A CONSISTENT EXAMPLE OF A HEREDITARILY $\mathfrak{c}$-LINDELÖF FIRST COUNTABLE SPACE OF SIZE $> \mathfrak{c}$

ISTVÁN JUHÁSZ, LAJOS SOUKUP, AND ZOLTÁN SZENTMIKLÓSSY

Abstract. Answering a question raised by Anishkievič and Arhangelskiĭ, we show that if $V \models CH$ then there is an ω_1-closed and $\omega_2 - CC$ partial order P such that, in V^P, there exists a 0-dimensional, T_2, hereditarily $\mathfrak{c}$-Lindelöf, and first countable space of cardinality $\omega_2 = \mathfrak{c}^+$. The question if there is such a space (even with "hereditarily" dropped) in ZFC remains open.

At the 1998 Topology Colloquium in Gyula, Hungary, A. V. Arhangelskiĭ asked us the following question (that he attributed to Anishkievič and claimed that it was about 20 years old): Is it true that the cardinality of a $\mathfrak{c}$-Lindelöf first countable T_2 space cannot exceed $\mathfrak{c}$? Clearly, this would yield a natural strengthening of his celebrated result saying that the cardinality of a first countable Lindelöf T_2 space is at most $\mathfrak{c}$. In this note we present a consistent counterexample to this question in that we prove the following result.

Theorem 1. *If $V \models CH$ then there is an ω_1-closed and $\omega_2 - CC$ partial order P such that, in V^P, there is a hereditarily $\mathfrak{c}$-Lindelöf first countable 0-dimensional T_2 space of size $\omega_2 = \mathfrak{c}^+$.*

Proof. Let us start by defining our partial order P. Intuitively, P is the set of countable approximations to a "canonical" first countable, 0-dimensional T_2 topology on ω_2 that is also left separated by the natural well-ordering of ω_2, i. e. all final segments $[\alpha, \omega_2)$ of ω_2 will be open. It turns out that making the approximations left separated yields a technical advantage in finding appropriate amalgamations that are essential for the proof of the $\omega_2 - CC$ of P and especially the hereditary ω_1-Lindelöfness of our generic space.

Now formally the elements of P are couples $\langle A, U\rangle$ where $A \in [\omega_2]^{\le\omega}$ and $U \colon A \times \omega \to \mathcal{P}(A)$ are such that (using the index notation $U_{\alpha,n}$ instead $U(\alpha, n)$)

(1) $\alpha = \min U_{\alpha,n}$ for $\langle\alpha, n\rangle \in A \times \omega$;
(2) $U_{\alpha,n+1} \subset U_{\alpha,n}$ for $\langle\alpha, n\rangle \in A \times \omega$;
(3) $\cap\{U_{\alpha,n} \colon n \in \omega\} = \{\alpha\}$ for $\alpha \in A$;
(4) if $\beta \in U_{\alpha,n}$ then $U_{\beta,k} \subset U_{\alpha,n}$ for some $k \in \omega$;
(5) if $\beta \in A \setminus U_{\alpha,n}$ then $U_{\beta,k} \cap U_{\alpha,n} = \emptyset$ for some $k \in \omega$.

If $p = \langle A, U\rangle \in P$ then we usually write $A = A^p$ and $U = U^p$. It is obvious that the family $\{U^p_{\alpha,n} \colon \langle\alpha, n\rangle \in A^p \times \omega\}$ generates a 0-dimensional T_2 topology τ^p on A^p such that for every fixed $\alpha \in A^p$ the collection $\{U^p_{\alpha,n} \colon n \in \omega\}$ forms a decreasing clopen neighbourhood base.

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Next we define a partial order $\leq$ on P as follows. Let $p = \langle A, U\rangle$ and $p' = \langle A', U'\rangle$ be elements of P, we say that p extends p', i.e. $p \leq p'$, if conditions (i)–(iv) below are satisfied:

(i) $A \supset A'$;
(ii) $U_{\alpha,n} \cap A' = U'_{\alpha,n}$ for $\langle \alpha, n\rangle \in A' \times \omega$;
(iii) $U'_{\beta,k} \subset U'_{\alpha,n}$ implies $U_{\beta,k} \subset U_{\alpha,n}$;
(iv) $U'_{\beta,k} \cap U'_{\alpha,n} = \emptyset$ implies $U_{\beta,k} \cap U_{\alpha,n} = \emptyset$.

It is straightforward to check that $\leq$ is indeed a partial order on P, and it is not much harder to show that it is also ω_1-closed:

Lemma 1. *$\langle P, \leq\rangle$ is ω_1-closed.*

Proof. Indeed, let $p_n = \langle A^n, U^n\rangle \in P$ be such that $p_{n+1} \leq p_n$ for all $n \in \omega$. Set $A = \cup\{A^n : n \in \omega\}$ and for $\langle \alpha, k\rangle \in A \times \omega$ let

$$U_{\alpha,k} = \cup\{U^n_{\alpha,k} : \alpha \in A^n\}.$$

It is straightforward to check then that $p = \langle A, U\rangle \in P$ and $p \leq p_n$ for every $n \in \omega$. $\square$

Our next lemma will, in particular, imply that $\langle P, \leq\rangle$ is also $\omega_2 - CC$, however its main use will be in showing that our generic space is hereditarily ω_1-Lindelöf. First, however, we need a definition.

Definition 1. Let $p = \langle A^p, U^p\rangle$ and $q = \langle A^q, U^q\rangle$ be two members of P. We say that p, q are twins, in short $tw(p, q)$, if

$$\Delta = A^p \cap A^q < A^p \setminus \Delta < A^q \setminus \Delta,$$

moreover $tp(A^p) = tp(A^q)$ and for $\sigma : A^p \to A^q$, the (unique) order preserving map of A^p onto A^q, we have

$$U^q_{\sigma(\alpha),n} = \sigma[U^p_{\alpha,n}]$$

whenever $\langle \alpha, n\rangle \in A^p \times \omega$. Note that as $\sigma \restriction \Delta = id_\Delta$, we clearly have $U^p_{\alpha,n} \cap \Delta = U^q_{\alpha,n} \cap \Delta$ for $\langle \alpha, n\rangle \in \Delta \times \omega$.

The following result says much more than that if p, q are twins then they are compatible. So, as CH will be shown to imply that among any ω_2 members of P there are always two twins, we obtain that $\langle P, \leq\rangle$ is $\omega_2 - CC$.

Lemma 2. *Assume that $p = \langle A^p, U^p\rangle$ and $q = \langle A^q, U^q\rangle$ are twins (more precisely, $tw(p, q)$ holds) and $\alpha \in A^p \setminus \Delta$, $n \in \omega$ are given. Set $A = A^p \cup A^q$ and define $U : A \times \omega \to \mathcal{P}(A)$ as follows (for simplicity, we shall write below $\beta' = \sigma(\beta)$ for $\beta \in A^p \setminus \Delta$, where $\sigma : A^p \to A^q$ is the order preserving bijection between A^p and A^q):*

(a) *if $\delta \in \Delta$ then $U_{\delta,k} = U^p_{\delta,k} \cup U^q_{\delta,k}$;*

(b) $U_{\alpha,k} = \begin{cases} U^p_{\alpha,k} \cup U^q_{\alpha',n} & \textit{for } k \leq n, \\ U^p_{\alpha,k} & \textit{for } k > n; \end{cases}$

(c) *for $\beta \in A^p \setminus (\Delta \cup \{\alpha\})$*

$$U_{\beta,k} = \begin{cases} U^p_{\beta,k} \cup U^q_{\alpha',n} & \textit{if } U^p_{\alpha,n} \subset U^p_{\beta,k}, \\ U^p_{\beta,k} & \textit{otherwise}; \end{cases}$$

(d) *for* $\gamma \in A^q \setminus \Delta$ *we let*

$$U_{\gamma,k} = U^q_{\gamma,k}.$$

Then $r = r(p,q;\alpha,n) = \langle A, U\rangle \in P$ *and* $r \le p, q$.

Proof. To see that $r \in P$, note first that (1) and (2) are obvious, moreover (3), (4) and (5) hold because $U_{\beta,k} = U^p_{\beta,k}$ is satisfied for each large enough $k \in \omega$ if $\beta \in A^p \setminus \Delta$, while the case of other $\beta \in A$ is trivial.

To show that $r \le q$ is again obvious. Finally, proving that $r \le p$ requires a somewhat tedious but straightforward procedure of checking several cases. Thus, for instance, if $\delta \in \Delta$, $\beta \in A^p \setminus \Delta$ and $U^p_{\beta,k} \subset U^p_{\delta,\ell}$ then $tw(p,q)$ implies $U^q_{\beta',k} \subset U^q_{\delta,\ell}$, hence either $U_{\beta,k} = U^p_{\beta,k}$ or $U_{\beta,k} = U^p_{\beta,k} \cup U^q_{\alpha',n} \subset U^p_{\beta,k} \cup U^q_{\beta',k}$ so in both cases $U_{\beta,k} \subset U_{\delta,\ell}$, and similarly $U^p_{\beta,k} \cap U^p_{\delta,\ell} = \emptyset$ implies $U_{\beta,k} \cap U_{\delta,\ell} = \emptyset$.

Next, assume that $\beta, \gamma \in A^p \setminus \Delta$. If $U^p_{\beta,k} \subset U^p_{\gamma,\ell}$ then $U^p_{\alpha,n} \subset U^p_{\beta,k}$ implies $U_{\beta,k} = U^p_{\beta,k} \cup U^q_{\alpha',n} \subset U^p_{\gamma,\ell} \cup U^q_{\alpha',n} = U_{\gamma,\ell}$, while $U^p_{\alpha,n} \not\subset U^p_{\beta,k}$ implies $U_{\beta,k} = U^p_{\beta,k} \subset U_{\gamma,\ell}$. If $U^p_{\beta,k} \cap U^p_{\gamma,\ell} = \emptyset$, then either $U^p_{\alpha,n} \not\subset U^p_{\beta,k}$ and so $U_{\beta,k} = U^p_{\beta,k}$, or $U^p_{\alpha,n} \subset U^p_{\beta,k}$ and then $U^p_{\alpha,n} \not\subset U^p_{\gamma,\ell}$, so clearly in both cases $U_{\beta,k} \cap U_{\gamma,\ell} = \emptyset$. The remaining cases for checking (iii) and (iv) are even easier, while (i) and (ii) are both obvious. □

The following easy "extension lemma" is needed to show that in a P-generic extension of V we indeed obtain a suitable topology on ω_2.

Lemma 3. *For each* $\alpha \in \omega_2$ *let us set*

$$D_\alpha = \{p \in P \colon \alpha \in A^p\}.$$

Then D_α *is dense in* $\langle P, \le\rangle$.

Proof. Assume $p = \langle A^p, U^p\rangle \in P$ and $\alpha \notin A^p$. Then let $A = A^p \cup \{\alpha\}$ and $U \colon A \times \omega \to \mathcal{P}(A)$ be defined by $U_{\beta,k} = U^p_{\beta,k}$ for $\beta \in A^p$ and $U_{\alpha,k} = \{\alpha\}$ for all $k \in \omega$. It is trivial to check then that $q = \langle A, U\rangle \in P$, $q \le p$ and, of course, $q \in D_\alpha$. □

Let us now turn to the proof of our theorem. If G is a P-generic set over V we define in $V[G]$ a map $U \colon \omega_2 \times \omega \to \mathcal{P}(\omega_2)$ by

$$U_{\alpha,k} = \cup\{U^p_{\alpha,k} \colon p \in G \cap D_\alpha\}.$$

It follows from Lemma 3 that U is well-defined and it is straightforward to check that the family $\{U_{\alpha,k} \colon \langle \alpha, k\rangle \in \omega_2 \times \omega\}$ generates a 0-dimensional, T_2, and left separated topology τ on ω_2 such that for each $\alpha \in \omega_2$ the family $\{U_{\alpha,n} \colon n \in \omega\}$ forms a decreasing, clopen neighbourhood base at α, hence τ is also first countable.

That CH will remain valid in $V[G]$ and cardinals are not collapsed follow from the fact that $\langle P, \le\rangle$ is both ω_1-closed and $\omega_2 - CC$, see e.g. [4]. Thus we have $\mathfrak{c}^+ = \omega_2$ in $V[G]$. Consequently, it only remains to show that $\langle \omega_2, \tau\rangle$ is hereditarily ω_1-Lindelöf, i.e. that this space has no right separated subspace of size ω_2, see [2].

Assume, indirectly, that $S \subset \omega_2$ with $|S| = \omega_2$ is right separated in τ. Using the case $\omega_2 \to (\omega_2, \omega)$ of the Erdős–Dushnik–Miller theorem (see [1] or [2]), we can assume that S is actually right separated by the natural ordering on ω_2, moreover that there is a fixed $n \in \omega$ such that if $\alpha, \beta \in S$, $\alpha < \beta$ then $\beta \notin U_{\alpha,n}$.

Let $f \colon \omega_2 \to S$ be the enumerating function of S and $\dot f$ be a P-name for f, then there is a condition $q \in G$ such that $q \Vdash$ "if $\alpha < \beta < \omega_2$ then $\dot f(\beta) \notin U_{\dot f(\alpha),n}$".

Now, working in the ground model V, we may choose for every $\alpha \in \omega_2$ an ordinal $\nu(\alpha) \in \omega_2$ and a condition $p_\alpha \le q$ such that $\nu(\alpha) \in A^{p_\alpha}$ and $p_\alpha \Vdash \dot{f}(\alpha) = \nu(\alpha)$. Then a standard Δ-system and counting argument using CH will yield a set $N \in [\omega_2]^{\omega_2}$ such that if $\alpha, \beta \in N$ and $\alpha < \beta$ then $tw(p_\alpha, p_\beta)$ holds and, moreover, $\sigma(\nu(\alpha)) = \nu(\beta)$ where $\sigma\colon A^{p_\alpha} \to A^{p_\beta}$ is the order preserving bijection between A^{p_α} and A^{p_β}.

Fix $\alpha, \beta \in N$ with $\alpha < \beta$, then we may apply Lemma 2 to $p_\alpha, p_\beta, \nu(\alpha)$ and n and thus obtain $r = r(p_\alpha, p_\beta; \nu(\alpha), n) \le p_\alpha, p_\beta$. But then it is clear from the definition of r that $r \Vdash \nu(\beta) \in U_{\nu(\alpha),n}$, consequently $r \Vdash \dot{f}(\beta) \in U_{\dot{f}(\alpha),n}$. This, however, contradicts that

$$q \Vdash \forall\, \alpha < \beta < \omega_2[\dot{f}(\beta) \notin U_{\dot{f}(\alpha),n}],$$

and thus the proof is completed. □

To finish, let us point out that our theorem yields only a partial solution to the original problem. It remains an open question whether one can provide a ZFC example of a first countable T_2 (or T_3) space of cardinality $> \mathfrak{c}$ that is (hereditarily) $\mathfrak{c}$-Lindelöf.

We have realized only after giving this construction that a completely different consistent example with the same properties as in our theorem had been "almost" constructed in Theorem 1.1 of [3]. What we have there is a first countable GO space S with $s(S) = \omega_1 < d(S)$, obtained from the assumption that both CH holds and there is an ω_2-Suslin line.

However, it is well-known that for any GO space S we have $hL(S) = s(S)$, hence the above space S is actually hereditarily ω_1-Lindelöf as well.

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MTA Rényi Institute of Math., P. O. Box 127, 1364 Budapest, Hungary
E-mail address: juhasz@renyi.hu

MTA Rényi Institute of Math., P. O. Box 127, 1364 Budapest, Hungary
E-mail address: soukup@renyi.hu

Z. Szentmiklóssy, ELTE TTK, 1053 Budapest, Kecskeméti u. 10–12.
E-mail address: zoli@renyi.hu