Boolean algebras with prescribed topological densities*

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Abstract

We give a simplified proof of a theorem of M. Rabus and S. Shelah claiming that for each cardinal μ there is a c.c.c Boolean algebra $\mathcal{B}$ with topological density μ.

The *topological density* $\mathrm{d}(B)$ of a Boolean algebra B is the minimal cardinal μ such that $B \setminus \{0_B\}$ can be covered by μ ultrafilters.

Theorem 1 (M. Rabus and S. Shelah). *For each cardinal μ there is a c.c.c Boolean algebra $\mathcal{B}$ with topological density μ.*

The simplest way to guarantee $\mathrm{d}(B) \geq \mu$ for some Boolean algebra B is to find a family $\mathcal{A} \subset B \setminus \{0_B\}$ of size μ such that $a \wedge a' = 0$ for each $\{a, a'\} \in \left[\mathcal{A}\right]^2$. Obviously if B satisfies c.c.c then this argument can not work for $\mu \geq \omega_1$. However, instead of finding μ elements of B with pairwise empty intersections it is enough to require that B contains sequences with "prescribedly small pairwise intersections":

Observation 2. *Let B be a Boolean algebra, μ be a cardinal. Assume that there is a set $X \subset B \setminus \{0\}$ such that X is not the union of finitely many centered sets and for each $\nu < \mu$ we have*

$$\forall \left\langle x_\xi : \xi < \nu^+ \right\rangle \subset X \ \forall F : \left[\nu^+\right]^2 \xrightarrow{1-1} \nu^+$$
$$\exists \left\langle y_\xi : \xi < \nu^+ \right\rangle \subset X \ \forall \{\xi, \zeta\} \in \left[\nu^+\right]^2 \ y_\xi \wedge y_\zeta \leq x_{F(\xi,\zeta)}. \quad (\dagger)$$

Then the topological density of B is at least μ.

Proof of the observation. Assume on the contrary that $\mathrm{d}(B) < \mu$. Let ν be the minimal cardinal such that X can be covered by ν many ultrafilters, $\{U_\zeta : \zeta < \nu\}$. Clearly $\omega \leq \nu \leq \mathrm{d}(B) < \mu$.

Let $F : \left[\nu^+\right]^2 \longrightarrow \nu^+$ be an injective function such that $(F(\alpha, \gamma) \mod \nu) \neq (F(\beta, \gamma) \mod \nu)$ for $\alpha < \beta < \gamma < \nu^+$. Then for each $I \in \left[\nu^+\right]^{\nu^+}$ and $\sigma < \nu$ there is $\{\alpha, \beta\} \in \left[I\right]^2$ such that $\sigma < (F(\alpha, \beta) \mod \nu)$.

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For each $\xi < \nu^+$ pick an element $x_\xi \in X \setminus \bigcup\{U_\zeta : \zeta < (\xi \mod \nu)\}$. By (†) we can find $\langle y_\xi : \xi < \nu^+\rangle \subset X$ such that $y_\xi \wedge y_\zeta \le x_{F(\xi,\zeta)}$ for each $\{\xi, \zeta\} \in \left[\nu^+\right]^2$.

Since $X \subset \bigcup_{\zeta<\nu} U_\zeta$, there is $\rho < \nu$ such that $I = \{\xi : y_\xi \in U_\rho\}$ is of size ν^+. Then there is $\{\alpha, \beta\} \in \left[I\right]^2$ such that $\rho < (\ F(\alpha, \beta) \mod \rho)$. Thus $x_{F(\alpha,\beta)} \notin U_\rho$ by the choice of $x_{F(\alpha,\beta)}$. On the other hand, $y_\alpha \wedge y_\beta \in U_\rho$, which contradicts $y_\alpha \wedge y_\beta \le x_{F(\alpha,\beta)}$. □

Unfortunately (†) is still too strong to be held in a c.c.c Boolean algebra of topological density $\le \mu$. But, as it turns out, it is not necessary to consider all the sequences $\langle x_\xi : \xi < \nu^+\rangle$ in (†) to derive that the topological density of B is at least μ. We will introduce the index set T in the proof below in order to construct a manageable, but still large enough family of sequences.

A simplified proof of theorem 1. The length of a sequence τ is denoted by $\ell(\tau)$. If $\ell(\tau) = \alpha + 1$ then put $\mathbf{b}(\tau) = \tau \restriction \alpha$. Given sequences ρ and τ we write $\rho \lhd \tau$ to mean that ρ is an initial segment of τ. Denote by $\mathbf{2}$ the trivial Boolean algebra $\{0, 1\}$.

For each $\nu < \mu$ choose a function $h_{\nu^+} : \left[\nu^+\right]^2 \longrightarrow \nu^+$ such that h_{ν^+} is 1–1 and $(h_{\nu^+}(\alpha, \gamma) \mod \nu) \ne (h_{\nu^+}(\beta, \gamma) \mod \nu)$ for $\alpha < \beta < \gamma < \nu^+$. Then for each $I \in \left[\nu^+\right]^{\nu^+}$ and $\sigma < \nu$ there is $\{\alpha, \beta\} \in \left[I\right]^2$ such that $\sigma < (h_{\nu^+}(\alpha, \beta) \mod \nu)$.

Definition 3. We define, by induction on $\alpha \le \mu^+$, T_α as follows:

(1) $T_0 = \{\emptyset\}$,

(2) if α is limit, then $T_\alpha = \bigcup\{T_\beta : \beta < \alpha\}$,

(3) if $\alpha = \beta + 1$ then let
$$T_\alpha = T_\beta \cup \big\{\rho\colon \rho \text{ is a sequence of length} < \mu^+ \text{ and for each } \zeta < \ell(\rho) \text{ either } \rho(\zeta) \in \mu \text{ or } \big(\rho(\zeta) \in T_\beta \text{ and } \rho \restriction \zeta \lhd \rho(\zeta)\big)\big\}.$$

Let $T = T_{\mu^+}$. For $\eta \in T$ let $\operatorname{rank}(\eta) = \min\{\alpha : \eta \in T_\alpha\}$. For $\rho \in T$ we say that ρ is a *sequence with nice tail* iff we can write $\rho = \rho^\downarrow {}^\frown \rho^\uparrow$, where $(\ell(\rho^\downarrow) \mod \mu) = 0$, $\ell(\rho^\uparrow) = \nu^+$ for some $\nu < \mu$ and $\rho \restriction \zeta \lhd \rho(\zeta)$ for each $\ell(\rho^\downarrow) \le \zeta < \ell(\rho)$.

Let $T_{\rm nt} = \{\rho \in T : \rho \text{ is sequence with nice tail }\}$. For $\rho \in T_{\rm nt}$ put $E(\rho) = \{\rho {}^\frown \langle i\rangle : i < \ell(\rho)\}$. The sets $E(\rho)$ are pairwise disjoint.

Definition 4. Define the function F as follows. Let

$$\operatorname{dom}(F) = \bigcup\{\left[E(\rho)\right]^2 : \rho \in T_{\rm nt}\}.$$

For $\{\tau_0, \tau_1\} \in \left[E(\rho)\right]^2$ write $\tau_i = \rho {}^\frown \langle k_i\rangle$ and $\rho = \rho^\downarrow {}^\frown \rho^\uparrow$ and put

$$F(\tau_0, \tau_1) = \rho^\uparrow(h_{\ell(\rho^\uparrow)}(k_0, k_1))$$

Definition 5. For $\eta \in T$ let

$$D(\eta) = \big\{\ \{\eta {}^\frown \langle\emptyset\rangle {}^\frown \langle\omega n + i\rangle\,, \eta {}^\frown \langle\emptyset\rangle {}^\frown \langle\omega n + j\rangle\}\ :\ i < j < n < \omega\big\}$$

and $D = \bigcup\{D(\eta) : \eta \in T\}$.

Definition 6. Let $\mathcal{B}_\mu$ be the Boolean algebra generated by $\{x_\eta : \eta \in T\}$ freely, except the relations in the following set $\Gamma \cup \Delta$:

$$\Gamma = \big\{x_{\tau_0} \wedge x_{\tau_1} \leq x_{F(\tau_0,\tau_1)} : \{\tau_0, \tau_1\} \in \mathrm{dom}(F)\big\},$$

$$\Delta = \big\{x_{\tau_0} \wedge x_{\tau_1} = 0 : \{\tau_0, \tau_1\} \in D\big\}.$$

Lemma 7. $\mathrm{d}(\mathcal{B}_\mu) \geq \mu$.

Proof. Let

$$\nu = \min\big\{\nu' : \text{there are } \eta \in T \text{ and ultrafilters } \langle U_j : j < \nu'\rangle \text{ such that } \{x_\tau : \eta \triangleleft \tau\} \subset \bigcup\{U_j : j < \nu'\}\big\}.$$

It is enough to show that $\nu = \mu$. Assume on the contrary that $\nu < \mu$ witnessed by $\eta \in T$ and ultrafilters $\langle U_j : j < \nu\rangle$. We can assume that $\ell(\eta) \equiv 0 \mod \mu$. First observe that $\nu \geq \omega$ because $D(\eta)$ can not be covered by finitely many ultrafilters either.

Construct a sequence ρ of length ν^+ such that for each $\zeta < \nu^+$:

$$\eta^\frown(\rho \restriction \zeta) \triangleleft \rho(\zeta) \in T \text{ and } x_{\rho(\zeta)} \notin \bigcup\{U_j : j < (\zeta \mod \nu)\}. \tag{$\star$}$$

Assume we have constructed $\rho(\xi)$ for $\xi < \zeta$ satisfying ($\star$). Then $\eta^\frown(\rho \restriction \zeta) \in T$ by definition. Since ν was minimal we have

$$\{x_\tau : \eta^\frown(\rho \restriction \zeta) \triangleleft \tau\} \not\subset \bigcup\big\{U_j : j < (\zeta \mod \nu)\big\}$$

and so we can choose a suitable $\rho(\zeta)$.

Now $\eta^\frown\rho \in T_{\mathrm{nt}}$. For $i < \nu^+$ write $\tau_i = \eta^\frown\rho^\frown\langle i\rangle$. Since $\{\tau_i : i < \nu^+\} \subset \bigcup\{U_\alpha : \alpha < \nu\}$ there are $I \in \big[\nu^+\big]^{\nu^+}$ and $\alpha < \nu$ such that $\tau_i \in U_\alpha$ for each $i \in I$. Pick $\{i,j\} \in \big[I\big]^2$ such that $h_{\nu^+}(i,j) > (\alpha \mod \nu)$. Then $x_{F(\tau_i,\tau_j)} = x_{\rho(h_{\nu^+}(i,j))} \notin U_\alpha$ by ($\star$). But $x_{\tau_i} \wedge x_{\tau_j} \in U_\alpha$ which contradicts $x_{\tau_i} \wedge x_{\tau_j} \leq x_{F(\tau_i,\tau_j)}$. □

Definition 8. We say that $X \subset T$ is *closed* provided:

(i) if $\rho \in X$ and $\ell(\rho) = \alpha + 1$ then $\mathbf{b}(\rho) = \rho \restriction \alpha \in X$,

(ii) if $\{\tau_1, \tau_2\} \in \mathrm{dom}(F) \cap \big[X\big]^2$ then $F(\tau_1, \tau_2) \in X$,

(iii) if $\{\tau_1, \tau_2\} \in D$ and $\tau_1 \in X$ then $\tau_2 \in X$.

Lemma 9. *Every finite $X \subset T$ is contained in a finite closed $Y \subset T$.*

Proof. Observing $D \cap \mathrm{dom}(F) = \emptyset$ close X first for *i* and *ii*, then close for *iii*. □

Definition 10. Let $X \subset T$ be closed and $f : X \longrightarrow \mathbf{2}$. We say that $(*)_f$ holds iff

(1) $f(\tau_1) \wedge f(\tau_2) = 0$ for each $\{\tau_1, \tau_2\} \in D \cap \big[\mathrm{dom}(f)\big]^2$,

(2) $f(\tau_1) \wedge f(\tau_2) \le f(F(\tau_1,\tau_2))$ for each $\{\tau_1,\tau_2\} \in \operatorname{dom}(F) \cap \big[\operatorname{dom}(f)\big]^2$.

The following lemma is a special case of a well-known fact.

Lemma 11. *Assume that* $f : T \longrightarrow \mathbf{2}$. *Then there is a (unique) homomorphism* φ_f *from* $\mathcal{B}_\mu$ *into* $\mathbf{2}$ *such that* $\varphi_f(x_\tau) = f(\tau)$ *iff* $(*)_f$ *holds.*

For each $a \in \mathcal{B}_T \setminus \{0\}$ fix a homomorphism $\psi_a : \mathcal{B}_T \longrightarrow \mathbf{2}$ with $\psi_a(a) = 1$, and finite, closed set $X_a \subset T$ such that a is a Boolean combination of $\{x_\tau : \tau \in X_a\}$. Define $f_a : X_a \longrightarrow \mathbf{2}$ by $f_a(\tau) = \psi_a(x_\tau)$.

Lemma 12. $\mathcal{B}_\mu$ *has precaliber* κ *for each* $\kappa = \operatorname{cf}(\kappa) > \aleph_0$.

Proof. Let $\{a_\alpha : \alpha < \kappa\} \subset \mathcal{B}_\mu \setminus \{0\}$.

It is enough to define a map $f : T \longrightarrow \mathbf{2}$ satisfying $(*)_f$ such that $|\{\alpha : f_{a_\alpha} \subset f\}| = \kappa$ because $f_{a_\alpha} \subset f$ implies $1 = \psi_{a_\alpha}(a_\alpha) = \varphi_{f_{a_\alpha}}(a) = \varphi_f(a_\alpha)$.

By thinning out $\{a_\alpha : \alpha < \kappa\}$ we can assume that $\{X_\alpha : \alpha < \kappa\}$ is a Δ-system with kernel X and that $f_\alpha \restriction X = f'$.

A pair $\{\tau_1,\tau_2\} \in \operatorname{dom}(F)$ is called *crossing pair* if there are $\alpha \ne \beta < \kappa$ such that $\tau_1 \in X_\alpha \setminus X$ and $\tau_2 \in X_\beta \setminus X$. The family of crossing pairs is denoted my $\mathcal{CP}$.

For $\gamma \in \omega_1$ let

$$\mathcal{B}^{(1)}_\gamma = \big\{\{\tau_o,\tau_1\} \in \mathcal{CP} : F(\tau_0,\tau_1) \in X_\gamma\big\}.$$

Claim 12.1. $|\mathcal{B}^{(1)}_\gamma| \le |X||X_\gamma|$.

Proof of the claim 12.1. Let $\{\tau_0,\tau_1\} \in \mathcal{B}^{(1)}_\gamma$, $\tau_i = \tau^\frown\langle k_i\rangle$, $\tau \in X$, $\eta = F(\tau_0,\tau_1) \in X_\gamma$. Then the pair $\langle\tau,\eta\rangle$ determines the pair $\{\tau_0,\tau_1\}$. Indeed, $F\restriction\big[E(\tau)\big]^2$ is 1–1, so $\{\tau_0,\tau_1\}$ is the unique pair $\{\tau_0',\tau_1'\} \in \big[E(\tau)\big]^2$ with $F(\tau_0',\tau_1') = \eta$. □

We say that τ_0 and τ_1 are *twins* iff $\tau_0 \ne \tau_1$ but $\mathbf{b}(\tau_0) = \mathbf{b}(\tau_1)$.

$$\mathcal{B}^{(2)}_\gamma = \big\{\{\tau_o,\tau_1\} \in \mathcal{CP} : \exists\eta \in X_\gamma\ F(\tau_0,\tau_1) \text{ and } \eta \text{ are twins}\big\}.$$

Claim 12.2. $|\mathcal{B}^{(2)}_\gamma| \le |X||X_\gamma|$.

Proof of claim 12.2. Let $\{\tau_0,\tau_1\} \in \mathcal{B}^{(2)}_\gamma$, $\tau_i = \tau^\frown\langle k_i\rangle$, $\tau \in X$. Fix $\eta \in X_\gamma$ such that $F(\tau_0,\tau_1)$ and η are twins. Now the pair $\langle\tau,\eta\rangle$ determines the pair $\{\tau_0,\tau_1\}$. Indeed, $F(\tau_0,\tau_1) = \tau(\xi)$ for some ξ, and there is at most one ξ such that $\tau(\xi)$ and η are twins. But τ and ξ determine $\{\tau_0,\tau_1\}$ because $F\restriction\big[E(\tau)\big]^2$ is 1–1. □

Let

$$\mathcal{B}^{(3)} = \big\{\ \{\{\tau_0,\tau_1\},\{\tau_2,\tau_3\}\} \in \big[\mathcal{CP}\big]^2 :\ F(\tau_0,\tau_1) \text{ and } F(\tau_2,\tau_3) \text{ are twins}\big\}.$$

Claim 12.3. $|\mathcal{B}^{(3)}| \le |X||X|$.

Proof of claim 12.3. Assume that $\big\{\{\tau_0,\tau_1\},\{\tau_2,\tau_3\}\big\} \in \big[\mathcal{CP}\big]^2$. Then $\mathbf{b}(\tau_0)=\mathbf{b}(\tau_1)=\eta\in X$ and $\mathbf{b}(\tau_2)=\mathbf{b}(\tau_3)=\rho\in X$. Moreover $F(\tau_0,\tau_1)=\eta(\xi)$ and $F(\tau_2,\tau_3)=\rho(\zeta)$ for some ξ and ζ. But for given $\eta,\rho\in X$ there is at most one pair $\{\xi,\zeta\}$ such that $\eta(\xi)$ and $\rho(\zeta)$ are twins. Since there is at most one $\{\tau_0',\tau_1'\}\in E(\eta)$ with $F(\tau_0',\tau_1')=\eta(\xi)$ and there is at most one $\{\tau_2',\tau_3'\}\in E(\rho)$ with $F(\tau_2',\tau_3')=\rho(\xi')$, we are done. □

So applying Lázár's free set mapping theorem we can thin out our sequence such that $\mathcal{B}^{(1)}_\gamma=\mathcal{B}^{(2)}_\gamma=\mathcal{B}^{(3)}=\emptyset$ for each $\gamma\in\kappa$.

Let $f^-=\bigcup\{f_\alpha:\alpha<\kappa\}$. Define $f:T\longrightarrow\mathbf{2}$ as follows. Let $f(\eta)=1$ iff either $f^-(\eta)=1$ or $\eta=F(\tau_1,\tau_2\})$ for some $\{\tau_0,\tau_1\}\in\mathcal{CP}$ such that $f^-(\tau_1)=f^-(\tau_2)=1$.

Since $\bigcup\limits_{\gamma<\kappa}\mathcal{B}^{(1)}_\gamma=\emptyset$ we have $f^-\subset f$.

We show that $(*)_f$ holds. Assume that τ_0 and τ_1 are twins and $f(\tau_0)=f(\tau_1)=1$. Since $\mathcal{B}^{(3)}=\emptyset$ and $\mathcal{B}^{(2)}_\gamma=\emptyset$ for each $\gamma<\kappa$, it follows that $\tau_0,\tau_1\in\bigcup\{X_\alpha:\alpha<\kappa\}=\operatorname{dom}(f^-)$. So $\{\tau_0,\tau_1\}\in D$ is impossible because $\tau_0\in X_\gamma$ implies $\tau_1\in X_\gamma$. Thus $\{\tau_0,\tau_1\}\in\operatorname{dom}(F)$ and $f^-(\tau_0)=f^-(\tau_1)=1$ and so $f(F(\tau_0,\tau_1))=1$ by the construction of f. □

Lemma 13. $\mathrm{d}(\mathcal{B}_\mu)\le\mu$.

Proof. First fix a well-ordering $\prec$ of T such that if $\operatorname{rank}(\tau)<\operatorname{rank}(\tau')$ then $\tau\prec\tau'$.

Let $T^-=T\setminus\bigcup\{E(\eta):\eta\in T_{\mathrm{nt}}\}$. Consider the product space

$$\mathcal{X}=2^{T^-}\times(D_{[\mu]^{<\omega}})^{T_{\mathrm{nt}}},$$

where $D_{[\mu]^{<\omega}}$ denotes the discrete topological space of size μ whose underlying set is $\big[\mu\big]^{<\omega}$ instead of μ. Applying $\mathrm{d}((D_\mu)^{2^\mu})=\mu$ and $|T|=2^\mu$ we can fix a dense family $\{g_\xi:\xi<\mu\}\subset\mathcal{X}$. Write $g_\xi=\left\langle g^-_\xi,g^*_\xi\right\rangle$. For $\xi<\mu$ define $s_\xi:T\longrightarrow\mathbf{2}$ as follows: $s_\xi\restriction T^-=g^-_\xi$ and if $\tau\in T\setminus T^-$, then pick the unique $\eta\in T_{\mathrm{nt}}$ with $\tau\in E(\eta)$, $\tau=\eta^\frown\langle i\rangle$, and let $s_\xi(\tau)=1$ iff $i\in g^*_\xi(\eta)$. Let

$$\mathcal{S}=\{s_\xi:\xi<\mu\}.$$

For $\xi<\mu$ define $s^*_\xi:T\longrightarrow\mathbf{2}$ by recursion on $\prec$ as follows. Let $s^*_\xi(\tau)=1$ iff $s_\xi(\tau)=1$ and for each $\{\tau,\tau'\}\in D$ with $\tau'\prec\tau$ we have $s^*_\xi(\tau')=0$ and for each $\{\tau',\tau\}\in\operatorname{dom}(F)$ with $\tau'\prec\tau$ we have $s^*_\xi(\tau')\le s^*_\xi(F(\tau,\tau'))$.

By induction on $\prec$ it is clear that $(*)_{s^*_\xi}$ holds.

Now let $a\in\mathcal{B}_T\setminus\{0\}$. By construction of $\mathcal{S}$ we can find $s_\xi\in\mathcal{S}$ such that $f_a\subset s_\xi$, moreover for each $\eta\in T_{\mathrm{nt}}\cap X_a$ if $\tau\in E(\eta)\setminus X_a$ then $s_\xi(\tau)=0$.

Claim 13.1. $s^*_\xi\supset f_a$.

Proof. By induction on $\prec$. Assume that the claim holds for $\tau'\in X_a$ provided $\tau'\prec\tau$. We can assume $f_a(\tau)=1$. If $\{\tau',\tau\}\in D$, $\tau'\prec\tau$ then $\tau'\in X_a$ so $f_a(\tau')=0$ as $(*)_{f_a}$ holds. So, by the induction hypothesis, $s^*_\xi(\tau')=f_a(\tau')=0$. Assume $\{\tau',\tau\}\in\operatorname{dom}(F)$, $\tau'\prec\tau$. If $\tau'\in X_a$ then $F(\tau',\tau)\in X_a$ and $F(\tau',\tau)\prec\tau$ so $f_a(\tau')=s^*_\xi(\tau')$ and $f_a(F(\tau',\tau))=s^*_\xi(F(\tau',\tau))$. Thus $s^*_\xi(\tau')\le s^*_\xi(F(\tau',\tau))=1$

because $f_a(\tau') = f_a(\tau) \wedge f_a(\tau') \leq f_a(F(\tau', \tau))$ as $(*)_{f_a}$ holds. If $\tau' \notin X_a$ then $s_\xi(\tau') = 0$ by the assumption about s_ξ and so $s^*_\xi(\tau') = 0$. Thus $s^*_\xi(\tau) = 1$ by the construction of f^*. $\square$

Thus $f_a(a) = s^*_\xi(a)$, i.e. $\mathcal{B}_T \setminus \{\emptyset\} = \bigcup\{\varphi^{-1}_{s^*_\xi}\{1\} : \gamma < \mu\}$, which was to be proved, so the lemma holds. $\square$

$\mathcal{B}_\mu$ is c.c.c by lemma 12 and $\mathrm{d}(\mathcal{B}_\mu) = \mu$ by lemmas 13 and 7 so the theorem is proved. $\square$

References

[1] M. Rabus, S. Shelah, *Topological density of ccc Boolean algebras - every cardinality occurs*, PROC. AM. MATH. SOC. Vol 127 (1999). No 9. pp. 2573-2581.