NAGATA'S CONJECTURE AND COUNTABLY COMPACT HULLS IN GENERIC EXTENSIONS

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Abstract. Nagata conjectured that every M-space is homeomorphic to a closed subspace of the product of a countably compact space and a metric space. Although this conjecture was refuted by Burke and van Douwen, and A. Kato, independently, but we can show that there is a c.c.c. poset P of size 2^ω such that in V^P Nagata's conjecture holds for each first countable regular space from the ground model (i.e. if a first countable regular space $X \in V$ is an M-space in V^P then it is homeomorphic to a closed subspace of the product of a countably compact space and a metric space in V^P). By a result of Morita, it is enough to show every first countable regular space from the ground model has a first countable countably compact extension in V^P. As a corollary, we also obtain that every first countable regular space from the ground model has a maximal first countable extension in model V^P.

1. Introduction

A topological space X is called an *M-space* (see [8]) if there is a countable collection of open covers $\{\mathcal{U}_n : n \in \omega\}$ of X, such that:

(i) $\mathcal{U}_{n+1}$ star-refines $\mathcal{U}_n$, for all n.
(ii) If $x_n \in St(x,\mathcal{U}_n)$, for all n, then the set $\{x_n : n \in \omega\}$ has an accumulation point.

Nagata, [8], conjectured that every M-space is homeomorphic to a closed subspace of the product of a countably compact space and a metric space.

To attack this problem the notion of countably-compactifiable spaces was introduced and studied in [6]. We say that a space S is a *countably compact extension* of a space X provided S is countably compact and

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X is a dense subspaces of S. A space X is *countably-compactifiable* if it has a *countably compact hull*, i.e. X has a countably compact extension S such that every countably compact closed subset of X is closed in S. (In [6] a countably compact hull was called *countably compactification.*)

Theorem (Morita, [6]). *An M-space satisfies Nagata's conjecture (i.e. it is homeomorphic to a closed subspace of the product of a countably compact space and a metric space) if and only if it is countably-compactifiable.*

Consider a special case of Nagata's Conjecture, namely just for first countable spaces. A countably compact space is not necessarily a countably compact hull of a dense subspace, but since a countably compact subspace of a first countable space is closed we have

Fact 1.1. *A first countable countably compact space is a countably compact hull of a dense subspace.*

The following question, as we will see soon, is related to Nagata's Conjecture. A first-countable space Y is said to be a *maximal first-countable extension* of a space X provided X is a dense subspace of Y and Y is closed in any first countable space $Z \supset Y$. In [9] the authors considered which first-countable spaces have first-countable maximal extensions and whether all do. Since a countably compact subspace Y of a first countable space Z is closed in Z we have that

Fact 1.2. *A first-countable countably compact space Y is a maximal first-countable extension of any dense subspace X.*

So if you want to construct maximal first-countable extensions or to prove Nagata's Conjecture for first countable spaces the following seems to be a natural idea: *Embed the first countable spaces into first countable, countably compact spaces!*

Unfortunately, generally this is not possible in ZFC because of the following results. Burke and van Douwen in [3], and independently Kato in [5] showed that there are normal, first countable M-spaces which are not countably-compactifiable, hence Nagata's conjecture was refuted. Moreover, in [9], Terada and Terasawa gave three first-countable spaces without maximal first-countable extensions.

Although examples from [3], [5] and [9] are really sophisticated it is easy to construct a ZFC example of a first countable space which can not be embedded into a first countable, countably compact space:

Proposition 1.3. *A Ψ-space does not have a first-countable countably compact extension.*

Proof. The underlying set of a Ψ-space X is $\omega \cup \{x_A : A \in \mathcal{A}\}$, where $\mathcal{A}$ is a maximal almost disjoint family on ω, and A converges to x_A in X for $A \in \mathcal{A}$.

Assume on the contrary that a first countable, countably compact space Y contains X as a dense subspace. Let $\{A_n : n \in \omega\}$ be distinct elements of $\mathcal{A}$. Then $\{x_{A_n} : n \in \omega\}$ has an accumulation point d in Y. Since $\{x_{A_n} : n \in \omega\}$ is closed in X we have $d \in Y \setminus X$. Since ω is dense in X, and so in Y, as well, there is a sequence $D = \{d_n : n \in \omega\} \subset \omega$ converging to d in Y because Y is first-countable. But $\mathcal{A}$ was maximal so there is $A \in \mathcal{A}$ with $|D \cap A| = \omega$. Hence x_A is an accumulation point of D in Y and so $d = x_A$ because Y is T_2. Contradiction. $\square$

However the situation changes dramatically if we want to find a first countable countably compact extension of X in some generic extension of the ground model: in theorem 2.5 we show there is a c.c.c. poset P of size 2^ω such that every first countable regular space from the ground model has a first countable countably compact regular extension in V^P. Hence, by Corollary 2.6, in V^P Nagata's conjecture holds for each first countable regular space from the ground model.

The cardinality of a Ψ-space is at least $\mathfrak{a}$. In theorem 2.1 we show that under Martin's Axiom every first countable regular space of cardinality $< \mathfrak{c}$ can be embedded, as a dense subspace, into a first countable countably compact regular space. Hence, under Martin's Axiom, Nagata's conjecture holds for first countable regular spaces of size less than $\mathfrak{c}$.

The proof of the key lemma 2.3 uses Theorem 2.4 which was proved by Brendle, [2], and by Balcar and Pazak, [1], independently. To make this paper self-contained we include Brendle's proof with his kind permission in Section 3.

The author is grateful to the referee for the formulation of Lemma 2.3 which made possible to unify the proof of Theorems 2.1 and 2.5.

2. First countable, countably compact extensions

Theorem 2.1. *If $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$ then every first countable regular space X of cardinality $< \mathfrak{c}$ can be embedded, as a dense subspace, into a first countable countably compact regular space Y.*

By Morita's results from [6] and [7] Theorem 2.1 yields the following corollary:

Corollary 2.2. *If Martin's Axiom holds then Nagata's conjecture holds for every first countable regular space X of cardinality $< \mathfrak{c}$, i.e. if X is an M-space, then X is homeomorphic to a closed subspace of the* *product of a countably compact space and a metric space. Moreover, X has maximal first-countable extension.*

Proof of Theorem 2.1. The proof is based on the following lemma.

Lemma 2.3. *Let $M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3$ be models of (a large enough fragment of) ZFC such that $M_1 \setminus M_0$ contains a new real, $M_2 \setminus M_1$ contains a real which is not split by the reals of M_1, and $M_3 \setminus M_2$ contains a dominating real over M_2. Let $X \in M_0$, be a first countable regular space, $X \subset M_0$, and for each $x \in X$ let $\{U(x,n) : n < \omega\}$ be a neighbourhood base of x such that $\overline{U(x,n+1)} \subset U(x,n)$, $\langle U(x,n) : x \in X, n < \omega\rangle \in M_0$. Then there is a first countable regular space Y in M_3 such that X is a dense subspace of Y, and for each $y \in Y$ there is a neighbourhood base $\{U'(y,n) : n < \omega\}$ of y with $\overline{U'(y,n+1)} \subset U'(y,n)$ such that*

(i) $U'(x,n) \cap X = U(x,n)$ for $x \in X$ and $n \in \omega$,
(ii) if $U(x,n) \cap U(x',m) = \emptyset$ then $U'(x,n) \cap U'(x',m) = \emptyset$ for $x, x' \in X$ and $n, m \in \omega$,
(iii) if $U(x,n) \subset U(x',m)$ then $U'(x,n) \subset U'(x',m)$ for $x, x' \in X$ and $n, m \in \omega$,
(iv) every $A \in \left[X\right]^{\omega} \cap M_0$ has an accumulation point in Y.

Remark . If X is 0-dimensional then so is Y, and in this case the proof could be simplified a bit.

Proof of lemma 2.3. To start with let us recall a result which was proved by Brendle, and a few weeks later by Balcar and Pazak, independently.

Theorem 2.4 (Brendle [2], Balcar-Pazak [1])**.** *If $M_0 \subset M_1$ are models of (a large enough fragment of) ZFC such that $M_1 \setminus M_0$ contains a real then there is an almost disjoint family $\mathcal{B} \subset \left[\omega\right]^{\omega}$ in M_1 which refines $\left[\omega\right]^{\omega} \cap M_0$.*

Remark . Since Brendle does not intend to publish his proof, which is quite different from the argument of Balcar and Pazak, his proof is included with his kind permission in Section 3.

Let $\mathcal{S} \subset \left[X\right]^{\omega} \cap M_0$ be a maximal almost disjoint family in M_0. By Theorem 2.4 above for each $S \in \mathcal{S}$ there is a maximal almost disjoint family $\mathcal{T}_S \subset \left[S\right]^{\omega}$ in M_1 which refines $\left[S\right]^{\omega} \cap M_0$. Then $\mathcal{T} = \cup\{\mathcal{T}_S : S \in \mathcal{S}\}$ is almost disjoint and refines $\left[X\right]^{\omega} \cap M_0$.

Let

$$\mathcal{A} = \{A \in \left[X\right]^{\omega} \cap M_0 : A \text{ is closed discrete in } X\}.$$

Put

$$\mathcal{B} = \{B \in \mathcal{T} : B \text{ is closed discrete in } X\}.$$

Then $\mathcal{B} \in M_1$ refines $\mathcal{A}$ because $\mathcal{T}$ refines $\left[X\right]^\omega \cap M_0$..

Next, in M_2, for each $B \in \mathcal{B}$ there is $E_B \in \left[B\right]^\omega$ such that for each $x \in X$ and for each $n < \omega$ either $E_B \subset^* U(x,n)$ or $E_B \cap U(x,n)$ is finite.

Let $\mathcal{E} = \{E_B : B \in \mathcal{B}\}$ and take $Y = X \cup \{y_E : E \in \mathcal{E}\}$, where y_E are new points.

We will define the topology on Y in M_3 as follows.

Let d be the dominating real over M_2. For $x \in X$ and $n < \omega$ let

$$U'(x,n) = U(x,n) \cup \{y_E : E \subset^* U(x,n)\}.$$

For $E \in \mathcal{E}$ let $\vec{E}$ be a 1-1 enumeration of E in M_2, and for $n \in \omega$ let

$$U'(y_E,n) = \{y_E\} \cup \bigcup \{U'(\vec{E}(k), d(k)+n) : n \le k < \omega\}.$$

The family $\{U'(y,n) : y \in Y, n \in \omega\}$ clearly satisfies (i), (ii) and (iii).

We intend to define the topology on Y as the one induced by the neighbourhood base $\{U'(y,n) : y \in Y, n \in \omega\}$. First we prove that $\{U'(y,n) : y \in Y, n \in \omega\}$ is a neighbourhood base of a topology.

Claim 2.4.1. *If $t \in U'(v,n)$ then there is m such that $U'(t,m) \subset U'(v,n)$.*

Proof of the claim.
Case 1: $v, t \in X$.
Then there is m such that $U(t,m) \subset U(v,n)$ and so $U'(t,m) \subset U'(v,n)$.
Case 2: $t \in X$ and $v = y_E \in Y \setminus X$.
Then $t \in U'(\vec{E}(k), d(k)+n)$ for some $k \ge n$ and so $t \in U(\vec{E}(k), d(k)+n)$. Thus there is m such that $U(t,m) \subset U(\vec{E}(k), d(k)+n)$ and so $U'(t,m) \subset U'(\vec{E}(k), d(k)+n) \subset U'(v,n)$.
Case 3: $t = y_E \in Y \setminus X$ and $v \in X$.
Then $E \subset^* U(v,n)$. Fix k such that $\vec{E}(k') \in U(v,n)$ for $k' \ge k$. Pick a function $g : \omega \setminus k \longrightarrow \omega$ in M_2 such that $U(\vec{E}(k'), g(k')) \subset U(v,n)$. Then there is $m \ge k$ such that $d(m') \ge g(m')$ for $m' \ge m$. Then $U(\vec{E}(m'), d(m')+m) \subset U(v,n)$ for $m' \ge m$, hence $U'(\vec{E}(m'), d(m')+m) \subset U'(v,n)$ for $m' \ge m$, and so $U'(y_E, m) \subset U'(v,n)$.
Case 4: $t = y_D$ and $v = y_E$ for some $D, E \in \mathcal{E}$.
Then $t \in U'(\vec{E}(n'), d(n')+n)$ for some $n' \ge n$. Since $\vec{E}(n') \in X$ we can apply Case 3 to get an m such that $U'(t,m) \subset U'(\vec{E}(n'), d(n')+n)$ and so $U'(t,m) \subset U'(v,n)$. □

Hence the family $\{U'(y,n) : y \in Y, n \in \omega\}$ can be considered as the neighbourhood base of a topology on Y. Clearly X is dense in Y.

Claim 2.4.2. *If $t \notin U'(v,n)$ then there is m such that $U'(t,m) \cap U'(v,n+1) = \emptyset$.*

Proof of the claim.
Case 1: $v, t \in X$.
Since $\overline{U(v,n+1)} \subset U(v,n)$ there is m such that $U(t,m) \cap U(v,n+1) = \emptyset$. Then $U'(t,m) \cap U'(v,n+1) = \emptyset$ by (ii).
Case 2: $t = y_E \in Y \setminus X$ and $v \in X$.
Since $y_E \notin U'(v,n)$ we have that $E \cap U'(v,n)$ is finite and so there is ℓ such that $\{\vec{E}(i) : i \geq \ell\} \cap U(v,n) = \emptyset$. Then $\{\vec{E}(i) : i \geq \ell\} \cap \overline{U(v,n+1)} = \emptyset$ and so we can find a function $g : \omega \setminus \ell \longrightarrow \omega$ in M_2 such that $U(\vec{E}(i), g(i)) \cap U(v,n+1) = \emptyset$ for $i \geq \ell$. Then there is $m \geq \ell$ such that $d(i) \geq g(i)$ for $i \geq m$. Thus $U(\vec{E}(i), d(i)) \cap U(v,n+1) = \emptyset$ and so $U'(\vec{E}(i), d(i)) \cap U'(v,n+1) = \emptyset$ as well for $i \geq m$. Thus $U'(t,m) \cap U'(v,n+1) = \emptyset$.
Case 3: $t \in X$ and $v = y_E \in Y \setminus X$.
Since E is closed discrete and $t \notin U'(v,n)$ there is ℓ such that $\overline{U(t,\ell)} \cap \{\vec{E}(i) : i \geq n\} = \emptyset$. Fix a function $g : \omega \setminus n \longrightarrow \omega$ in M_2 such that $U(t,\ell) \cap U(\vec{E}(i), g(i)) = \emptyset$ for $i \geq n$. There is $k \geq n$ such that $d(i) \geq g(i)$ for $i \geq k$. Thus

$$(\star) \qquad U(t,\ell) \cap \bigcup \{U(\vec{E}(i), d(i)) : i \geq k\} = \emptyset$$

Since $t \notin U'(v,n)$ we have $t \notin U(\vec{E}(i), d(i)+n)$ for $i \geq n$. Thus $t \notin \overline{U(\vec{E}(i), d(i)+n+1)}$ for $i \geq n$. Fix $m \geq \ell$ such that

$$(\star\star) \qquad U(t,m) \cap \cup \{U(\vec{E}(i), d(i)+n+1) : n \leq i < \ell\} = \emptyset.$$

Putting together $(\star)$ and $(\star\star)$ we obtain $U(t,m) \cap U'(v,n+1) = \emptyset$ and so $U'(t,m) \cap U'(y,n+1) = \emptyset$ as well.
Case 4: $t = y_D$ and $v = y_E$ for some $D, E \in \mathcal{E}$.
Since D and E are closed discrete and $E \cap D$ is finite there is $\ell < \omega$ and a function $g : \omega \setminus \ell \longrightarrow \omega$ in M_2 such that

$$\bigcup \{U(\vec{D}(i), g(i)) : i \geq \ell\} \cap \bigcup \{U(\vec{E}(i), g(i)) : i \geq \ell\} = \emptyset.$$

There is $k \geq \ell$ such that $g(i) \leq d(i)$ for $i \geq k$. Then

$$(*) \qquad \bigcup \{U(\vec{D}(i), d(i)) : i \geq k\} \cap \bigcup \{U(\vec{E}(i), d(i)) : i \geq k\} = \emptyset.$$

Since $t \notin U(\vec{E}(i), d(i)+n)$ for $i \geq n$, by Case 2 for each $i \geq n$ there is j_i such that $U'(t, j_i) \cap U'(\vec{E}(i), d(i)+n+1) = \emptyset$. Let $m_0 = \max\{j_i : n \leq i < k\}$. Then

$$(**) \qquad U'(t, m_0) \cap \bigcup \{U'(\vec{E}(i), d(i)+n+1) : n \leq i < k\} = \emptyset.$$

Let $m = \max\{m_0, k\}$. Then

$$U'(t,m) \setminus \{t\} \subset \bigcup\{U'(\vec{E}(i), d(i)) : i \geq k\}$$

and so by $(*)$ we have

$$(***) \qquad U'(t,m) \cap \bigcup\{U'(\vec{E}(i), d(i)) : i \geq k\} = \emptyset.$$

Since

$$U'(v, n+1) \setminus \bigcup\{U'(\vec{E}(i), d(i)) : i \geq k\} \subset \{U'(\vec{E}(i), d(i)+n+1) : n \leq i < k\}$$

$(**)$ and $(***)$ together yields $U'(t,m) \cap U'(v,n+1) = \emptyset$. $\square$

By claim 2.4.2 we have $\overline{U'(y,n+1)} \subset U'(y,n)$ for $y \in Y$ and $n < \omega$. Since $\bigcap\{U'(y,n) : n < \omega\} = \{y\}$ for $y \in Y$ this yields that that Y is a regular space. Since $X_0 = X$ is a dense subspace $\square_{2.3}$

Using Lemma 2.3 above we can easily prove the theorem. For each $x \in X$ let $\{U(x,n) : n < \omega\}$ be a neighbourhood base of x such that $\overline{U(x,n+1)} \subset U(x,n)$.

For $\alpha \leq 2^\omega$ we will construct first countable spaces X_α with bases $\{U_\alpha(x,n) : x \in X_\alpha, n < \omega\}$ satisfying $\overline{U_\alpha(x,n+1)} \subset U_\alpha(x,n)$, and sets $A_\alpha \in \left[X_\alpha\right]^\omega$ such that

(1) $X_0 = X$, $U_0(x,n) = U(x,n)$,
(2) $|X_\alpha| \leq |X| + |\alpha|$,
(3) $U_\beta(x,n) \cap X_\alpha = U_\alpha(x,n)$ for $\alpha < \beta$, $x \in X_\alpha$ and $n < \omega$,
(4) if $U_\alpha(x,n) \cap U_\alpha(y,m) = \emptyset$ then $U_\beta(x,n) \cap U_\beta(y,m) = \emptyset$ for $\alpha < \beta$,
(5) if $U_\alpha(x,n) \subset U_\alpha(y,m)$ then $U_\beta(x,n) \subset U_\beta(y,m)$ for $\alpha < \beta$,
(6) X_α is a dense subspace of X_β for $\alpha < \beta$,
(7) A_α has an accumulation point in $X_{\alpha+1}$,
(8) $\{A_\alpha : \alpha < 2^\omega\} = \left[X_{2^\omega}\right]^\omega$

For limit α take $X_\alpha = \bigcup\{X_\zeta : \zeta < \alpha\}$ and $U_\alpha(x,n) = \bigcup\{U_\zeta(x,n) : x \in X_\zeta\}$.

If $\alpha = \beta + 1$ then we will apply lemma 2.3 as follows.

Let $X = X_\beta$ and $U(x,n) = U_\beta(x,n)$ for $x \in X_\beta$ and $n < \omega$.

Let N_0 be a model of a large enough fragment of ZFC such that $X, A_\beta \in N_0$, $X \subset N_0$, $\langle U(x,n) : x \in X, n \in \omega\rangle \in N_0$ and $|N_0| = |X|$.

Since $|N_0| < \mathfrak{c}$ the model N_0 can not contain all the reals. Let $N_1 \supset N_0$ be a a model of a large enough fragment of ZFC such that N_1 contains a new real, but $|N_1| = |N_0|$.

Since $|X| < \mathfrak{c} = \mathfrak{s}$ there is a real r which is not split by the reals of N_1. Let $N_2 \supset N_1$ be a a model of a large enough fragment of ZFC such that $r \in N_2$ and $|N_1| = |N_0| = |X|$.

Since $|X| < \mathfrak{c} = \mathfrak{b}$ there is a real d which is a dominating real over N_2. Let $N_3 \supset N_2$ be a a model of a large enough fragment of ZFC such that $d \in N_2$ and $|N_3| = |N_2| = |X|$.

Now applying Lemma 2.3 we obtain a regular space Y in N_3 with a base $\langle U'(y,n) : y \in Y, n < \omega \rangle$ satisfying 2.3.(i)-(iv). Now A_β has an accumulation point z in Y because $A_\beta \in N_0$. Hence the subspace $Z = X \cup \{z\}$ of Y works as $X_{\beta+1}$.

So the inductive construction can be carried out.

By (8) every countable subset of $Y = X_{2^\omega}$ appears as A_α in some intermediate step and so it will have an accumulation point in $X_{\alpha+1}$. So the space Y will be countably compact.

$\square_{2.1}$

Theorem 2.5. *There is a c.c.c. poset of size 2^ω such that every first countable regular space X from the ground model can be embedded, as a dense subspace, into a first countable countably compact regular space Y from the generic extension, and so X has a countably compact hull in the generic extension.*

By Morita's results from [6] and [7] and by Facts 1.1 and 1.2, Theorem 2.5 above yields immediately the following corollary:

Corollary 2.6. *There is a c.c.c. poset P of size 2^ω such that for every first countable regular space X from the ground model V Nagata's conjecture holds for X in V^P, i.e. the following holds in V^P: if X is an M-space, then X is homeomorphic to a closed subspace of the product of a countably compact space and a metric space. Moreover, X has maximal first-countable extension in V^P.*

Proof of Theorem 2.5. We can easily get the theorem from Lemma 2.3. The poset P is obtained by a finite support iteration $\langle P_\alpha : \alpha \le \omega_1 \rangle$ of length ω_1, $P_{\alpha+1} = P_\alpha * \mathcal{C} * R_{\mathcal{F}_\alpha} * \mathcal{D}_\alpha$, where $\mathcal{C}$ is the Cohen-poset, $\mathcal{F}_\alpha$ is a non-principal ultrafilter on ω in $V^{P_\alpha * \mathcal{C}}$, $R_{\mathcal{F}_\alpha}$ introduces a pseudo intersection of the elements of $\mathcal{F}$, and $\mathcal{D}_\alpha$ is the standard c.c.c poset which adds a dominating real to $V^{P_\alpha * \mathcal{C} * R_{\mathcal{F}_\alpha}}$.

Let X be a regular first countable space from the ground model. For each $x \in X$ let $\{U(x,n) : n < \omega\}$ be a neighbourhood base of x such that $\overline{U(x,n+1)} \subset U(x,n)$.

We will construct first countable regular spaces X_α with neighbourhood bases $\{U_\alpha(x,n) : x \in X_\alpha, n < \omega\}$ satisfying $\overline{U_\alpha(x,n+1)} \subset U_\alpha(x,n)$ such that

(1) $X_0 = X$ and $U_0(x,n) = U(x,n)$ for $x \in X$ and $n \in \omega$,
(2) $X_\alpha, \langle U_\alpha(x,n) : x \in X_\alpha, n \in \omega\rangle \in V^{P_\alpha}$,
(3) $U_\beta(x,n) \cap X_\alpha = U_\alpha(x,n)$ for $\alpha < \beta$, $x \in X_\alpha$ and $n < \omega$,
(4) if $\alpha < \beta$ and $U_\alpha(x,n) \cap U_\alpha(y,m) = \emptyset$ then $U_\beta(x,n) \cap U_\beta(y,m) = \emptyset$,
(5) if $\alpha < \beta$ and $U_\alpha(x,n) \subset U_\alpha(y,m)$ then $U_\beta(x,n) \subset U_\beta(y,m)$,
(6) X_α is a dense subspace of X_β for $\alpha < \beta$,
(7) every $A \in \left[X_\alpha\right]^\omega \cap V^{P_\alpha}$ has an accumulation point in $X_{\alpha+1}$.

The construction is straightforward: take $X_\alpha = \bigcup\{X_\zeta : \zeta < \alpha\}$, and $U_\alpha(x,n) = \bigcup\{U_\zeta(x,n) : x \in X_\zeta\}$ for limit α; and apply lemma 2.3 in successor steps as follows: if $\alpha = \nu+1$ then let $M_0 = V^{P_\nu}$, $M_1 = V^{P_\nu * \mathcal{C}}$, $M_2 = V^{P_\nu * \mathcal{C} * \mathcal{R}_{\mathcal{F}_\nu}}$ and $M_3 = V^{P_\nu * \mathcal{C} * \mathcal{R}_{\mathcal{F}_\nu} * \mathcal{D}_\nu}$, $X = X_\nu$ and $U(x,n) = U_\nu(x,n)$ to get the space Y with the base $\langle U'(y,n) : y \in Y, n \in \omega\rangle$, and take $X_\alpha = Y$ and $U_\alpha(x,n) = U'(x,n)$ for $x \in X_\alpha$ and $n \in \omega$.

Clearly X_α and $\{U_\alpha(x,n) : x \in X_\alpha, n < \omega\}$ satisfy the inductive requirements (1)–(7).

Since every countable subset of $Y = X_{\omega_1}$ appears in some intermediate model V^{P_α} the space Y will be countably compact by (6). $\square_{2.5}$

Remark . The inductive construction of the proof of Theorem 2.5 is based on a method which was developed in [4] to show that after adding ω_1-many dominating reals inductively there are locally compact, locally countable, countably compact spaces of arbitrarily large size.

3. Almost disjoint refinement of ground model sets

The proof below is due to J. Brendle, [2], and it is included here with his kind permission.

Proof of Theorem 2.4. Working in M_0 for each countable subset X of ω let T_X be a perfect set of almost disjoint subsets of X. For $Y \in \left[\omega\right]^\omega \cap M_0$ let

$$B^Y_X = \{b \in T_X : |b \cap Y| = \aleph_0\}.$$

Since B^Y_X is a G_δ-set it follows that either $B^Y_X \subset M_0$ is at most countable or B^Y_X contains a perfect set, and in this case in M_1 we have

$$(\star) \qquad |B^Y_X \setminus M_0| = 2^\omega.$$

In M_1 let $\{x_\alpha : \alpha < \kappa\}$ be an enumeration of $[\omega]^\omega \cap M_0$, $\kappa \le \mathfrak{c}$.

By transfinite induction on α we construct an almost disjoint family $\{a_\alpha : \alpha < \kappa\}$ with $a_\alpha \subset x_\alpha$ as follows:

Stage α: Let $\gamma \le \alpha$ be minimal such that $B^{x_\alpha}_{x_\gamma}$ contains a perfect subset. Choose $b_\alpha \in B^{x_\alpha}_{x_\gamma} \setminus M_0$ such that b_α is not chosen earlier as b_ν for some $\nu < \alpha$, and let $a_\alpha = b_\alpha \cap x_\alpha$.

Since $B^{x_\alpha}_{x_\alpha} = T_{x_\alpha}$ the ordinal γ is defined. Hence $|B^Y_X \setminus M_0| = 2^\omega$ by $(\star)$, and so b_α is also defined. Thus $a_\alpha \subset x_\alpha$ is infinite.

Finally we should show that if $\alpha \neq \alpha' < \kappa$ then $|a_\alpha \cap a_{\alpha'}| < \aleph_0$. Assume that γ was chosen for α, and γ' was chosen for α'.

Case 1. $\gamma = \gamma'$.

Then b_α and $b_{\alpha'}$ are different elements of T_{x_γ}, so $b_\alpha \cap b_{\alpha'}$ is finite. Since $a_\alpha \subset b_\alpha$ and $a_{\alpha'} \subset b_{\alpha'}$ we have $|a_\alpha \cap a_{\alpha'}| < \aleph_0$.

Case 2. $\gamma \neq \gamma'$.

We can assume that $\gamma < \gamma'$. By the minimality of γ' we have that $B^{x_{\alpha'}}_{x_\gamma}$ is at most countable and so $B^{x_{\alpha'}}_{x_\gamma} \subset M_0$. Since $b_\alpha \notin M_0$ we have $b_\alpha \notin B^{x_{\alpha'}}_{x_\gamma}$, i.e. $b_\alpha \cap x_{\alpha'}$ is finite.

But $a_\alpha \subset b_\alpha$ and $a_{\alpha'} \subset x_{\alpha'}$. So $|a_\alpha \cap a_{\alpha'}| < \aleph_0$. □

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