alpha < \omega_1) \ \forall i < n \ \forall j < m \ x_{f(\alpha,i)}(g(\beta, j)) = H(i, j).$$

X is *strong* HFD_w (*strong* HFC_w) iff it is HFD^n_w (HFC^n_w) for each $n < \omega$.

Definition 3.2. For $X = \{x_\nu : \nu < \omega_1\} \subset 2^{\omega_1}$ define two graphs $G^<_X$ and $G^>_X$ as follows. Fix a countable dense subset D of 2^{ω_1}, let $K = \left[\omega_1\right]^{<\omega} \times D$ and

$$G^<_X = \Big\{ \{\langle \nu_0, \langle a_0, d_0\rangle\rangle, \langle \nu_1, \langle a_1, d_1\rangle\rangle\} \in \left[\omega_1 \times K\right]^2 :$$
$$(\nu_0 \cap a_0 \ne \emptyset \vee \nu_1 \cap a_1 \ne \emptyset \vee d_0 \ne d_1) \vee (\nu_0 < \nu_1 \wedge x_{\nu_0} \restriction a_1 = d_1 \restriction a_1 \Big\}$$

and

$$G^>_X = \Big\{ \{\langle \nu_0, \langle a_0, d_0\rangle\rangle, \langle \nu_1, \langle a_1, d_1\rangle\rangle\} \in \left[\omega_1 \times K\right]^2 :$$
$$(\nu_0 \cap a_0 \ne \emptyset \vee \nu_1 \cap a_1 \ne \emptyset \vee d_0 \ne d_1) \vee (\nu_0 > \nu_1 \wedge x_{\nu_0} \restriction a_1 = d_1 \restriction a_1 \Big\}.$$

Lemma 3.3. *X is HFD^n_w if and only if $G^<_X$ is n-solid. Similarly, X is HFC^n_w iff $G^>_X$ is n-solid.*

Proof. We prove only the first equivalence because second one can be obtained by the same arguments.

Assume first that X is HFD^n_w and fix a dom-disjoint sequence $\langle s_\xi : \xi < \omega_1\rangle \subset \mathrm{Fn}_n(\omega_1, K)$. For each $\xi < \omega_1$ write $\operatorname{dom} s_\xi = \{\nu_{\xi,i} : i < n\}$ and $s_\xi(\nu_{\xi,i}) = \langle a_{\xi,i}, d_{\xi,i}\rangle$. We can assume that $\nu_{\xi,i} \cap a_{\xi,i} = \emptyset$ for each $i < n$ and $\xi < \omega_1$.

Let $a_\xi = \bigcup\{\alpha_{\xi,i} : i < n\}$ and write $a_\xi = \{\alpha_{\xi,j} : j < m_\xi\}$. Define the function $H_\xi : n \times m_\xi \longrightarrow 2$ by the stipulation

$$(\circ) \qquad H_\xi(i, j) = d_{\xi,i}(\alpha_{\xi,j}).$$

By thinning out our sequence we can assume that $m_\xi = m$, $H_\xi = H$ and $d_{\xi,i} = d_i$ for each $i < n$ and $\xi < \omega_1$, moreover $\max a_\zeta < \min a_\xi$ for $\zeta < \xi < \omega_1$.

Since X is a HFD^n_w there are $\zeta < \xi < \omega_1$ such that

$$(\star) \qquad \forall i < n \ \forall j < m \ x_{\nu_{\zeta,i}}(\alpha_{\xi,j}) = H(i, j).$$

Let $i_0, i_1 < n$ and prove $\{\langle \nu_{\zeta,i_0}, \langle a_{\zeta,i_0}, d_{i_0}\rangle\rangle, \langle \nu_{\xi,i_1}, \langle a_{\xi,i_1}, d_{i_1}\rangle\rangle\} \in G^<_X$. Putting ($\star$) and ($\circ$) together, we have $x_{\nu_{\zeta,i_0}} \restriction a_\xi = d_{i_0} \restriction a_\xi$. Since by the definition of $G^<_X$ we can assume $d_{i_0} = d_{i_1}$

it follows that $x_{\nu_{\zeta,i_0}} \restriction a_{\xi,i_1} = d_{i_0} \restriction a_{\xi,i_1} = d_{i_1} \restriction a_{\xi,i_1}$ which was to be proved. Thus $G^<_X$ is really n-solid.

Assume now that $G^<_X$ is n-solid. Let $f : \omega_1 \times n \xrightarrow{1-1} \omega_1$, $m \in \omega$, $g : \omega_1 \times m \xrightarrow{1-1} \omega_1$ and $H : n \times m \longrightarrow 2$. Put $a_\alpha = \{g(\alpha, j) : j < m\}$ and $t_\alpha = \{f(\alpha, i) : i < n\}$ for $\alpha < \omega_1$. By thinning out our sequences we can assume that that $\max(a_\alpha \cup t_\alpha) < \min(a_\beta \cup t_\beta)$ for $\alpha < \beta < \omega_1$.

For $\alpha < \omega_1$ define the function $s_\alpha \in \mathrm{Fn}_n(\omega_1, K)$ as follows: let $\mathrm{dom}(s_\alpha) = t_\alpha$, for each $i < n$ pick $d_{\alpha,i} \in D$ such that $d_{\alpha,i}(g(\alpha+1, j)) = H(i, j)$ for each $j < m$ and let $s_\alpha(f(\alpha, i)) = \langle a_{\alpha+1}, d_{\alpha,i}\rangle$. By thinning out our sequences we can assume that $d_{\alpha,i} = d_i$ for each $i < n$ and $\alpha < \omega_1$. Since $G^<_X$ is n-solid there are $\alpha < \beta < \omega_1$ such that $[s_\alpha, s_\beta] \subset G^<_X$. Especially, for each $i < n$ we have $\{\langle f(\alpha, i), \langle a_{\alpha+1}, d_i\rangle\rangle, \langle f(\beta, i), \langle a_{\beta+1}, d_i\rangle\rangle\} \in G^<_X$, i.e. $x_{f(\alpha,i)} \restriction a_{\beta+1} = d_i \restriction a_{\beta+1}$. Thus, by the choice of $d_i = d_{\beta,i}$ we have $x_{f(\alpha,i)}(g(\beta+1, j)) = H(i, j)$ for each $i < n$ and $j < m$, so α and $\beta+1$ satisfy the requirements of 3.1. Thus X is HFD^n_w. □

Theorem 3.4. *Assume that $2^{\omega_1} = \omega_2$. If $X = \{x_\nu : \nu < \omega_1\} \subset 2^{\omega_1}$ is a strong HFD_w, then for each natural number m we have a c.c.c-poset P_m of cardinality ω_2 such that*

$$V^{P_m} \models \text{“}X \text{ is a c.c.c-indestructible } HFD^m_w\text{”}.$$

Proof. By lemma 3.3, $G^<_X$ is strongly solid . Since $2^{\omega_1} = \omega_2$ we can apply theorem 2.2 to obtain a c.c.c-poset P_m of cardinality ω_2 such that

$$V^{P_m} \models G^<_X \text{ is c.c.c-indestructibly } m\text{-solid.”}$$

By lemma 3.3 this implies that

$$V^{P_m} \models \text{“the space } X \text{ is a c.c.c-indestructible HFD}^m_w\text{”},$$

that is, the poset P_m satisfies the requirements. □

The same argument gives the following theorem.

Theorem 3.5. *Assume that $2^{\omega_1} = \omega_2$. If $X = \{x_\nu : \nu < \omega_1\} \subset 2^{\omega_1}$ is a strong HFC_w, then for each natural number m we have a c.c.c-poset P_m of cardinality ω_2 such that*

$$V^{P_m} \models \text{“}X \text{ is a c.c.c-indestructible } HFC^m_w\text{”}.$$

K. Kunen [7] proved that under MA there are no strong S- and L-spaces. Theorems 3.4 and 3.5 above clearly yield corollary 3.6 below which shows that Kunen's result is sharp. It should be mentioned that this corollary is not new: by folklore it was known that the method of [1] can be applied to prove it but its proof was never published.

Corollary 3.6. *If ZF is consistent then so is ZFC + Martin's Axiom + “for each natural number n there are topological spaces spaces X_n and Y_n such that $(X_n)^n$ is an S-space and $(Y_n)^n$ is an L-space”.*

The next application of theorem 2.2 is less straightforward.

Theorem 3.7. *A first countable O-space can exist under Martin's Axiom.*

Proof of theorem 3.7. First we sketch the idea of our proof. Assume that $2^{\omega_1} = \omega_2$ in the ground model. We construct a c.c.c poset $\mathcal{Q}$ of size ω_1 to get a 0-dimensional, first countable space $X = \langle \omega_1, \tau\rangle$ in $V^{\mathcal{Q}}$. Then we define a graph G on $\omega_1 \times K$, where $K = \omega \times \omega$, and in lemma 3.8 we show that G is strongly solid.

Since $(2^{\omega_1})^{V^{\mathcal{Q}}} = (2^{\omega_1})^V = \omega_2$, we can apply theorem 2.2 to get a c.c.c poset R in $V^{\mathcal{Q}}$ such that

$$V^{\mathcal{Q}*R} \models \text{“}G \text{ is c.c.c-indestructibly 2-solid”}.$$

So introducing Martin's axiom by forcing with a c.c.c poset P we obtain

$$V^{\mathcal{Q}*R*P} \models \text{"}MA_{\aleph_1} + G \text{ is 2-solid."}$$

Now to complete the proof of the theorem we prove in lemma 3.9 that if G is 2-solid, then X is an O-space.

To start with we define the poset $\mathcal{Q} = \langle Q, \leq \rangle$ as follows.

The underlying set of $\mathcal{Q}$ consists of triples $q = \langle I, n, u \rangle$, where $I \in [\omega_1]^{<\omega}$, $n \in \omega$ and $u : I \times n \longrightarrow \mathcal{P}(I)$ such that $\alpha \in u(\alpha, k) \subset u(\alpha, 0) = I \cap (\alpha + 1)$ for each $\alpha \in I$ and $k < n$.

If $q = \langle I, n, u \rangle, q' = \left\langle I', n', u' \right\rangle \in Q$ put

$$\begin{aligned} q \leq q' \quad \text{iff} \quad & I' \subset I, \\ & n' \leq n, \\ & u'(\alpha, k) = u(\alpha, k) \cap I', \\ & \text{if } u'(\alpha, i) \cap u'(\beta, j) = \emptyset \text{ then } u(\alpha, i) \cap u(\beta, j) = \emptyset, \\ & \text{if } u'(\alpha, i) \subset u'(\beta, j) \text{ then } u(\alpha, i) \subset u(\beta, j) \\ & \quad \text{for each } \alpha, \beta \in I' \text{ and } 1 \leq i, j, k < n'. \end{aligned}$$

Write $q = \langle I^q, n^q, u^q \rangle$ for $q \in \mathcal{Q}$. The set $D_\alpha = \{q \in Q : \alpha \in I^q\}$ is dense in $\mathcal{Q}$ for each $\alpha \in \omega_1$.

If $\mathcal{G}$ is a $\mathcal{Q}$-generic filter let

$$U^{\mathcal{G}}(\alpha, k) = \bigcup \{u^q(\alpha, k) : q \in \mathcal{G}, \alpha \in I^q, k < n^q\}$$

for $\alpha < \omega_1$ and $k < \omega$ and let

$$\mathcal{B}^{\mathcal{G}} = \{U^{\mathcal{G}}(\alpha, k) : \alpha \in \omega_1, k < \omega\}$$

and

$$\mathcal{B}^{\mathcal{G}}_+ = \{U^{\mathcal{G}}(\alpha, k) : \alpha \in \omega_1, 1 \leq k < \omega\}.$$

We show that

(∗) $\mathcal{B}^{\mathcal{G}}_+$ is a clopen base of a T_2 topological space $X^{\mathcal{G}}$ on ω_1.

Let $D = \{q \in Q : n^q \geq 2 \text{ and } u^q(\alpha, n^q - 1) = \{\alpha\} \text{ for each } \alpha \in I^q\}$. If $q \in D$ then for each $\{\alpha, \beta\} \in [I^q]^2$ and $1 \leq k < n^q$ we have

$$q \Vdash U^{\mathcal{G}}(\alpha, n^q - 1) \subset U^{\mathcal{G}}(\beta, k) \text{ or } U^{\mathcal{G}}(\alpha, n^q - 1) \cap U^{\mathcal{G}}(\beta, k) = \emptyset, \text{ and}$$
$$U^{\mathcal{G}}(\alpha, n^q - 1) \cap U^{\mathcal{G}}(\beta, n^q - 1) = \emptyset$$

by the definition of the order on $\mathcal{Q}$. Since the set D is dense in $\mathcal{Q}$, it follows that (∗) holds.

A standard density argument gives that ω is dense in $X^{\mathcal{G}}$.

Let $K = \omega \times \omega$, $\mathcal{J} = \{\langle \alpha, \langle k, d \rangle \rangle \in \omega_1 \times K : d \in U^{\mathcal{G}}(\alpha, k)\}$ and

$$G^{\mathcal{G}} = \left([\omega_1 \times K]^2 \setminus [\mathcal{J}]^2 \right) \cup \Big\{ \{\langle \alpha_0, \langle k_0, d_0 \rangle \rangle, \langle \alpha_1, \langle k_1, d_1 \rangle \rangle\} \in [\mathcal{J}]^2 : d_0 \neq d_1 \vee \alpha_0 \in U^{\mathcal{G}}(\alpha_1, k_1) \vee \alpha_1 \in U^{\mathcal{G}}(\alpha_0, k_0) \Big\}.$$

The following lemma yields that $\mathcal{Q}$ is c.c.c and $G^{\mathcal{G}}$ is n-solid in $V^{\mathcal{Q}}$.

Lemma 3.8. *If $n \in \omega$, $\{q_\alpha : \alpha < \omega_1\} \subset \mathcal{Q}$, $\{s_\alpha : \alpha < \omega_1\} \subset \mathrm{Fn}_n(\omega_1, K)$ is dom-disjoint, then there are $\{\alpha, \beta\} \in [\omega_1]^2$ and $q \in \mathcal{Q}$ such that $q \leq q_\alpha, q_\beta$ and $q \Vdash [s_\alpha, s_\beta] \subset G^{\mathcal{G}}$.*

Proof of lemma 3.8. Write $q_\alpha = \langle I_\alpha, n_\alpha, u_\alpha \rangle$. We can assume that $s_\alpha \subset I_\alpha \times (n_\alpha \times n_\alpha)$. By the definition of $G^{\mathcal{G}}$ we can also assume that $s_\alpha \subset \mathcal{J}$ because of $1_{\mathcal{Q}} \Vdash [s_\alpha \setminus \mathcal{J}, \omega_1 \times K] \subset G^{\mathcal{G}}$. By standard Δ-system and counting arguments we can find $\{\alpha, \beta\} \in [\omega_1]^2$ such that

(1) $\alpha < \beta$,

(2) $I_\alpha \cap I_\beta < I_\alpha \setminus I_\beta < I_\beta \setminus I_\alpha$,
(3) $|I_\alpha| = |I_\beta|$ and $n_\alpha = n_\beta = n$,
(4) the natural bijection σ between I_α and I_β gives an isomorphism between q_α and q_β, and between s_α and s_β in the following sense:
  (i) $(\forall \nu \in I_\alpha)\ (\forall k < n)\ \sigma'' u_\alpha(\nu, k) = u_\beta(\sigma(\nu), k)$,
  (ii) $s_\beta = \{\langle \sigma(\nu), x\rangle : \langle \nu, x\rangle \in s_\alpha\}$.

Now define the condition $q = \langle I, n, u\rangle$ as follows. Roughly speaking, q will be the minimal amalgamation of q_α and q_β which may force "$[s_\alpha, s_\beta] \subset G^{\mathcal{G}}$". Let $I = I_\alpha \cup I_\beta$. For $\nu \in I_\alpha$ and $m < n$ let $u(\nu, k) = u_\alpha(\nu, k)$. For $\nu \in I_\beta \setminus I_\alpha$ let $u(\nu, 0) = I \cap (\nu + 1)$ and for $1 \le m < n$ put

$$u(\nu, m) = u_\beta(\nu, m) \cup \{\xi \in \operatorname{dom} s_\alpha : \exists \zeta \in \operatorname{dom} s_\beta\ \exists d \in \omega\\ s_\alpha(\xi) = \langle k_\xi, d\rangle \wedge s_\beta(\zeta) = \langle k_\zeta, d\rangle \wedge u_\beta(\zeta, k_\zeta) \subset u_\beta(\nu, m)\}.$$

Then $q \in \mathcal{Q}$ and $q \Vdash [s_\alpha, s_\beta] \subset G^{\mathcal{G}}$, so we need to check only $q \le q_\alpha, q_\beta$. First observe that $q \le q_\alpha$ is straightforward because I_α is an initial segment of I and $u \restriction I_\alpha \times n = u_\alpha$.

To check $q \le q_\beta$ assume that $\nu_0, \nu_1 \in I_\beta \setminus I_\alpha$ and $m_0, m_1 < n$ such that $u(\nu_0, m_0) \cap u(\nu_1, m_1) \cap (I_\alpha \setminus I_\beta) \ne \emptyset$. We need to show $u_\beta(\nu_0, m_0) \cap u_\beta(\nu_1, m_1) \ne \emptyset$. If $m_0 = 0$ or $m_1 = 0$ then it is clear because $u_\beta(\nu, 0) = (\nu+1) \cap I_\beta$. So assume $m_0, m_1 \ge 1$ and pick $\xi \in u(\nu_0, m_0) \cap u(\nu_1, m_1) \cap (I_\alpha \setminus I_\beta)$. Then $\xi \in \operatorname{dom} s_\alpha$ and there are $\zeta_0, \zeta_1 \in \operatorname{dom} s_\beta$ such that $s_\alpha(\xi) = \langle k_\xi, d\rangle$, $s_\beta(\zeta_i) = \langle k_{\zeta_i}, d\rangle$ and $u_\beta(\zeta_i, k_{\zeta_i}) \subset u_\beta(\nu_i, m_i)$ for $i = 0, 1$.

Since $s_\beta \subset \mathcal{J}$, it follows that $d \in u_\beta(\nu_0, m_0) \cap u_\beta(\nu_1, m_1)$, i.e. $u_\beta(\nu_0, m_0) \cap u_\beta(\nu_1, m_1) \ne \emptyset$.

Since $u_\beta(\nu_0, m_0) \subset u_\beta(\nu_1, m_1)$ clearly implies $u(\nu_0, m_0) \subset u(\nu_1, m_1)$ by the construction of u it follows that $q \le q_\beta$. □

Carrying out our plan we apply theorem 2.2 to find a c.c.c extension $V^{\mathcal{Q}*R*P}$ of $V^{\mathcal{Q}}$ such that

$$V^{\mathcal{Q}*R*P} \models \text{“}MA_{\aleph_1} \text{ holds} + G^{\mathcal{G}} \text{ is 2-solid.”}$$

Thus the following lemma completes the proof of theorem 3.7.

Lemma 3.9. *If $G^{\mathcal{G}}$ is 2-solid then every open set in $X^{\mathcal{G}}$ is either countable or co-countable.*

Proof of the lemma 3.9. Let $V \subset X^{\mathcal{G}}$ be an uncountable open set and $Y \in \left[X\right]^{\omega_1}$. To show $V \cap Y \ne \emptyset$ pick pairwise disjoint, infinite ordinals $\{\nu_\alpha, \mu_\alpha : \alpha < \omega_1\}$ and natural numbers $\{k_\alpha, d_\alpha : \alpha < \omega_1\}$ such that

(i) $\nu_\alpha \in V$ and $U^{\mathcal{G}}(\nu_\alpha, k_\alpha) \subset V$,
(ii) $d_\alpha \in U^{\mathcal{G}}(\nu_\alpha, k_\alpha)$,
(iii) $\mu_\alpha \in Y$,
(iv) $\nu_\alpha < \mu_\alpha < \nu_\beta < \mu_\beta$ for $\alpha < \beta < \omega_1$.

By thinning out our sequence we can assume that $d_\alpha = d$. Let $s_\alpha = \left\{\langle \nu_\alpha, \langle k_\alpha, d\rangle\rangle, \langle \mu_\alpha, \langle 0, d\rangle\rangle\right\}$ for $\alpha < \omega_1$. Then $s_\alpha \in \mathcal{J}$ because of $d \in \omega \subset U^{\mathcal{G}}(\mu_\alpha, 0)$.

Since G is 2-solid there are $\alpha < \beta < \omega_1$ such that $[s_\alpha, s_\beta] \subset G$, especially,

$$\{\langle \mu_\alpha, \langle 0, d\rangle\rangle, \langle \nu_\beta, \langle k_\beta, d\rangle\rangle\} \in G.$$

Since $\nu_\beta \notin \mu_\alpha + 1 = U^{\mathcal{G}}(\mu_\alpha, 0)$ it follows that $\mu_\alpha \in U^{\mathcal{G}}(\nu_\beta, k_\beta) \subset V$ and so $V \cap Y \ne \emptyset$ which was to be proved. □

Theorem 3.7 is proved. □

Denote by $\mathbf{2}^{\omega_1}$ the ω_1^{th} power of the discrete, additive topological group $\mathbf{2} = \{0, 1\}$.

Theorem 3.10. *If GCH holds then there is a c.c.c poset P such that*

$$V^P \models \text{“}\mathbf{2}^{\omega_1} \text{ contains an } S\text{-group and } MA_{\aleph_1} \text{ holds”}.$$

Proof. Since $2^\omega = \omega_1$ we have a strong HFD_w $X = \{x_\nu : \nu < \omega_1\} \subset 2^{\omega_1}$.

We can assume that $x_\nu(\nu+1) = 0$ and $x_\nu(\xi) = 1$ for each $\xi \le \nu < \omega_1$. Let A be the subgroup of $\mathbf{2}^{\omega_1}$ generated by X. We show that A will be a c.c.c-indestructible S-group in a certain generic extension.

If $a \in \left[\omega_1\right]^{<\omega} \setminus \{\emptyset\}$ write $x_a = \sum\{x_\nu : \nu \in a\}$ and let $x_\emptyset$ be the unit element of $\mathbf{2}^{\omega_1}$. Since $x_a + x_b = x_{a\triangle b}$ and so $-(x_a) = x_a$, we have $A = \{x_a : a \in \left[\omega_1\right]^{<\omega}\}$.

Fix a countable dense subset D of 2^{ω_1}. Let $K = \left[\omega_1\right]^{<\omega} \times \left[\omega_1\right]^{<\omega} \times D$ and

$$\mathcal{J} = \{\langle \nu, \langle a, t, d\rangle\rangle \in \omega_1 \times K : \nu = \min a \le \min t, x_a \restriction t = d \restriction t\}.$$

Put

$$G = (\left[\omega_1 \times K\right]^2 \setminus \left[\mathcal{J}\right]^2) \cup \Big\{\{\langle \nu_0, \langle a_0, t_0, d_0\rangle\rangle, \langle \nu_1, \langle a_1, t_1, d_1\rangle\rangle\} \in \left[\mathcal{J}\right]^2 : d_0 \ne d_1 \vee (\nu_0 < \nu_1 \wedge x_{a_0} \restriction t_1 = d_1 \restriction t_1)\Big\}.$$

Lemma 3.11. *G is strongly solid.*

Proof. Let $n \in \omega$. Fix a dom-disjoint sequence $\langle s_\alpha : \alpha < \omega_1\rangle \subset \mathrm{Fn}_n(\omega_1, K)$. For each $\alpha < \omega_1$ write $\operatorname{dom} s_\alpha = \{\sigma_{\alpha,i} : i < n\}$ and let $s_\alpha(\sigma_{\alpha,i}) = \langle a_{\alpha,i}, t_{\alpha,i}, d_{\alpha,i}\rangle$.

By the definition of G we can assume that $s_\alpha \subset \mathcal{J}$.

Let $a_\alpha = \cup\{a_{\alpha,i} : i < n\}$ and $t_\alpha = \cup\{t_{\alpha,i} : i < n\}$ and fix increasing enumerations $a_\alpha = \{\gamma_{\alpha,j} : j < k_\alpha\}$ and $t_\alpha = \{\tau_{\alpha,\ell} : \ell < m_\alpha\}$.

By thinning out our sequence we may assume that

(i) $k_\alpha = k$ and $m_\alpha = m$ for each $\alpha < \omega_1$,
(ii) there are elements $\{d_i : i < m\} \subset B$ such that $d_{\alpha,i} = d_i$ for each $\alpha < \omega_1$ and $i < n$,
(iii) $\max a_\alpha \cup t_\alpha < \min a_\beta \cup t_\beta$ for $\alpha < \beta < \omega_1$.
(iv) $\gamma_{\alpha,j} \in a_{\alpha,i}$ iff $\gamma_{\beta,j} \in a_{\beta,i}$ for each $\alpha < \beta < \omega_1$ and $i < n, j < k$.

Define functions $H_\alpha : k \times m \longrightarrow 2$ as follows. Fix $\ell < m$. We will determine the values of $H_\alpha(k-1, \ell)$, $H_\alpha(k-2, \ell), \dots,$ $H(0, \ell)$ successively. Assume $j < k$ and $H_\alpha(j', \ell)$ is defined for $j < j' < k$. If $\gamma_{\alpha,j} \notin \operatorname{dom} s_\alpha$, then let $H_\alpha(j, \ell)$ be arbitrary. If $\gamma_{\alpha,j} = \sigma_{\alpha,i} \in \operatorname{dom} s_\alpha$ then let

$$(*) \qquad H_\alpha(j, \ell) = d_i(\tau_{\alpha,\ell}) - \sum \{H_\alpha(j', \ell) : j < j' < k, \gamma_{\alpha,j'} \in a_{\alpha,i}\}.$$

Observe that for each $i < n$ and $\ell < m$ by $(*)$ and by $\sigma_{\alpha,i} = \min a_{\alpha,i}$ we have

$$(\star) \qquad d_i(\tau_{\alpha,\ell}) = \sum \{H_\alpha(j, \ell) : \gamma_{\alpha,j} \in a_{\alpha,i}\}.$$

We can assume that $H_\alpha = H$ for each $\alpha < \omega_1$.

Since X is a HFD^k_w there are $\alpha < \beta < \omega_1$ such that

$$\forall j < k\ \forall \ell < m\ x_{\gamma_{\alpha,j}}(\tau_{\beta,\ell}) = H(j, \ell).$$

We claim that $[s_\alpha, s_\beta] \subset G$.

Fix $i_0, i_1 < n$ and check $\{\langle \sigma_{\alpha,i_0}, \langle a_{\alpha,i_0}, t_{\alpha,i_0}, d_{i_0}\rangle\rangle, \langle \sigma_{\beta,i_1}, \langle a_{\beta,i_1}, t_{\beta,i_1}, d_{i_1}\rangle\rangle\} \in G$. By the definition of G we can assume $d_{i_0} = d_{i_1}$. To show $x_{a_{\alpha,i_0}} \restriction t_{\beta,i_1} = d_{i_1} \restriction t_{\beta,i_1}$ let $\tau_{\beta,\ell} \in t_{\beta,i_1}$. Then

$$x_{a_{\alpha,i_0}}(\tau_{\beta,\ell}) = \sum \{x_{\gamma_{\alpha,j}}(\tau_{\beta,\ell}) : \gamma_{\alpha,j} \in a_{\alpha,i_0}\} = \sum \{H(j, \ell) : \gamma_{\alpha,j} \in a_{\alpha,i_0}\} = \sum \{H(j, \ell) : \gamma_{\beta,j} \in a_{\beta,i_0}\} = d_{i_0}(\tau_{\beta,\ell}) = d_{i_1}(\tau_{\beta,\ell}),$$

where the first equality holds by definition, the second is satisfied by the choice of α and β, the third is fulfilled by (iv) , and the fourth holds by ($\star$). This completes the proof of the lemma. $\square$

Lemma 3.12. *If G is 1-solid in some model $W \supset V$ then $W \models$ "A is hereditarily separable".*

Proof. Assume on the contrary that $\{x_{a_\alpha} : \alpha < \omega_1\} \subset A$ is a left separated subspace witnessed by basic open sets $[c_\alpha]$, i.e. $c_\alpha \in \mathrm{Fn}(\omega_1, 2)$ such that $c_\alpha \subset x_{a_\alpha}$ and $c_\gamma \not\subset x_{a_\alpha}$ for $\gamma < \alpha$. Pick $d_\alpha \in D$ with $c_\alpha \subset d_\alpha$.

We can assume that

(a) $d_\alpha = d$ for $\alpha < \omega_1$,
(b) $\{a_\alpha : \alpha < \omega_1\}$ forms a Δ-system with kernel a,
(c) $a = \emptyset$ because the mapping $g \longrightarrow g + x_a$ is a homeomorphism of A,
(d) $\max(a_\alpha) < \min(a_\beta)$ for $\alpha < \beta < \omega_1$,
(e) $\{\mathrm{dom}(c_\alpha) : \alpha < \omega_1\}$ forms a Δ-system with kernel r,
(f) $c_\alpha \restriction r = c$ and so $r = \emptyset$ because c_α can be replaced by $c_\alpha \restriction (\mathrm{dom}\, c_\alpha \setminus r)$,
(g) $\max((\mathrm{dom}\, c_\alpha) \cup a_\alpha) < \min((\mathrm{dom}\, c_\beta) \cup a_\beta)$ for $\alpha < \beta < \omega_1$.

For $\alpha < \omega_1$ let $\sigma_\alpha = \min a_\alpha$ and $s_\alpha = \{\langle \sigma_\alpha, \langle a_\alpha, \mathrm{dom}\, c_{\alpha+1}, d\rangle\rangle\}$.

Since G is 1-solid, there are $\alpha < \beta < \omega_1$ such that $[s_\alpha, s_\beta] \subset G$, i.e. $x_{a_\alpha} \restriction \mathrm{dom}\, c_{\beta+1} = d \restriction \mathrm{dom}\, c_{\beta+1} = c_{\beta+1}$ which contradicts the choice of $c_{\beta+1}$. $\square$

By lemma 3.11, G is strongly solid , so applying theorem 2.2 we obtain a c.c.c. poset P such that

$$V^P \models \text{"}G \text{ is 1-solid} + MA_{\aleph_1} \text{ holds"}.$$

Then, by lemma 3.12,

$$V^P \models \text{"}A \text{ is hereditarily separable"}.$$

Finally we show that A is not Lindelöf. Indeed, take $U_\nu = \{f \in A : f(\nu) = 0\}$ for $\nu < \omega_1$. Then $\mathcal{U} = \{U_\nu : \nu < \omega_1\}$ is an open cover of A, because for each $a \in [\omega_1]^{<\omega} \setminus \{\emptyset\}$, taking $\alpha = \min a$, we have $x_a(\alpha) + x_a(\alpha+1) = 1$ and so $x_a \in U_\alpha \cup U_{\alpha+1}$. On the other hand, $\mathcal{U}$ does not contain a countable subcover, because $x_\nu \notin \bigcup_{\zeta \le \nu} U_\zeta$. Thus the group A and the poset P satisfy the requirements of the theorem. $\square$

Using similar arguments we can also get the following result.

Theorem 3.13. *If GCH holds then there is a c.c.c poset P such that*

$$V^P \models \text{"}\mathbf{2}^{\omega_1} \text{ contains an L-group and } MA_{\aleph_1} \text{ holds"}.$$

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