# "Scattered spaces"

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## 1 Basic definitions and notions

All *topological spaces* will be assumed to be *Hausdorff* in this entry.

A topological space is called **scattered** (**dispersed**, **clairsemé**) if its every non-empty subspace has an isolated point. A space is scattered iff it is *right-separated*, so $|X| \le w(X)$ for scattered spaces.

Given a topological space $X$, for each ordinal number $\alpha$, the **$\alpha$-th derived set** of $X$, $X^{(\alpha)}$, is defined as follows: $X^{(0)} = X$, $X^{(\alpha+1)}$ is the *derived set* of $X^{(\alpha)}$, i.e. the collection of all limit points of $X^{(\alpha)}$, and if $\alpha$ is limit then $X^{(\alpha)} = \cap_{\nu<\alpha} X^{(\nu)}$. Since $X^{\alpha} \supseteq X^{\beta}$ for $\alpha < \beta$ we have a minimal ordinal $\alpha$ such that $X^{\alpha} = X^{\alpha+1}$. This ordinal $\alpha$, denoted by $ht(X)$, is called the **Cantor-Bendixson height**, or the **height** of $X$. Clearly the subspace $X^{(\alpha)}$ does not have any *isolated points*, it is *dense-in-itself*. The derived sets are all closed, so $X^{(\alpha)}$ is *perfect*. Moreover, $Y = X \setminus X^{(\alpha)}$ is scattered and so it has cardinality $\le w(X)$. This yields the classical Cantor-Bendixson theorem: every space of countable weight can be represented as the union of two disjoint sets, of which one is *perfect* and the other is countable. G. Cantor and I. Bendixson proved this fact independently in 1883 for subsets of the real line.

Historically, the investigation of scattered spaces was started by G. Cantor. He proved, in [3], that if the partial sums of a *trigonometric series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converge to zero except possibly on a set of points of finite scattered height, then all coefficients of the series must be zero.

Denote by $I(Y)$ the isolated points of a topological space $Y$. For each ordinal $\alpha$ define the $\alpha$-th **Cantor-Bendixson level** of a topological space $X$, $I_\alpha(X)$, as follows:

$$I_\alpha(X) = I\big(X \setminus \cup\{I_\beta(X) : \beta < \alpha\}\big).$$

Clearly $I_\alpha(X) = X^{(\alpha)} \setminus X^{(\alpha+1)}$ and $ht(X) = \min\{\alpha : I_\alpha(X) = \emptyset\}$.

Observe that $X$ is scattered iff $X = \bigcup\{I_\alpha(X) : \alpha < ht(X)\}$ iff $X^{ht(X)} = \emptyset$.

Given a scattered space $X$, define the **width** of $X$, $wd(X)$, as follows: $wd(X) = \sup\{|I_\alpha(X)| : \alpha < ht(X)\}$. The **cardinal sequence** of a scattered space $X$, $CS(X)$, is the sequence of the cardinalities of its Candor-Bendixson levels, i.e.

$$CS(X) = \big\langle\ |I_\alpha(X)| : \alpha < ht(X)\ \big\rangle.$$

Axioms of separation The coarsest refinement of $\beta\omega$ in which $\beta\omega \setminus \omega$ is discrete is a scattered Hausdorff space which is not *regular*. There are regular, but not completely regular scattered spaces: see e.g. [E, Example 1.5.9]. A *completely regular* scattered space is not necessarily *0-dimensional*, counterexamples were constructed e.g. by M. Rajagopalan, R. C. Solomon, and J. Terasawa. A scattered space is *hereditarily disconnected.* Since in a compact space the *component* and the *quasi-component* of a point are the same, locally compact scattered spaces are *totally disconnected* and so they are *0-dimensional.* A 0-dimensional scattered space need not be normal. If $\mathcal{A} \subset [\omega]^{\omega}$ is a *Luzin-gap* then the $\Psi$-*like* space $X(\mathcal{A}) = \langle \omega \cup \mathcal{A}, \tau \rangle$ is an uncountable, locally compact scattered space of height 2, which is not normal.

DUAL ALGEBRAS A *Boolean space* is a compact, 0-dimensional, Hausdorff space. The *Stone duality* establishes a 1–1 correspondence between Boolean spaces and *Boolean algebras.*

A Boolean algebra $B$ is **superatomic** iff every homomorphic image of $B$ is atomic. $B$ is superatomic iff every quotient algebra of $B$ is atomic iff every subalgebra of $B$ is atomic.

Write *clopen*$(X)$ for the Boolean algebra of the *clopen* subsets of a Boolean space $X$. Under Stone duality, *closed subspaces* of $X$ correspond to *quotient algebras* of *clopen*$(X)$, and isolated points of a closed subspace correspond to atoms of the corresponding quotient algebra of *clopen*$(X)$. Since a space is scattered iff every closed subspace of $X$ has an isolated point, a Boolean space $X$ is scattered iff *clopen*$(X)$ is superatomic.

Let $B$ be a Boolean algebra; then the Boolean algebra $B^*$ is a **free complete extension** of $B$ if $B^*$ is *complete* and $B$ can be embedded in $B^*$ in such a way that homomorphisms of $B$ into complete Boolean algebras can be extended to complete homomorphisms on $B^*$. G. Day, in [4], proved that a Boolean algebra is superatomic iff it has a free complete extension. He also showed that every Boolean algebra generated by a finite number of superatomic subalgebras is superatomic.

PRESERVATION THEOREMS Subspaces of scattered spaces are scattered. The product of finitely many scattered spaces is scattered, but $2^{\omega}$ is dense in itself. In [13], V. Kannan and M. Rajagopalan proved that a closed continuous image of a scattered zero-dimensional Hausdorff space need not be scattered although it is zero-dimensional and Hausdorff as well. Under CH they constructed a locally compact, locally countable, scattered, first countable, sequentially compact, separable Hausdorff space, that admits a closed continuous map onto the closed unit interval $[0, 1]$.

SCATTERED COMPACTIFICATION The first example of a completely regular scattered space with no scattered compactification was given by P. Nyikos. For more details see entry Compactification.

VARIATIONS OF SCATTEREDNESS A topological space is $C$**-scattered** if for every closed subspace $F \neq \emptyset$ there is a point $x$ with a compact neighborhood contained in $F$. The notion of $C$-scatteredness is a simple simultaneous generalization of scatteredness and of local compactness. A **rim-scattered** space has a base each of whose elements has a scattered boundary. A space $X$ is called $\sigma$**-scattered** if $X = \bigcup\{X_n : n \in \omega\}$, where each $X_n$ is scattered. A space is $G_{\omega}$**-scattered** if every subspace contains a point which is a (relative) $G_{\delta}$. A space is $N$**-scattered** iff nowhere dense subsets are all scattered.

## 2 Theorems on scattered spaces

Mrówka, Rajagopalan and Soundararajan proved in [22] that compact scattered spaces are *pseudo-radial* (*chain net*) spaces. A completely regular, scattered, *countably compact* space need not be pseudo-radial. They also proved that a space is compact and scattered iff it is *chain compact*; i.e., every chain net (=net with a totally ordered directed set) has a convergent cofinal subnet.

Gerlits and Juhász, in [6], proved that that a left-separated compact $T_2$ space is both scattered and sequential. Tkačenko generalized this theorem showing that every countably compact Hausdorff space which is a union of $\omega$ left-separated subspaces is scattered.

Let $X$ be a compact space. Then $X$ is scattered iff $C_p(X)$ is Frèchet-Urysohn (Gerlits-Nagy, [7]) iff the *Pixley-Roy hyperspace* of $X$ is normal (Przymusiński, [23]) iff the tightness of the product $C(X) \times Y$ is countable for every $Y$ of countable tightness (Uspenskiĭ [27]). V. I. Malykhin showed in [16] that if $X$ is compact and $C_p(X)$ is *subsequential*, then $X$ is scattered.

Nyikos and Purisch characterized monotonically normal compact scattered spaces as continuous images of compact ordinal spaces. There are *monotonically normal* (MN) spaces which are not *acyclic monotonically normal* (AMN) as was shown by M. E. Rudin in 1992. P. J. Moody proved that any scattered MN space is AMN.

A topological space $X$ **omits** $\kappa$ if $|X| > \kappa$ and $|F| = \kappa$ for no closed subset $F$ of $X$. Juhász and Nyikos proved that scattered regular spaces do not omit $\kappa$ with $\kappa = \kappa^{<\kappa}$. On the other hand, there is a model of set theory in which there is a Lindelöf scattered space $Y$ of cardinality $2^{\omega_1} > 2^{\omega}$ which has no closed (nor even Lindelöf) subset of cardinality $2^{\omega}$.

In [2] Y. Bregman, A. Shostak and A. Shapirovskiĭ, proved that under some additional set-theoretical assumption (e. g. $V = L$), every space $X$ can be represented as the union of two subspaces $X_1$ and $X_2$ in such a way that, if $Y \subset X$ is compact and either $Y \subset X_1$ or $Y \subset X_2$, then $Y$ is scattered. Using large cardinal assumptions, S. Shelah showed that it is consistent that there is a space $X$ such that if $X = X_1 \cup X_2$ then $X_1$ or $X_2$ contains a copy of the *Cantor set*.

S. Banach raised the following question: determine the topological spaces admitting coarser compact $T_2$ topology. Katětov, in [14], showed that a countable regular space possesses the above property iff it is scattered. I. Juhász and Z. Szentmiklóssy, in [10], proved that there is a Tychonov scattered space of size $\omega_3$ with no smaller compact $T_2$ topology. They also gave a consistent (counter)example of cardinality $\omega_1$.

## 3 Cardinal sequences of scattered spaces

Regular spaces. Since $|X| \leq 2^{|I(X)|}$ for a regular, scattered space $X$, we have $ht(X) < (2^{|I(X)|})^+$. S. Shelah, in [1], constructed, for each $\gamma < (2^{\omega})^+$ a 0-dimensional, scattered space of height $\gamma$ and width $\omega$.

For an infinite cardinal $\kappa$, let $S_\kappa$ be the following family of sequences of cardinals:

$$S_\kappa = \big\{ \langle \kappa_i : i < \delta \rangle : \delta < (2^\kappa)^+,\ \kappa_0 = \kappa \text{ and } \kappa \le \kappa_i \le 2^\kappa \text{ for each } i < \delta \ \big\}.$$

Building on the method of Shelah, it was observed by Juhász and Soukup that $s = CS(X)$ for some regular scattered space $X$ iff $s = CS(X)$ for some 0-dimensional scattered space $X$ iff for some natural number $m$ there are infinite cardinals $\kappa_0 > \kappa_1 > \ldots > \kappa_m$ and sequences $s_i \in S_{\kappa_i}$ such that $s = s_0{}^\frown s_1{}^\frown \ldots {}^\frown s_m$ or $s = s_0{}^\frown s_1{}^\frown \ldots {}^\frown s_m{}^\frown \langle n \rangle$ for some natural number $n > 0$.
Compact spaces The question concerning the cardinal sequences of (locally) compact scattered spaces is much harder. If $X$ is a compact scattered space, then $ht(X)$ should be a successor ordinal, $ht(X) = \beta + 1$, and $I_\beta(X)$ is finite. The subspace $Y = X \setminus I_\beta(X)$ is a locally compact, noncompact, scattered space and $CS(X) = CS(Y)^\frown \langle |I_\beta(X)| \rangle$. On the other hand, , if $Y$ is an arbitrary locally compact, noncompact scattered space, then $X = \alpha Y$, the one-point compactification of $Y$, is a compact scattered space and $CX(X) = CS(Y)^\frown \langle 1 \rangle$. Hence, instead of compact, scattered spaces we can study the cardinal sequences of locally compact, scattered (**LCS**) spaces.

It is a classical result of S. Mazurkiewicz and J. Sierpiński, [19], that a countable compact scattered space is determined completely by the ordinal $\beta$, where $\beta + 1 = ht(X)$, and by the natural number $n = |I_\beta(X)|$: $X$ is homeomorphic to the compact ordinal space $\omega^\beta \cdot n + 1$.

An LCS space $X$ is called $\omega_1$**-thin thick** iff $ht(X) = \omega_1 + 1$, $|I_\alpha(X)| = \omega_1$ for $\alpha < \omega_1$, but $|I_{\omega_1}(X)| = \omega_2$. It is **very thin thick** iff $ht(X) = \omega_1 + 1$, $|I_\alpha(X)| = \omega$ for $\alpha < \omega_1$, but $|I_{\omega_1}(X)| = \omega_2$. It is **thin tall** iff $ht(X) = \omega_1$ and $wd(X) = \omega$. It is **thin very tall** iff $ht(X) = \omega_2$ and $wd(X) = \omega$.

The following problem was first posed by R. Telgarsky in 1968 (unpublished): Does there exist a thin tall LCS space? After some consistency results Rajagopalan, in [24], constructed such a space in ZFC. In [11] Juhász and Weiss showed that for each $\alpha < \omega_2$ there is an LCS space with height $\alpha$ and width $\omega$.

W. Just proved, in [12], that the result of [11] is sharp in the following sense. Add $\omega_2$ *Cohen reals* to a ZFC model satisfying *CH*. Then, in the generic extension, CH fails and there are neither thin very tall nor very thin thick LCS spaces. The first part of this result was improved in [9] by I. Juhász, S. Shelah, L. Soukup and Z. Szentmiklóssy: if we add Cohen reals to a model of set theory satisfying CH, then, in the new model, every LCS space has at most $\omega_1$ many countable levels.

Let $f : [\omega_2]^2 :\to [\omega_2]^{\le\omega}$ be a function with $f(\{\alpha, \beta\}) \subset \alpha \cap \beta$ for $\{\alpha, \beta\} \in [\omega_2]^2$. We say that two finite subsets $x$ and $y$ of $\omega_2$ are *good for* $f$ provided that for $\alpha \in x \cap y$, $\beta \in x \setminus y$ and $\gamma \in y \setminus x$ we always have (i) if $\alpha < \beta, \gamma$ then $\alpha \in f(\{\beta, \gamma\})$, (ii) if $\alpha < \beta$ then $f(\{\alpha, \gamma\}) \subset f(\{\beta, \gamma\})$, (iii) if $\alpha < \gamma$ then $f(\{\alpha, \beta\}) \subset f(\{\gamma, \beta\})$. We say that $f$ is a **$\Delta$-function** if every uncountable family of finite subsets of $\omega_2$ contains two sets $x$ and $y$ which are good for $f$. The notion of $\Delta$-function was introduced in [1]. Baumgartner and Shelah proved that (a) the existence of a $\Delta$-function is consistent with ZFC, (b) if there is a $\Delta$-function then a thin very tall LCS space can be obtained by a natural c.c.c forcing. In this way they obtained the consistency of the existence of a thin very tall LCS space.

The consistency of the existence of an LCS space of height $\omega_3$ and width $\omega_1$ is open. Step (a) of the proof of Baumgartner and Shelah can be generalized: a "$\Delta$-function" $f : [\omega_3]^2 :\to [\omega_3]^{\leq\omega_1}$ can be constructed. However, it is unclear how to step up in part (b) without collapsing cardinals.

J. Roitman, [25], proved that the existence of a very thin thick LCS spaces is consistent with ZFC. On the other hand, J. Baumgartner and S. Shelah, in [1], proved that in Mitchell's model there are no $\omega_1$-thin thick LCS space.

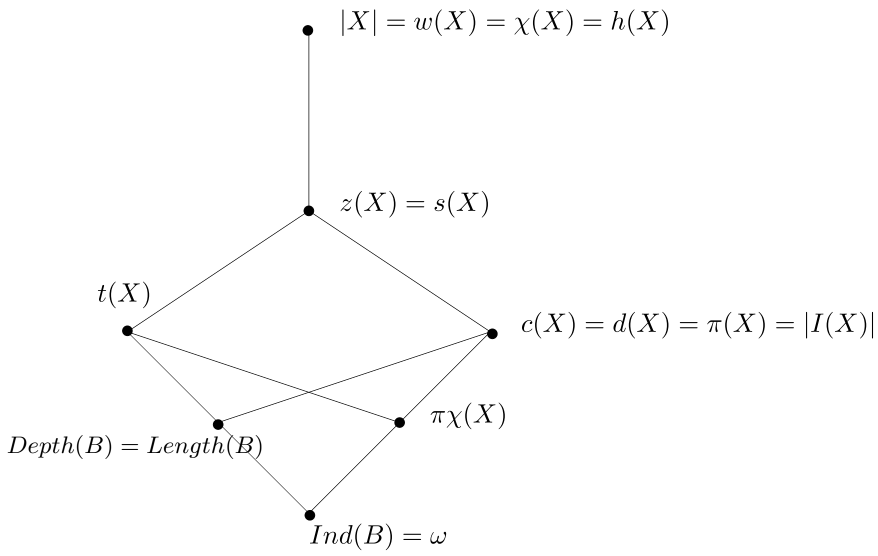
La Grange, in [15], characterized the countable cardinal sequences of LCS spaces: a countable sequence $\langle \kappa_\alpha : \alpha < \delta \rangle$ of infinite cardinals is the cardinal sequence of an LCS space iff $\kappa_\beta \leq {\kappa_\alpha}^\omega$ for each $\alpha < \beta < \delta$. Juhász and Weiss proved that LaGrange's characterization holds even for $\delta = \omega_1$. For $\delta > \omega_1$ we need more assumptions because of the consistency result of Just. The strongest known theorem is the following (unpublished) result of Juhász and Weiss: If $\delta < \omega_2$ and $s = \langle \kappa_\alpha : \alpha < \delta \rangle$ is a sequence of infinite cardinals such that $(*)$ $\kappa_\beta \leq {\kappa_\alpha}^\omega$ for each $\alpha < \beta < \delta$, and $(**)$ $\kappa_\alpha \in \{\omega, \omega_1\}$ for each $\alpha < \delta$ with $cf(\alpha) = \omega_1$, then there is an LCS space $X$ with cardinal sequence $s$.

The cardinal sequences do not characterize uncountable LCS-spaces. An autohomeomorphism of a scattered space is called trivial if it is the identity on some (infinite) Cantor-Bendixson level (and then also on all higher levels). The quotient of the full autohomeomorphism group of a scattered space $X$ by its normal subgroup of trivial autohomeomorphisms is denoted $G(X)$. A. Dow and P. Simon in [5] showed in ZFC that for each countable group $G$, there are $2^{\omega_1}$ pairwise nonisomorphic thin-tall LCS space $X$ such that $G(X)$ is isomorphic to $G$.

The above mentioned estimate $ht(X) < (2^{|I(X)|})^+$ is sharp for LCS spaces with countably many isolated points : it is easy to construct an LCS space with countable "bottom" and of height $\alpha$ for each $\alpha < (2^\omega)^+$. (see [9]). Much less is known about LCS spaces with $\omega_1$ isolated points, for example it is a long standing open problem whether there is, in ZFC, an LCS space of height $\omega_2$ and width $\omega_1$. In fact, as was noticed by Juhász in the mid eighties, even the much simpler question if there is a ZFC example of an LCS space of height $\omega_2$ with only $\omega_1$ isolated points, turned out to be surprisingly difficult. Z. Szentmiklóssy proved in 1983 that if GCH holds in $V$ and we add $\aleph_{\omega_1}$ Cohen reals to $V$ then in the generic extension there is no compact scattered space $X$ such that $|X| = 2^{\omega_1} = \aleph_{\omega_1+1}$ and $I(X) = \omega_1$. Martínez in [17] proved that it is consistent that for each $\alpha < \omega_3$ there is a LCS space of height $\alpha$ and width $\omega_1$. In [9] I. Juhász, S. Shelah, L. Soukup and Z. Szentmiklóssy gave an affirmative answer to the above question of Juhász: they construct, in ZFC, an LCS space of height $\omega_2$ with $\omega_1$ isolated points. However, it is unknown whether there is, in ZFC, an LCS space $X$ of height $\omega_3$ having $\omega_2$ isolated points.

## 4 Cardinal functions on compact scattered spaces

The results mentioned in this section can be found in [20]. Let $X$ be a compact, scattered spaces and $B = clopen(X)$ be the corresponding superatomic Boolean algebra. The following diagram summarizes the relationship between cardinal functions.

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