How to drive our families mad

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November 23, 2006

Keywords: `cardinal invariants, almost disjoint number, Cohen model, destructible maximal almost disjoint family`

Abstract

Given a family $\mathcal{F}$ of pairwise almost disjoint (ad) sets on a countable set S, we study families $\tilde{\mathcal{F}}$ of maximal almost disjoint (mad) sets extending $\mathcal{F}$.

We define $\mathfrak{a}^+(\mathcal{F})$ to be the minimal possible cardinality of $\tilde{\mathcal{F}} \setminus \mathcal{F}$ for such $\tilde{\mathcal{F}}$ and $\mathfrak{a}^+(\kappa) = \max\{\mathfrak{a}^+(\mathcal{F}) : |\mathcal{F}| \leq \kappa\}$. We show that all infinite cardinal less than or equal to the continuum $\mathfrak{c}$ can be represented as $\mathfrak{a}^+(\mathcal{F})$ for some ad $\mathcal{F}$ (Theorem 21) and that the inequalities $\aleph_1 = \mathfrak{a} < \mathfrak{a}^+(\aleph_1) = \mathfrak{c}$ (Corollary 19) and $\mathfrak{a} = \mathfrak{a}^+(\aleph_1) < \mathfrak{c}$ (Theorem 20) are both consistent.

We also give a several constructions of mad families with some additional properties.

1 Introduction

Given a family $\mathcal{F}$ of pairwise almost disjoint countable sets, we can ask how the maximal almost disjoint (mad) families extending $\mathcal{F}$ look like. In this and forthcoming note [5] we address some instances of this question and other related problems.

[0)] The first author is partially supported by Chubu University grant 16IS55A. The second author is supported by Center of Excellence grant from the European Union. The third author is supported by Bolyai Grant and OTKA grant no T61600. The research of this paper was began when the first and second authors visited Alfréd Rény Institute in Budapest in 2002. The research was then resumed when the first author visited Centre de Recerca Matemàtica in Barcelona and the third author Barcelona University at the same time in 2006. The first and third authors would like to thank Joan Bagaria and Juan-Carlos Martínez for the arrangement of the visit and their hospitality during the stay in Barcelona. The authors also would like to thank K. Kunen for allowing them to include his unpublished results.

Let us begin with the definition of some notions and notation about almost disjointness we shall use here. Two countable sets A, B are said to be *almost disjoint* (*ad* for short) if $A \cap B$ is finite. A family $\mathcal{F}$ of countable sets is said to be *pairwise almost disjoint* (*ad* for short) if any two distinct A, $B \in \mathcal{F}$ are ad.

If $\mathcal{X} \subseteq [S]^{\aleph_0}$ and $S = \bigcup \mathcal{X}$, $\mathcal{F} \subseteq \mathcal{X}$ is said to be *mad in* $\mathcal{X}$ if $\mathcal{F}$ is ad and there is no ad $\mathcal{F}'$ such that $\mathcal{F} \subsetneqq \mathcal{F}' \subseteq \mathcal{X}$. Thus an ad family $\mathcal{F}$ is mad in $\mathcal{X}$ if and only if there is no $X \in \mathcal{X}$ which is ad from every $Y \in \mathcal{F}$. If $\mathcal{F}$ is mad in $[S]^{\aleph_0}$ for $S = \bigcup \mathcal{F}$, we say simply that $\mathcal{F}$ is a mad family (on S). S is called the *underlying set* of $\mathcal{F}$.

Let

$$\text{(1.1)} \quad \mathfrak{a}(\mathcal{X}) = \min\{|\mathcal{F}| : |\mathcal{F}| \geq \aleph_0 \text{ and } \mathcal{F} \text{ is mad in } \mathcal{X}\}.$$

Clearly, the cardinal invariant $\mathfrak{a}$ known as the almost disjoint number ([2]) can be characterized as:

Example 1 $\mathfrak{a} = \mathfrak{a}([S]^{\aleph_0})$ *for any countable* S.

In this paper we concentrate on the case where the underlying set $S = \bigcup \mathcal{X}$ (or $S = \bigcup \mathcal{F}$) is countable. In [5] we will deal with the cases where S may be also uncountable.

As a countable $S = \bigcup \mathcal{X}$, we often use ω or $T = {}^{\omega>}2$ where T is considered as a tree growing downwards. That is, for b, $b' \in T$, we write $b' \leq_T b$ if $b \subseteq b'$. Each $f \in {}^{\omega}2$ induces the (maximal) branch

$$\text{(1.2)} \quad B(f) = \{f \restriction n : n \in \omega\} \subseteq T$$

in T.

In Section 2, we consider several cardinal invariants of the form $\mathfrak{a}(\mathcal{X})$ for some $\mathcal{X} \subseteq [T]^{\aleph_0}$.

For $\mathcal{X} \subseteq [S]^{\aleph_0}$ with $S = \bigcup \mathcal{X}$, let

$$\text{(1.3)} \quad \mathcal{X}^{\perp} = \{Y \in [S]^{\aleph_0} : \forall X \in \mathcal{X} \ |X \cap Y| < \aleph_0\}.$$

If $Y \in \mathcal{X}^{\perp}$ we shall say that Y is *almost disjoint* (ad) *to* $\mathcal{X}$.

For an ad family $\mathcal{F}$, let

$$\text{(1.4)} \quad \mathfrak{a}^+(\mathcal{F}) = \mathfrak{a}(\mathcal{F}^{\perp}).$$

For a cardinal κ, let

$$\text{(1.5)} \quad \mathfrak{a}^+(\kappa) = \sup\{\mathfrak{a}^+(\mathcal{F}) : \mathcal{F} \text{ is an ad family on } \omega \text{ of cardinality } \leq \kappa\}.$$

Clearly, $\mathfrak{a}^+(\omega) = \mathfrak{a}$ and $\mathfrak{a}^+(\kappa) \leq \mathfrak{a}^+(\lambda) \leq \mathfrak{c}$ for any $\kappa \leq \lambda \leq \mathfrak{c}$. In Section 3 we give several construction of ad families $\mathcal{F}$ for which $\mathcal{F}^\perp$ has some particular property. Using these constructions, we show in Section 4 that $\mathfrak{a}^+(\mathfrak{c}) = \mathfrak{c}$ (actually we have $\mathfrak{a}^+(\bar{\mathfrak{o}}) = \mathfrak{c}$, see Theorem 17) and the consistency of the inequalities $\mathfrak{a} = \aleph_1 < \mathfrak{a}^+(\aleph_1) = \mathfrak{c}$ (actually we have $\mathfrak{a}^+(\bar{\mathfrak{o}}) = \mathfrak{c}$, see Corollary 19). We also show the consistency of $\mathfrak{a}^+(\aleph_1) < \mathfrak{c}$ (Theorem 20).

For undefined notions connected to the forcing, the reader may consult [7] or [8]. We mostly follow the notation and conventions set in [7] and/or [8]. In particular, the forcing is denoted in such a way that stronger conditions are smaller. We assume that $\mathbb{P}$-names are constructed just as in [8] for a poset $\mathbb{P}$ but different from [8] we use symbols with tilde below them like $\underset{\sim}{a}$, $\underset{\sim}{b}$ etc. to denote the $\mathbb{P}$-names corresponding to the sets a, b etc. in the generic extension. V denotes the ground model (in which we live). For poset $\mathbb{P}$ (in V) we use $V^{\mathbb{P}}$ to denote a "generic" generic extension $V[G]$ of V by some $(V, \mathbb{P})$-generic filter G. Thus $V^{\mathbb{P}} \models \cdots$ is synonymous to $\Vdash_{\mathbb{P}}$ " $\cdots$ " or $V \models \Vdash_{\mathbb{P}}$ " $\cdots$ " and a phrase like: "Let $W = V^{\mathbb{P}}$" is to be interpreted as saying: "Let W be a generic extension of V by some/any $(V, \mathbb{P})$-generic filter".

For the notation connected to the set theory of reals see [1] and [2]. With $\mathfrak{c}$ we denote the size of the continuum $2^{\aleph_0}$. $\mathcal{M}$ and $\mathcal{N}$ are the ideals of meager sets and null sets (e.g. over the Cantor space ${}^{\omega}2$) respectively. For $I = \mathcal{M}, \mathcal{N}$ etc., $\mathsf{cov}(I)$ and $\mathsf{non}(I)$ are *covering number* and *uniformity* of I.

For a cardinal κ let $\mathcal{C}_\kappa = \mathrm{Fn}(\kappa, 2)$ or, more generally $\mathcal{C}_X = \mathrm{Fn}(X, 2)$ for any set X. $\mathcal{C}_\kappa$ is the Cohen forcing for adding κ many Cohen reals. $\mathcal{R}_\kappa$ denotes the random forcing for adding κ many random reals. $\mathcal{R}_\kappa$ is the poset consisting of Borel sets of positive measure in ${}^{\kappa}2$ which corresponds to the homogeneous measure algebra of Maharam type κ.

For a poset $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$, $X \subseteq \mathbb{P}$ and $p \in \mathbb{P}$, let

$$X \downarrow p = \{q \in X \,:\, q \leq_{\mathbb{P}} p\}.$$

2 Mad families and almost disjoint numbers

One of the advantages of using $T = {}^{\omega>}2$ as the countable underlying set is that we can define some natural subfamily of $[T]^{\aleph_0}$.

For $X \subseteq T$, let

(2.1) $\lfloor X \rfloor = \{f \in {}^{\omega}2 \,:\, B(f) \subseteq X\}$, and

(2.2) $\lceil X \rceil = \{f \in {}^{\omega}2 \,:\, |\, B(f) \cap X \,| = \aleph_0\}$.

Clearly, we have $[X] \subseteq \lceil X \rceil$. For $X \subseteq T$, let $X^{\uparrow}$ be the upward closure of X, that is:

$$X^{\uparrow} = \{t \restriction n \,:\, t \in X,\, n \le \ell(t)\}. \tag{2.3}$$

Then we have $\lceil X \rceil = [X^{\uparrow}]$ for any $X \subseteq T$.

Definition 1 (Off-binary sets, [9]) *Let*

$$\mathcal{O}_T = \{X \in [T]^{\aleph_0} \,:\, \lceil X \rceil = \emptyset\}.$$

Leathrum [9] called elements of $\mathcal{O}_T$ off-binary sets. Note that $\lceil X \rceil = \emptyset$ if and only if there is no branch in T with infinite intersection with X.

Definition 2 (Antichains) *Let*

$$\mathcal{A}_T = \{X \in [T]^{\aleph_0} \,:\, X \textit{ is an antichain in } T\}.$$

Clearly, we have $\mathcal{A}_T \subseteq \mathcal{O}_T$.

Using the notation above, the cardinal invariant $\mathfrak{o}$ and $\bar{\mathfrak{o}}$ introduced by Leathrum [9] can be characterized as:

$$\mathfrak{o} = \mathfrak{a}(\mathcal{O}_T), \tag{2.4}$$

$$\bar{\mathfrak{o}} = \mathfrak{a}(\mathcal{A}_T) \tag{2.5}$$

(see [9]). Leathrum also showed $\mathfrak{a} \le \mathfrak{o} \le \bar{\mathfrak{o}}$. Brendle [3] proved that $\mathsf{non}(\mathcal{M}) \le \mathfrak{o}$.

Definition 3 (Sets without infinite antichains) *Let*

$$\mathcal{B}_T = \{X \in [T]^{\aleph_0} \,:\, X \textit{ does not contain any infinite antichain}\}.$$

Elements of $\mathcal{B}_T$ are those infinite subsets of T which can be covered by finitely may branches:

Lemma 4 (K. Kunen) *Let $X \in [T]^{\aleph_0}$. Then $X \in \mathcal{B}_T$ if and only if X is covered by finitely may branches in T.*

Proof. If X is covered by finitely many branches in T then X clearly does not contain any infinite antichain since otherwise one of the finitely many branches would contain an infinite antichain.

Suppose now that X can not be covered by finitely many branches. By induction on n, we choose $t_n \in 2^n$ such that $t_0 = \emptyset$, $t_{n+1} = t_n {}^\frown i$ for some $i \in 2$ and

$$X_{n+1} = X \downarrow t_{n+1} \text{ can not be covered by finitely many branches.} \tag{2.6}$$

Since $X_0 = X$ and $X_n = (X_n \downarrow (t_n{}^\frown 0)) \cup (X_n \downarrow (t_n{}^\frown 1)) \cup \{t_n\}$ this construction can be carried out.

By (2.6), the branch $B = \{t_n : n < \omega\}$ does not cover X_n for each $n \in \omega$. So we can pick $s_n \in X_n \setminus B$. Let $S = \{s_n : n \in \omega\}$. S is an infinite set since $\ell(s_n) \geq n$ for all $n \in \omega$. If C is a branch in T different from B then $t_n \notin C$ for some $n \in \omega$ and so $s_m \notin C$ for all $m \geq n$. Hence $S \cap C$ is finite. Moreover $S \cap B = \emptyset$. So we have $\lceil S \rceil = \emptyset$. Thus $S \subseteq X$ should contain an infinite antichain by König's Lemma. ❑ (Lemma 4)

Theorem 5 (K. Kunen) $\mathfrak{a}(\mathcal{B}_T) = \mathfrak{c}$.

Proof. Suppose that $\mathcal{F} \subseteq \mathcal{B}_T$ is an ad family of cardinality $< \mathfrak{c}$. We show that $\mathcal{F}$ is not mad. For each $X \in \mathcal{F}$ there is $b_X \in [{}^\omega 2]^{<\aleph_0}$ such that $X \subseteq \bigcup_{f \in b_X} B(f)$ by Lemma 4. Since $\mathcal{S} = \bigcup \{b_X : X \in \mathcal{F}\}$ has cardinality $\leq |\,\mathcal{F}\,| \cdot \aleph_0 < \mathfrak{c}$, there is $f^* \in {}^\omega 2 \setminus \mathcal{S}$. We have $B(f^*) \in \mathcal{B}_T$ and $B(f^*)$ is ad to $\mathcal{F}$. ❑ (Theorem 5)

Let us say $X \subseteq T$ is *nowhere dense* if $\lceil X \rceil$ is nowhere dense in the Cantor space ${}^\omega 2$. Thus X is nowhere dense if and only if

(2.7) $\quad \forall t \in T\ \exists t' \leq_T t\ \forall t'' \leq_T t'\ (t'' \notin X)$.

Note that, if $X \subseteq T$ is not nowhere dense, then X is dense below some $t \in T$ (in terms of forcing).

Definition 6 (Nowhere dense sets) *Let*

$$\mathcal{ND}_T = \{X \in [T]^{\aleph_0} : X \textit{ is nowhere dense}\}.$$

Note that, for $X \in [T]^{\aleph_0}$ with $X = \{t_n : n \in \omega\}$, we have

$$\lceil X \rceil = \bigcup_{n \in \omega} \bigcap_{m > n} \lceil T \downarrow t_m \rceil.$$

In particular $\lceil X \rceil$ is a G_δ subset of ${}^\omega 2$. Hence by Baire Category Theorem we have

$$\mathcal{ND}_T = \{X \in [T]^{\aleph_0} : \lceil X \rceil \text{ is a meager subset of } {}^\omega 2\}.$$

Lemma 7 *If $X \in [T]^{\aleph_0}$ then there is $X' \in [X]^{\aleph_0}$ such that $X' \in \mathcal{ND}_T$.*

Proof. If $\lceil X \rceil = \emptyset$ then $X \in \mathcal{ND}_T$. Thus we can put $X' = X$. Otherwise let $f \in \lceil X \rceil$ and let $X' = X \cap B(f)$. ❑ (Lemma 7)

Theorem 8 $\mathsf{cov}(\mathcal{M})$, $\mathfrak{a} \leq \mathfrak{a}(\mathcal{ND}_T)$.

Proof. For the inequality $\mathsf{cov}(\mathcal{M}) \le \mathfrak{a}(\mathcal{ND}_T)$, suppose that $\mathcal{F} \subseteq \mathcal{ND}_T$ is an ad family of cardinality $< \mathsf{cov}(\mathcal{M})$. Then $\bigcup\{\lceil X\rceil : X \in \mathcal{F}\} \ne {}^{\omega}2$. Let $f \in {}^{\omega}2 \setminus \bigcup\{\lceil X\rceil : X \in \mathcal{F}\}$. Then $B(f) \in \mathcal{ND}_T \subseteq \mathcal{ND}_T$ and $B(f)$ is ad from all $X \in \mathcal{F}$.

To show $\mathfrak{a} \le \mathfrak{a}(\mathcal{ND}_T)$ suppose that $\mathcal{F} \subseteq \mathcal{ND}_T$ is an ad family of cardinality $< \mathfrak{a}$. Then $\mathcal{F}$ is not a mad family in $[T]^{\aleph_0}$. Hence there is some $X \in [T]^{\aleph_0}$ ad to $\mathcal{F}$. By Lemma 7, there is $X' \subseteq X$ such that $X' \in \mathcal{ND}_T$. Since X' is also ad to $\mathcal{F}$, it follows that $\mathcal{F}$ is not mad in $\mathcal{ND}_T$. ❑ (Theorem 8)

Let σ be the measure on Borel sets of the Cantor space ${}^{\omega}2$ defined as the product measure of the probability measure on 2. For $X \subseteq T$, let $\mu(X) = \sigma(\lceil X\rceil)$.

Definition 9 (Null sets) *Let*

$$\mathcal{N}_T = \{X \in [T]^{\aleph_0} : \mu(X) = 0\}.$$

Theorem 10 *$\mathsf{cov}(\mathcal{N})$, $\mathfrak{a} \le \mathfrak{a}(\mathcal{N}_T)$.*

Proof. Similarly to the proof of Theorem 8. ❑ (Theorem 10)

Definition 11 (Nowhere dense null sets) *Let*

$$\mathcal{NDN}_T = \mathcal{ND}_T \cap \mathcal{N}_T.$$

Lemma 12 *$\mathfrak{a}(\mathcal{ND}_T) \le \mathfrak{a}(\mathcal{NDN}_T)$ and $\mathfrak{a}(\mathcal{N}_T) \le \mathfrak{a}(\mathcal{NDN}_T)$.*

Proof. For the first inequality, suppose that $\mathcal{F}$ is a mad family in $\mathcal{NDN}_T$. Then $\mathcal{F}$ is an ad family in $\mathcal{ND}_T$. It is also mad in $\mathcal{ND}_T$. Suppose not. Then there is an $X \in \mathcal{ND}_T$ ad to $\mathcal{F}$. Let $X' \in [X]^{\aleph_0}$ be as in Lemma 7. Then $X' \in \mathcal{NDN}_T$. Hence $\mathcal{F}$ is not mad in $\mathcal{NDN}_T$. This is a contradiction. The second inequality can be also proved similarly. ❑ (Lemma 12)

The diagram in Fig. 1 summarizes the inequalities obtained in this section integrated into the cardinal diagram given in Brendle [4]. "$\kappa \to \lambda$" in the diagram means that "$\kappa \le \lambda$ is provable in ZFC". There are still quite a few open questions concerning the (in)completeness of this diagram.

Problem 1 (a) *Is it consistent that $\mathfrak{a}(\mathcal{ND}_T)$, $\mathfrak{a}(\mathcal{N}_T)$, $\mathfrak{a}(\mathcal{ND}_T)$, $\mathfrak{a}(\mathcal{NDN}_T)$ are different?*

(b) *Are $\mathfrak{a}(\mathcal{ND}_T)$ etc. independent from $\mathfrak{o}$, $\bar{\mathfrak{o}}$, $\mathfrak{a}_{\mathfrak{s}}$?*

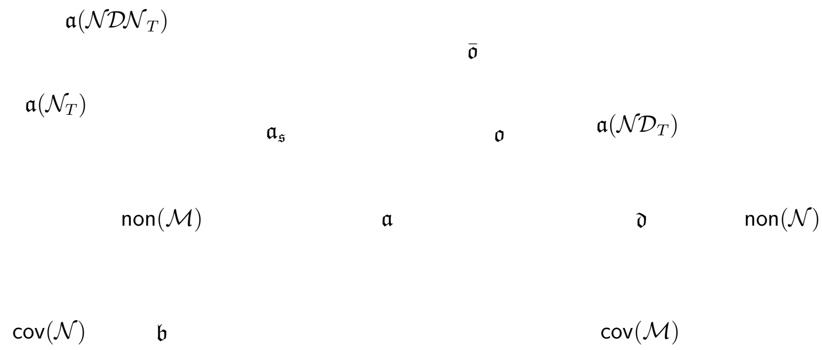

Figure 1:

3 Ad families $\mathcal{F}$ for which $\mathcal{F}^\perp$ is contained in a certain subfamily of $[T]^{\aleph_0}$

In this section we give several constructions of ad families with the property that the sets ad to them in a given generic extension are necessarily in a certain subfamily of $[T]^{\aleph_0}$. The constructions in this section are used in the proof of some results in the next sections.

Theorem 13 (CH) *There exists an ad family $\mathcal{F} \subseteq \mathcal{A}_T$ of size $\aleph_1$ such that for any cardinal κ we have*

$$(3.1) \quad V^{\mathcal{C}_\kappa} \models \mathcal{F}^\perp \subseteq \mathcal{ND}_T.$$

Proof. Let

$$(3.2) \quad \mathcal{S} = \{\langle p, \underset{\sim}{B}, t\rangle \,:\, p \in \mathcal{C}_\omega,\ \underset{\sim}{B} \text{ is a nice } \mathcal{C}_\omega\text{-name of a subset of } T,\ t \in T \text{ and } p \Vdash_{\mathcal{C}_\omega} \text{“}\underset{\sim}{B} \text{ is dense below } t\text{”.}\}$$

Note that this set is of cardinality $\aleph_1$ by CH. Let $\langle\langle p_\alpha, \underset{\sim}{B}_\alpha, t_\alpha\rangle \,:\, \alpha < \omega_1 \setminus \omega\rangle$ be an enumeration of $\mathcal{S}$.

By induction on $\alpha < \omega_1$, we construct $A_\alpha \subseteq T$, $\alpha < \omega_1$ such that

(3.3) $A_\alpha \in \mathcal{A}_T$ for all $\alpha < \omega_1$,

(3.4) A_n, $n \in \omega$ is a partition of T,

(3.5) $|\, A_\beta \cap A_\alpha \,| < \aleph_0$ for all $\beta < \alpha < \omega_1$, and

(3.6) if $\alpha \in \omega_1 \setminus \omega$, for each $q \leq_{\mathcal{C}_\omega} p_\alpha$ and $n \in \omega$, there are $r \leq_{\mathcal{C}_\omega} q$ and $t \in A_\alpha$ such that $|\, t \,| \geq n$ and $r \Vdash_{\mathcal{C}_\omega}$ “$t \in \underset{\sim}{B}_\alpha$” (in particular, $p_\alpha \Vdash_{\mathcal{C}_\omega}$ “$|\, A_\alpha \cap \underset{\sim}{B}_\alpha \,| = \aleph_0$”).

We show first that $\mathcal{F} = \{A_\alpha : \alpha < \omega_1\}$ with A_α's as above satisfies (3.1). Since every subset of T in $V^{\mathcal{C}_\kappa}$ is contained in $V^{\mathcal{C}_X}$ for some countable $X \subseteq \kappa$, it is enough to show (3.1) for $\kappa = \omega$. Assume for contradiction that for some $t^* \in T$, $p^* \in \mathcal{C}_\omega$ and $\mathcal{C}_\omega$-name $\underset{\sim}{B}^*$ of subset of T,

(3.7) $\quad p^* \Vdash_{\mathcal{C}_\omega}$ "$\underset{\sim}{B}^*$ is dense below t^* and $|\underset{\sim}{B}^* \cap A_\alpha| < \aleph_0$ for all $\alpha < \omega_1$".

We may assume that $\underset{\sim}{B}^*$ is a nice $\mathcal{C}_\omega$-name. Let $\alpha < \omega_1 \setminus \omega$ be such that $\langle p_\alpha, \underset{\sim}{B}_\alpha, t_\alpha\rangle = \langle p^*, \underset{\sim}{B}^*, t^*\rangle$. Then $p^* \Vdash_{\mathcal{C}_\omega}$ "$|A_\alpha \cap \underset{\sim}{B}^*| = \aleph_0$" by (3.6). This is a contradiction.

To see that the construction of A_α, $\alpha < \omega_1$ is possible, assume that $\langle A_\beta : \beta < \alpha\rangle$ satisfying (3.3), (3.4), (3.5) and (3.6) has been constructed for $\alpha \in \omega_1 \setminus \omega$.

For $q \leq_{\mathcal{C}_\omega} p_\alpha$ let

$$I(\underset{\sim}{B}_\alpha, q) = \{t \in T : t \leq_T t_\alpha \wedge \exists r \leq_{\mathcal{C}_\omega} q \left(r \Vdash_{\mathcal{C}_\omega} \text{“}\hat{t} \in \underset{\sim}{B}_\alpha\text{”}\right)\}.$$

Note that $I(\underset{\sim}{B}_\alpha, q)$ is dense below t_α by the definition (3.2) of $\langle p_\alpha, \underset{\sim}{B}_\alpha, t_\alpha\rangle \in \mathcal{S}$.

Fix an enumeration $\{\langle q_i, n_i\rangle : i < \omega\}$ of $\{\langle q, n\rangle : q \in \mathcal{C}_\omega \downarrow p_\alpha,\ n \in \omega\}$ and an enumeration $\{\beta_i : i < \omega\}$ of α.

By induction on $m \in \omega$ we choose $u_m \in T$ and $r_m \in \mathcal{C}_\omega$ as below and let

$$A_\alpha = \{u_m : m < \omega\}.$$

In the m'th step of the construction, let $u_m \in T$ and $r_m \in \mathcal{C}_\omega$ be such that

(3.8) $\quad \{u_i : i \leq m\}$ is an antichain in $T \downarrow t_\alpha$ which is not maximal below t_α;

(3.9) $\quad u_m \in I(\underset{\sim}{B}_\alpha, q_m) \setminus \bigcup\{A_{\beta_i} : i < m\}$;

(3.10) $\quad |u_m| \geq n_m$;

(3.11) $\quad r_m \leq_{\mathcal{C}_\omega} q_m$; and

(3.12) $\quad r_m \Vdash_{\mathcal{C}_\omega}$ "$\hat{u}_m \in \underset{\sim}{B}_\alpha$".

This can be carried out. Indeed, at the m'th step if $\{u_i : i < m\}$ has been chosen so that it is a non-maximal antichain below t_α, then we can find $u'_m \in T \downarrow t_\alpha$ distinct from all u_i, $i < m$ such that $\{u_i : i < m\} \cup \{u'_m\}$ is still a non-maximal antichain below t_α. We can also choose u'_m so that $|u'_m| \geq n_m$. Since $\{A_{\beta_i} : i < m\}$ are antichains we can find $u''_m \leq_T u'_m$ such that there is no $t \leq_T u''_m$ with $t \in \cup\{A_{\beta_i} : i < m\}$. Since $I(\underset{\sim}{B}_\alpha, q_m)$ is dense below t_α we can find a $u_m \leq_T u''_m$ such that $u_m \in I(\underset{\sim}{B}_\alpha, q_m)$. By the definition of $I(\underset{\sim}{B}_\alpha, q_m)$ there is an $r_m \leq_{\mathcal{C}_\omega} q_m$ such that $r_m \Vdash_{\mathcal{C}_\omega}$ "$\hat{u}_m \in \underset{\sim}{B}_\alpha$".

It is easy to see that A_α defined as above satisfies (3.3),(3.5) and (3.6): $A_\alpha \in \mathcal{A}_T$ by (3.8). $|\,A_\beta \cap A_\alpha\,| < \aleph_0$ for all $\beta < \alpha$ by (3.9). To show that A_α also satisfies (3.6), suppose that $q \leq_{\mathcal{C}_\omega} p_\alpha$ and $n \in \omega$. Let $m \in \omega$ be such that $\langle q, n\rangle = \langle q_m, n_m\rangle$. Then we have $r_m \leq_{\mathcal{C}_\omega} q$ by (3.11), $u_m \in A_\alpha$ by definition of A_α, $|\,u_m\,| \geq n$ by (3.10) and $r_m \Vdash_{\mathcal{C}_\omega}$ "$\hat{u}_m \in \underset{\sim}{B}_\alpha$" by (3.12). ❑ (Theorem 13)

We can obtain a slightly stronger conclusion than that of the theorem above if our ground model is a generic extension of some inner model by adding uncountably may Cohen reals. Note that CH need not to hold in such a model.

Theorem 14 *Let $W = V^{\mathcal{C}_{\omega_1}}$. Then, in W, there is an ad family $\mathcal{F}$ in $\mathcal{ND}_T$ of cardinality $\aleph_1$ such that, for any c.c.c. poset $\mathbb{P}$ with $\mathbb{P} \in V$, we have $W^{\mathbb{P}} \models \mathcal{F}^\perp \subseteq \mathcal{ND}_T$.*

Proof. Let G be a $(V, \mathcal{C}_{\omega_1})$-generic filter and $W = V[G]$. Working in W, let

$$f_\alpha^G = \{\langle n, i\rangle \,:\, \langle \omega\alpha + n, i\rangle \in p \text{ for some } p \in G\}$$

for $\alpha < \omega_1$. By genericity of G we have $f_\alpha^G \in {}^\omega 2$ and each f_α^G is a Cohen real over V. Let

$$\mathcal{F} = \{B(f_\alpha^G) \,:\, \alpha < \omega_1\}.$$

Clearly $\mathcal{F}$ is an ad family in $\mathcal{ND}_T$. We show that this $\mathcal{F}$ is as desired.

Suppose that $\mathbb{P}$ is c.c.c. (in W) and $\mathbb{P} \in V$. Let H be a $(W, \mathbb{P})$-generic filter. It is enough to show that, in $W[H]$, if $X \in [T]^{\aleph_0}$ is not nowhere dense then X is not ad to $\mathcal{F}$. So suppose that (in $W[H]$) $X \in [T]^{\aleph_0}$ is not nowhere dense. By the c.c.c. of $\mathcal{C}_{\omega_1} * \mathbb{P} \sim \mathcal{C}_{\omega_1} \times \mathbb{P}$, there is an $\alpha^* \in \omega_1 \setminus \omega$ such that $X \in V[(G \restriction \mathcal{C}_{\omega\alpha^*})][H]$. Let $t \in T$ be such that X is dense below t. Note that

$$D = \{p \in \mathcal{C}_{\omega_1} \,:\, \{\langle n, i\rangle \,:\, \langle \omega\alpha + n, i\rangle \in p\} \supseteq t \text{ for some } \alpha \in \omega_1 \setminus \alpha^*\}$$

is dense in $\mathcal{C}_{\omega_1}$. Hence, by the genericity of G, there is an $\alpha \in \omega_1 \setminus \alpha^*$ such that $t \subseteq f_\alpha^G$.

Since f_α^G is a $V[(G \restriction \mathcal{C}_{\omega\alpha^*})][H]$-generic Cohen real, it follows that

$$|\,B(f_\alpha^G) \cap X \downarrow t\,| = \aleph_0.$$

❑ (Theorem 14)

The measure version of Theorem 14 also holds:

Theorem 15 *Let $W = V^{\mathcal{C}_{\omega_1}}$. Then, in W, there is an ad family $\mathcal{F}$ in $\mathcal{N}_T$ of cardinality $\aleph_1$ such that for any c.c.c. poset $\mathbb{P}$ with $\mathbb{P} \in V$, we have $W^{\mathbb{P}} \models \mathcal{F}^\perp \subseteq \mathcal{O}_T$.*

For the proof of Theorem 15 we note first the following:

Lemma 16 *Suppose that $X \subseteq T$ is such that $X = \{t_k : k \in \omega\}$ for some enumeration t_k, $k \in \omega$ of X with $\ell(t_k) \geq k$ for all $k \in \omega$. Then $X \in \mathcal{N}_T$.*

Proof. For all $n \in \omega$, we have $\lceil X \rceil \subseteq \bigcup_{k\in\omega\setminus n} \lceil T \downarrow t_k \rceil$. Hence

$$\mu(X) = \sigma(\lceil X \rceil) \leq \textstyle\sum_{k\in\omega\setminus n} \sigma(\lceil T \downarrow t_k \rceil) \leq \sum_{k\in\omega\setminus n} 2^k = 2^{-n}.$$

It follows that $\mu(X) = 0$. ❑ (Theorem 16)

Proof of Theorem 15 : Let G be a $(V, \mathcal{C}_{\omega_1})$-generic filter and $W = V[G]$. In W, let $f^G_\alpha \in {}^\omega 2$, $\alpha < \omega_1$ be as in the proof of Theorem 14. For $\alpha < \omega_1$ let $g^G_\alpha \in {}^\omega\omega$ be the increasing enumeration of $\left(f^G_\alpha\right)^{-1}[\{1\}]$.

Further in W, we construct inductively $A_\alpha \in \mathcal{N}_T$, $\alpha < \omega_1$ as follows.

For $n \in \omega$, let $A_n \in \mathcal{N}_T$ be such that $\langle A_n : n \in \omega \rangle$ is a partition of T. This can be easily done by Lemma 16.

For $\omega \leq \alpha < \omega_1$, suppose that pairwise almost disjoint A_β, $\beta < \alpha$ have been constructed. Let $\langle B_\ell : \ell \in \omega \rangle$ be an enumeration of $\{A_\beta : \beta < \alpha\}$ and, for each $n \in \omega$, let $\langle b_{n,m} : m \in \omega \rangle$ be an enumeration of

(3.13) $C_n = T \setminus ({}^{n>}2 \cup \{B_\ell : \ell < n\})$.

Let

(3.14) $A_\alpha = \{b_{n,g^G_\alpha(n)} : n \in \omega\}$.

$A_\alpha \in \mathcal{N}_T$ by (3.13) and Lemma 16. By (3.13) and (3.14) A_α is ad to $\{A_\beta : \beta < \alpha\}$.

Suppose that $\mathbb{P}$ is c.c.c. (in W) and $\mathbb{P} \in V$. Let H be a $(W, \mathbb{P})$-generic filter. It is enough to show that, in $W[H]$, if $X \in [T]^{\aleph_0} \setminus \mathcal{O}_T$ then X is not ad to $\mathcal{F}$. Thus suppose that (in $W[H]$) $X \in [T]^{\aleph_0} \setminus \mathcal{O}_T$ and $f \in \lceil X \rceil$. Let $B = X \cap B(f)$. By the c.c.c. of $\mathcal{C}_{\omega_1} * \mathbb{P} \sim \mathcal{C}_{\omega_1} \times \mathbb{P}$, there is an $\alpha^* \in \omega_1 \setminus \omega$ such that $B \in V[(G \restriction \mathcal{C}_{\omega\alpha^*})][H]$. If $B \cap A_\alpha$ is infinite for some $\alpha < \alpha^*$ then we are done. So assume that B is ad to all A_α, $\alpha < \alpha^*$. Then $B \cap C_n$ is infinite for all $n \in \omega$.

Since f^G_α is a $V[(G \restriction \mathcal{C}_{\omega\alpha^*})][H]$-generic Cohen real, it follows that $B \cap A_{\alpha^*}$ is infinite. ❑ (Theorem 15)

4 Almost disjoint numbers over ad families

Theorem 17 (K. Kunen) $\mathfrak{a}^+(\bar{\mathfrak{o}}) = \mathfrak{c}$.

Proof. Let $\mathcal{F}$ be any mad family in $\mathcal{A}_T$ of cardinality $\bar{\mathfrak{o}}$. By maximality of $\mathcal{F}$ we have $\mathcal{F}^{\perp} = \mathcal{B}_T$. If $\mathcal{G} \subseteq [T]^{\aleph_0}$ is disjoint from $\mathcal{F}$ and $\mathcal{F} \cup \mathcal{G}$ is mad then $\mathcal{G}$ is mad in $\mathcal{B}_T$ and hence $|\,\mathcal{G}\,| = \mathfrak{c}$ by Theorem 5. ❑ (Theorem 17)

Theorem 18 *$V^{\mathcal{C}_\kappa} \models \mathfrak{a}^+(\aleph_1) \geq \kappa$ for all regular κ.*

Proof. If $\kappa = \omega_1$ this is trivial. So suppose that $\kappa > \omega_1$. Let $W = V^{\mathcal{C}_{\omega_1}}$. Then $V^{\mathcal{C}_\kappa} = W^{\mathcal{C}_{\kappa\setminus\omega_1}}$. Let $\mathcal{F}$ be as in the proof of Theorem 14. Suppose that $\tilde{\mathcal{F}} \supseteq \mathcal{F}$ is mad on T in $V^{\mathcal{C}_\kappa}$. Then $\tilde{\mathcal{F}} \subseteq (\mathcal{ND}_T)^{V^{\mathcal{C}_\kappa}}$. Since $V^{\mathcal{C}_\kappa} \models \mathsf{cov}(\mathcal{M}) \geq \kappa$, it follows that $|\,\tilde{\mathcal{F}}\,| \geq \kappa$ by Theorem 8. ❑ (Theorem 18)

Corollary 19 *The inequality $\mathfrak{a} = \aleph_1 < \mathfrak{a}^+(\aleph_1) = \mathfrak{c}$ is consistent.* ❑

Theorem 20 *The inequality $\mathfrak{a}^+(\aleph_1) < \mathfrak{c}$ is consistent.*

For the proof of the theorem we use the following forcing notions: for a family $\mathcal{I} \subseteq [\omega]^{\aleph_0} \setminus \{\omega\}$ closed under union, let $\mathbb{Q}_\mathcal{I} = \langle \mathbb{Q}_\mathcal{I}, \leq_{\mathbb{Q}_\mathcal{I}} \rangle$ be the poset defined by

$$\mathbb{Q}_\mathcal{I} = \mathcal{C}_\omega \times \mathcal{I}\,;$$

For all $\langle s, A\rangle$, $\langle s', A'\rangle \in \mathbb{Q}_\mathcal{I}$

$$(4.1) \qquad \langle s', A'\rangle \leq_{\mathbb{Q}_\mathcal{I}} \langle s, A\rangle \quad\Leftrightarrow\quad s \subseteq s',\ A \subseteq A' \text{ and } \forall n \in \mathrm{dom}(s') \setminus \mathrm{dom}(s)\ (n \in A \ \rightarrow\ s'(n) = 0).$$

Clearly $\mathbb{Q}_\mathcal{I}$ is σ-centered.

For a $(V, \mathbb{Q}_\mathcal{I})$-generic G, let

$$f_G = \bigcup\{s \,:\, \langle s, A\rangle \in G \text{ for some } A \in \mathcal{I}\} \text{ and}$$
$$A_G = f_G^{-1}[\{1\}].$$

Let $\tilde{\mathcal{I}}$ be the ideal in $[\omega]^{\aleph_0}$ generated from $\mathcal{I}$ (i.e. the downward closure of $\mathcal{I}$ with respect to $\subseteq$). By the genericity of G and the definition of $\leq_{\mathbb{Q}_\mathcal{I}}$ it is easy to see that,

$$(4.2) \qquad \text{for every } B \in ([\omega]^{\aleph_0})^V,\ A_G \text{ is almost disjoint from } B \ \Leftrightarrow\ B \in \tilde{\mathcal{I}}.$$

Proof of Theorem 20 : $V \models 2^{\aleph_0} = 2^{\aleph_1} = \aleph_3$. In V, let

$$\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \,:\, \alpha \leq \omega_2,\ \beta < \omega_2\rangle$$

be the finite support iteration of c.c.c. posets defined as follows: for $\beta < \omega_2$, let $\underset{\sim}{\mathbb{Q}}_\beta$ be the $\mathbb{P}_\beta$-name of the finite support (side-by-side) product of

(4.3) $\quad \mathbb{Q}_{\tilde{\mathcal{F}}},\ \tilde{\mathcal{F}} \in \Phi$

where

$$\Phi = \{\tilde{\mathcal{F}} : \tilde{\mathcal{F}} \text{ is an ideal in } [\omega]^{\aleph_0} \text{ generated from an ad family in } [\omega]^{\aleph_0} \text{ of cardinality } \aleph_1\}$$

in $V^{\mathbb{P}_\beta}$. We have

$$V^{\mathbb{P}_\beta} \models \underset{\sim}{\mathbb{Q}}_\beta \text{ satisfies the c.c.c.}$$

since $V^{\mathbb{P}_\beta} \models \mathbb{Q}_{\tilde{\mathcal{F}}}$ is σ-centered for all $\tilde{\mathcal{F}} \in \Phi$. By induction on $\alpha \leq \omega_2$, we can show that $\mathbb{P}_\alpha$ satisfies the c.c.c. and $|\,\mathbb{P}_\alpha\,| \leq 2^{\aleph_1} = \aleph_3$ for all $\alpha \leq \omega_2$. It follows that

(4.4) $\quad V^{\mathbb{P}_{\omega_1}} \models 2^{\aleph_0} = 2^{\aleph_1} = \aleph_3.$

Thus the following claim finishes the proof:

Claim 20.1 $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{a} = \mathfrak{a}^+(\aleph_1) = \aleph_2$.

$\vdash$ Working in $V^{\mathbb{P}_{\omega_2}}$, suppose that $\mathcal{F}$ is an ad family in $[\omega]^{\aleph_0}$ of cardinality $\aleph_1$. By the c.c.c. of $\mathbb{P}_{\omega_2}$, there is some $\alpha^* < \omega_2$ such that $\mathcal{F} \in V^{\mathbb{P}_{\alpha^*}}$. By (4.3) and (4.2), there A_α, $\alpha \in \omega_2 \setminus \alpha^*$ such that

(4.5) for every $B \in ([\omega]^{\aleph_0})^{V^{\mathbb{P}_\alpha}}$, A_α is ad from $B \Leftrightarrow B \in$ the ideal generated from $\mathcal{F} \cup \{A_\beta : \beta \in \alpha \setminus \alpha^*\}$.

Since $([\omega]^{\aleph_0})^{\mathbb{P}_{\omega_2}} = \bigcup_{\alpha<\omega_2}([\omega]^{\aleph_0})^{V^{\mathbb{P}_\alpha}}$, it follows that $\mathcal{F} \cup \{A_\alpha : \alpha \in \omega_1 \setminus \alpha^*\}$ is a mad family in $V^{\mathbb{P}_{\omega_2}}$. This shows that $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{a}^+(\aleph_1) = \aleph_2$. Similar argument also shows that $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{a} = \aleph_2$. $\dashv$ ❑ (Theorem 20)

Clearly, the method of the proof of Theorem 20 cannot produce a model of $\mathfrak{a}^+(\aleph_1) = \aleph_1 < \mathfrak{c}$.

Problem 2 *Is $\mathfrak{a}^+(\aleph_1) = \aleph_1 < \mathfrak{c}$ consistent?*

All infinite cardinal less than or equal to the continuum $\mathfrak{c}$ can be represented as $\mathfrak{a}^+(\mathcal{F})$ for some $\mathcal{F}$.

Theorem 21 *For any infinite $\kappa \leq \mathfrak{c}$, there is an ad family $\mathcal{F} \subseteq [T]^{\aleph_0}$ of cardinality $\mathfrak{c}$ such that $\mathfrak{a}^+(\mathcal{F}) = \kappa$.*

Proof. Let $\mathcal{F}'$ be a mad family in $\mathcal{A}_T$. Then by Lemma 4, we have

(4.6) $\quad \mathcal{F}'^{\perp} = \mathcal{B}_T.$

Let $\mathcal{X}''$ and $\mathcal{X}'''$ be disjoint with ${}^{\omega}2 = \mathcal{X}'' \cup \mathcal{X}'''$, $|\,\mathcal{X}''\,| = \mathfrak{c}$ and $|\,\mathcal{X}'''\,| = \kappa$. Let

$$\mathcal{F} = \mathcal{F}' \cup \{B(f) \,:\, f \in \mathcal{X}''\}.$$

Clearly $\mathcal{F}$ is an ad family. By (4.6) we have $\mathcal{F}^{\perp} \subseteq \mathcal{B}_T$. Thus every mad $\tilde{\mathcal{F}}$ extending $\mathcal{F}$ has the form $\mathcal{F} \cup \{\{B(f) \,:\, f \in b\} \,:\, b \in \mathcal{P}\}$ for a partition $\mathcal{P}$ of $\mathcal{G}'''$ into finite sets. It follows that we have always $|\,\tilde{\mathcal{F}} \setminus \mathcal{F}\,| = \kappa$. This shows $\mathfrak{a}^{+}(\mathcal{F}) = \kappa$. ❑ (Theorem 21)

5 Destructibility of mad families

For a poset $\mathbb{P}$, a mad family $\mathcal{F}$ in $[T]^{\aleph_0}$ is said to be $\mathbb{P}$-*destructible* if $V^{\mathbb{P}} \models \mathcal{F}$ is not mad in $[T]^{\aleph_0}$. Otherwise it is $\mathcal{P}$-*indestructible*.

The results in the previous section can also be reformulated in terms of destructibility of mad families.

Theorem 22 (1) (CH) *There is an ad family $\mathcal{F} \subseteq \mathcal{A}_T$ which cannot be extended to a $\mathcal{C}_\omega$-indestructible mad family in any generic extension of the ground model of the form $V^{\mathcal{C}_\kappa}$.*

(2) *Let $W = V^{\mathcal{C}_{\omega_1}}$. Then, in W, there is an ad family $\mathcal{F} \subseteq \mathcal{ND}_T$ of cardinality $\aleph_1$ such that, in any generic extension of W by a c.c.c. poset $\mathbb{P}$ with $\mathbb{P} \in V$, $\mathcal{F}$ cannot be extended to a $\mathcal{C}_\omega$-indestructible mad family.*

(3) *Let $W = V^{\mathcal{C}_{\omega_1}}$. Then, in W, there is an ad family $\mathcal{F} \subseteq \mathcal{N}_T$ of cardinality $\aleph_1$ such that, in any generic extension of W by a c.c.c. poset $\mathbb{P}$ with $\mathbb{P} \in V$, $\mathcal{F}$ cannot be extended to a $\mathcal{R}_\omega$-indestructible mad family.*

Proof. (1): The family $\mathcal{F}$ as in Theorem 13 will do. Since we have $\mathcal{F}' \subseteq \mathcal{ND}_T$ for any mad $\mathcal{F}'$ extending $\mathcal{F}$ in $V^{\mathcal{C}_\kappa}$, a further Cohen real over $V^{\mathcal{C}_\kappa}$ introduces a branch almost avoiding all elements of $\mathcal{F}'$. Thus $\mathcal{F}'$ is no more mad in $V^{\mathcal{C}_\kappa * \mathcal{C}_\omega}$.

(2): By Theorem 14 and by an argument similar to the proof of (1).

(3): In W, let $\mathcal{F}$ be as in the proof of Theorem 15. Then any mad $\mathcal{F}' \supseteq \mathcal{F}$ on T in any $W^{\mathbb{P}}$ for $\mathbb{P}$ as above is included in $\mathcal{N}_T$ by $\mathcal{O}_T \subseteq \mathcal{N}_T$. Hence, in $W^{\mathbb{P}*\mathcal{R}_\omega}$, the random real f over $W^{\mathbb{P}}$ introduces the branch $B(f)$ almost avoiding all elements of $\mathcal{F}'$. Thus $\mathcal{F}'$ is no more mad in $W^{\mathbb{P}*\mathcal{R}_\omega}$. ❑ (Theorem 22)

6 Mad families with some additional properties

We do not know if CH is really necessary in Theorem 13. We neither know if there can be a mad family $\mathcal{F}$ on T consisting of antichains and branches which is not

$\mathcal{C}_\omega$-destructible. However there can be a mad family $\mathcal{F}$ consisting of antichains and branches such that $\mathcal{F}^\perp$ in $V^{\mathcal{C}_\omega}$ contains a dense subset of T.

Theorem 23 (CH) *There is a mad family $\mathcal{F}$ on T consisting only of antichains and branches such that $V^{\mathcal{C}_\omega} \models$ there is a dense $D \subseteq T$ ad to $\mathcal{F}$.*

Proof. Note first that T is a dense subset of $\mathcal{C}_\omega$. Let

(6.1) $\quad \underset{\sim}{D} = \{\langle t^\frown p, p\rangle : p, t \in T,\, \ell(p) = \ell(t)\}.$

Then $\underset{\sim}{D}$ is a $\mathcal{C}_\omega$-name of a dense subset of T. For $p \in \mathcal{C}_\omega$ let

(6.2) $\quad J(p) = \{t \in T : \exists q \leq_{\mathcal{C}_\omega} p \ (q \Vdash_{\mathcal{C}_\omega} \text{“} t \in \underset{\sim}{D}) \text{”}\}.$

By the definition of $\underset{\sim}{D}$ and $J(p)$ for $p \in \mathcal{C}_\omega$, we have

(6.3) $\quad p \Vdash_{\mathcal{C}_\omega} \text{“} \underset{\sim}{D} \subseteq J(p) \text{”}$ for all $p \in \mathcal{C}_\omega$ and

(6.4) $\quad |\, J(p^\frown 0) \cap J(p^\frown 1)\,| < \aleph_0$ for all $p \in \mathcal{C}_\omega$.

We construct $A_\alpha \in \mathcal{A}_T$, $\alpha < \omega_1$ inductively as follows so that the desired mad family $\mathcal{F}$ will be obtained as $\{A_\alpha : \alpha < \omega_1\} \cup \{B(f) : f \in {}^\omega 2\}$.

Fix an enumeration B_α, $\alpha \in \omega_1$ of $[T]^\omega$ such that

(6.5) $\quad B_n \in \mathcal{A}_T$ for $n \in \omega$ and $\{B_n : n \in \omega\}$ is a partition of T.

The condition (6.5) is needed to guarantee that the construction does not stop during the first ω steps. Also fix an enumeration p_n, $n \in \omega$ of T $(\subseteq \mathcal{C}_\omega)$.

Assume that $\{A_\beta : \beta < \alpha\}$ has been constructed for some $\alpha < \omega_1$. Let $\mathcal{I}_\alpha$ be the ideal in $\mathcal{P}(T)$ generated by $\{A_\beta : \beta < \alpha\} \cup \mathcal{B}_T \cup [T]^{<\aleph_0}$.

(6.6) $\quad$ If $B_\alpha \in \mathcal{I}_\alpha$ then let $A_\alpha = A_0$.

Note that $B_0 \notin \mathcal{I}_0$ by (6.5) so that this does not happen at $\alpha = 0$.

Otherwise, let $C_n \in [T]^{\aleph_0}$, $t_n \in T$, $i_n \in 2$ be such that

(6.7) $\quad C_0 = B_\alpha$;

(6.8) $\quad C_n \notin \mathcal{I}_\alpha$;

(6.9) $\quad t_n \in C_n$;

(6.10) $\quad C_{n+1} = B_n \setminus (J(p_n{}^\frown i_n) \cup T \downarrow t_n)$.

Note that C_0 as in (6.7) satisfies (6.8) by the assumption on B_α. (6.10) in combination with (6.9) is possible since T is a disjoint union of

$$T \downarrow t \text{ for } t \in T \text{ with } \ell(t) = m \text{ and } \{t \in T : \ell(t) < m\}$$

for any $m \in \omega$, and by (6.4). Note that C_n, $n \in \omega$ build a decreasing sequence with respect to $\subseteq$ in $\mathcal{P}(T) \setminus \mathcal{I}_\alpha$. Let $A_\alpha = \{t_n : n \in \omega\}$. Then

(6.11) $A_\alpha \subseteq B_\alpha$

by (6.9).

We show that $\mathcal{F} = \{A_\alpha : \alpha < \omega_1\} \cup \{B(f) : f \in {}^\omega 2\}$ is as desired. Clearly $\mathcal{F}$ is ad. It is mad since otherwise there is $\alpha < \omega_1$ such that B_α is ad to $\mathcal{F}$. But then $B_\alpha \notin \mathcal{I}_\alpha$ and $A_\alpha \subseteq B_\alpha$ by (6.11). This is a contradiction. Thus it is enough to show that

(6.12) $\Vdash_{\mathcal{C}_\omega}$ "$\underset{\sim}{D}$ is ad to $\mathcal{F}$".

Suppose that this is not the case. Then there is some $n \in \omega$ and $\alpha < \omega_1$ such that

(6.13) $p_n \Vdash_{\mathcal{C}_\omega}$ "$\underset{\sim}{D} \cap A_\alpha$ is infinite".

We may assume that $B_\alpha \notin \mathcal{I}_\alpha$ since otherwise we reassign α to be 0 by (6.6). But then, by (6.10), (6.9) and (6.3), we have $p_n {}^\frown i_n \Vdash_{\mathcal{C}_\omega}$ "$\underset{\sim}{D} \cap A_\alpha$ is finite". This is a contradiction. ❏ (Theorem 23)

Given an ad family $\mathcal{F}$ on T let $\mathcal{I}(\mathcal{F})$ be the ideal on T generated by $\mathcal{F} \cup [T]^{<\omega}$, i.e. for $S \subset T$ we have $S \in \mathcal{I}(\mathcal{F})$ if $S \subset^* \cup\mathcal{F}'$ for some finite subfamily $\mathcal{F}'$ of $\mathcal{F}$.

Let $\mathcal{F}$ be mad family on T and $\mathcal{B} \subseteq \mathcal{F}$. Clearly $\mathcal{B}^\perp \supseteq \mathcal{I}(\mathcal{F} \setminus \mathcal{B})$. We say that $\mathcal{B}$ *almost decides* $\mathcal{F}$ if $\mathcal{B}^\perp = \mathcal{I}(\mathcal{F} \setminus \mathcal{B})$. A mad family $\mathcal{F}$ is said to be κ*-almost decided* if every $\mathcal{B} \in [\mathcal{F}]^\kappa$ almost decides $\mathcal{F}$.

Theorem 24 *Assume that* MA(σ*-centered*) *holds. Then there is a* $\mathfrak{c}$*-almost decided mad family* $\mathcal{F}$ *on* T.

Proof. Let $\langle B_\beta : \beta < \mathfrak{c}\rangle$ be an enumeration of $[T]^{\aleph_0}$. We define A_α, $\alpha < \mathfrak{c}$ inductively such that

(6.14) $\{A_n : n \in \omega\}$ is a partition of T into infinite subsets;

For all $\alpha \in \mathfrak{c} \setminus \omega$

(6.15) A_α is ad from A_β for all $\beta < \alpha$;

(6.16) For $\beta < \alpha$, if $B_\beta \notin \mathcal{I}(\{A_\delta : \delta < \alpha\})$ then $|\, A_\alpha \cap B_\beta \,| = \aleph_0$;

Claim 24.1 *The construction of* A_α, $\alpha < \mathfrak{c}$ *as above is possible.*

$\vdash$ Suppose that $\alpha \in \mathfrak{c} \setminus \omega$ and A_β, $\beta < \alpha$ have been constructed. Let

$$S_\alpha = \{\beta < \alpha \,:\, B_\beta \notin \mathcal{I}(\{A_\delta \,:\, \delta < \alpha\})\}.$$

Let $\mathbb{P}_\alpha = \{\langle s, f\rangle \,:\, s \in [\alpha]^{<\aleph_0},\ f \in \mathrm{Fn}(T, 2)\}$ be the poset with the ordering defined by

$$\begin{aligned}\langle s', f'\rangle \leq_{\mathbb{P}_\alpha} \langle s, f\rangle \ \Leftrightarrow \\ & s \subseteq s',\ f \subseteq f' \text{ and} \\ & \forall t \in \mathrm{dom}(f') \setminus \mathrm{dom}(f)\ (f'(t) = 1 \ \rightarrow\ t \notin A_\delta \text{ for all } \delta \in s)\end{aligned}$$

for $\langle s, f\rangle$, $\langle s', f'\rangle \in \mathbb{P}_\alpha$.

$\mathbb{P}_\alpha$ is σ-centered since $\langle s, f\rangle$, $\langle s', f'\rangle \in \mathbb{P}_\alpha$ are compatible if $f = f'$.

For $\beta < \alpha$, let

$$C_\beta = \{\langle s, f\rangle \in \mathbb{P}_\alpha \,:\, \beta \in s\}$$

and, for $\beta \in S_\alpha$ and $n \in \omega$, let

$$D_{\beta,n} = \{\langle s, f\rangle \in \mathbb{P}_\alpha \,:\, \exists t \in \mathrm{dom}(f)\ (\ell(t) \geq n \ \wedge\ f(t) = 1 \ \wedge\ t \in B_\beta)\}.$$

It is easy to see that C_β, $\beta < \alpha$ and $D_{\beta,n}$, $\beta \in S_\alpha$, $n \in \omega$ are dense in $\mathbb{P}_\alpha$. Further, if $|\, B_\alpha \cap A_\beta\,| < \aleph_0$ for all $\beta < \alpha$, let

$$E_n = \{\langle s, f\rangle \in \mathbb{P}_\alpha \,:\, \exists t \in \mathrm{dom}(f)\ (\ell(t) \geq n \ \wedge\ f(t) = 1 \ \wedge\ t \in B_\alpha)\}$$

for $n \in \omega$. Otherwise let $E_n = \mathbb{P}_\alpha$ for all $n \in \omega$. It is also clear that E_n, $n \in \omega$ are dense in $\mathbb{P}_\alpha$. Let

$$\mathcal{D} = \{C_\beta \,:\, \beta < \alpha\} \cup \{D_{\beta,n} \,:\, \beta \in S_\alpha,\ n \in \omega\} \cup \{E_n \,:\, n \in \omega\}.$$

Since $|\,\mathcal{D}\,| < \mathfrak{c}$, we can apply MA($\sigma$-centered) to obtain a $(\mathcal{D}, \mathbb{P}_\alpha)$-generic filter G. Let

$$A_\alpha = \{t \in T \,:\, f(f) = 1 \text{ for some } (s, f) \in G\}.$$

Then this A_α is as desired. $\dashv$

Let $\mathcal{F} = \{A_\alpha \,:\, \alpha < \mathfrak{c}\}$. $\mathcal{F}$ is infinite by (6.15) and mad by (6.16).

We show that $\mathcal{F}$ is $\mathfrak{c}$-almost decided. First, note that we have $\mathfrak{a} = \mathfrak{c}$ by the assumptions of the theorem. By (6.16), we have:

(6.17) For any $B \in [T]^{\aleph_0}$, if $B \notin \mathcal{I}(\{A_\alpha \,:\, \alpha < \mathfrak{c}\})$ then $|\,\{\alpha < \mathfrak{c} \,:\, |\,A_\alpha \cap B\,| < \aleph_0\}\,| < \mathfrak{c}$.

Suppose that $\mathcal{H} \in [\mathcal{F}]^{\mathfrak{c}}$ and $B \in \mathcal{H}^{\perp}$. Then $|\{\alpha < \mathfrak{c} : |A_\alpha \cap B| < \aleph_0\}| = \mathfrak{c}$ and so $B \in \mathcal{I}(\mathcal{F})$ by (6.17). Thus there is a finite $\mathcal{F}' \subset \mathcal{F}$ such that $B \subset^* \bigcup \mathcal{F}'$ and $F \cap B$ is infinite for each $F \in \mathcal{F}'$. But $B \in \mathcal{H}^{\perp}$ so $\mathcal{F}' \cap \mathcal{H} = \emptyset$. Thus $\mathcal{F}'$ witnesses that $B \in \mathcal{I}(\mathcal{F} \setminus \mathcal{H})$ which was to be proved. ❑ (Theorem 24)

For a mad family $\mathcal{F}$ on T, $\mathcal{C} \subseteq \mathcal{F}$ is said to be *minimal in* $\mathcal{F}$ if $\mathfrak{a}^+(\mathcal{F} \setminus \mathcal{C}) = |\mathcal{C}|$. A mad family $\mathcal{F}$ is said to be *κ-minimal* if every $\mathcal{C} \in [\mathcal{F}]^{\kappa}$ is minimal in $\mathcal{F}$.

Lemma 25 *Suppose that $\mathcal{F}$ is a mad family on T.*
(1) *If $\mathcal{F}$ is $|\mathcal{F}|$-minimal then $|\mathcal{F}| = \mathfrak{a}$.*
(2) *If $\mathcal{B} \subseteq \mathcal{F}$ almost decides $\mathcal{F}$ then $\mathcal{F} \setminus \mathcal{B}$ is minimal in $\mathcal{F}$.*
(3) *If $|\mathcal{F}| = \mathfrak{a}$ and $\mathcal{F}$ is κ-almost decided then $\mathcal{F}$ is κ-minimal.*
(4) *If $\mathcal{F}$ is κ-almost decided for $\kappa = |\mathcal{F}|$ then $\mathcal{F}$ is λ-minimal for all $\omega \leq \lambda < \kappa$.*

Proof. (1): If $\mathcal{F}$ is $|\mathcal{F}|$-minimal then $\mathcal{F}$ itself is minimal in $\mathcal{F}$. Thus $\mathfrak{a} = \mathfrak{a}^+(\emptyset) = \mathfrak{a}^+(\mathcal{F} \setminus \mathcal{F}) = |\mathcal{F}|$.

(2): First, note that, for any infinite ad $\mathcal{F}$, we have $\mathfrak{a}(\mathcal{I}(\mathcal{F})) = |\mathcal{F}|$.

Suppose that $\mathcal{F}$ is a mad family on T and $\mathcal{B} \subseteq \mathcal{F}$ almost decides $\mathcal{F}$. Then we have $\mathcal{B}^{\perp} = \mathcal{I}(\mathcal{F} \setminus \mathcal{B})$. Hence

$$\mathfrak{a}^+(\mathcal{F} \setminus (\mathcal{F} \setminus \mathcal{B})) = \mathfrak{a}^+(\mathcal{B}) = \mathfrak{a}(\mathcal{B}^{\perp}) = \mathfrak{a}(\mathcal{I}(\mathcal{F} \setminus \mathcal{B})) = |\mathcal{F} \setminus \mathcal{B}|.$$

(3): Suppose that $|\mathcal{F}| = \mathfrak{a}$ and $\mathcal{F}$ is κ-almost decided. Suppose that $\mathcal{C} \in [\mathcal{F}]^{\mathfrak{a}}$. If $|\mathcal{F} \setminus \mathcal{C}| < \mathfrak{a}$, then clearly $\mathfrak{a}^+(\mathcal{F} \setminus \mathcal{C}) = \mathfrak{a} = |\mathcal{C}|$. Hence $\mathcal{C}$ is minimal in $\mathcal{F}$. If $|\mathcal{F} \setminus \mathcal{C}| = \mathfrak{a}$ then $\mathcal{F} \setminus \mathcal{C}$ almost decides $\mathcal{F}$. Thus, by (2), $\mathcal{C} = \mathcal{F} \setminus (\mathcal{F} \setminus \mathcal{C})$ is again minimal in $\mathcal{F}$.

(4): Suppose that $\kappa = |\mathcal{F}|$ and $\mathcal{F}$ is κ-almost decided. If $\mathcal{C} \in [\mathcal{F}]^{\lambda}$ for some $\omega \leq \lambda < \kappa$ then $|\mathcal{F} \setminus \mathcal{C}| = \kappa$ and hence $\mathcal{F} \setminus \mathcal{C}$ almost decides $\mathcal{F}$. By (2) it follows that $\mathcal{C} = \mathcal{F} \setminus (\mathcal{F} \setminus \mathcal{C})$ is minimal in $\mathcal{F}$. ❑ (Lemma 25)

Corollary 26 *Assume that* MA(*σ-centered*) *holds. Then there is a mad family $\mathcal{F}$ on T which is λ-minimal for all $\omega \leq \lambda \leq \mathfrak{c}$.*

Proof. By Theorem 24 and Lemma 25, (3), (4). ❑ (Corollary 26)

Theorem 24 can be further improved to the following theorem:

Theorem 27 *Assume that* MA(*σ-centered*) *holds. Let $\kappa = \mathfrak{c}$. Then there is a $\mathcal{C}_\omega$-indestructible mad family $\mathcal{F}$ (of size κ) such that*

$$V^{\mathcal{C}_\omega} \models \mathcal{F} \textit{ is } \kappa\textit{-almost decided on } T.$$

Proof. Let $\langle\langle t_\beta, \underset{\sim}{B}_\beta\rangle : \beta < \kappa\rangle$ be an enumeration of

$$T \times \{\underset{\sim}{B} : \underset{\sim}{B} \text{ is a nice } \mathcal{C}_\omega\text{-name of an element of } [T]^{\aleph_0} \text{ in } V^{\mathcal{C}_\omega}\}.$$

Let A_α, $\alpha < \kappa$ be then defined inductively just as in the proof of Theorem 24 with

(6.16)′ For $\beta < \alpha$, if $p_\beta \Vdash_{\mathcal{C}_\omega}$ " $\underset{\sim}{B}_\alpha \notin \mathcal{I}(\{A_\delta : \delta < \alpha\})$ " then $p_\beta \Vdash_{\mathcal{C}_\omega}$ " $|\, A_\alpha \cap \underset{\sim}{B}_\beta\,| = \aleph_0$ "

in place of (6.16). ❑ (Theorem 27)

Corollary 28 *It is consistent that for arbitrary large uncountable $\kappa < \mathfrak{c}$ there is a κ-almost decided mad family $\mathcal{F}$ of size κ (in particular $\mathcal{F}$ is λ-minimal for all $\omega \le \lambda \le \kappa$).*

Proof. Start from a model V of $\kappa = \mathfrak{c}$ and MA(σ-centered) and let $\mathcal{F}$ be as in Theorem 27. Then $\mathcal{F}$ is as desired in $V^{\mathcal{C}_\mu}$ for any $\mu > \kappa$. The claim in the parentheses follows from Lemma 25, (3) and (4). ❑ (Corollary 28)

Theorem 27 also proves the following assertion which should have been nown under some other proof:

Corollary 29 *Suppose that* MA(σ*-centered*) *holds and $\kappa = \mathfrak{c}$. Then $V^{\mathcal{C}_\mu} \models \mathfrak{a} = \kappa$ for any μ.*

Proof. By Corollary 28 and Lemma 25, (1). ❑ (Corollary 29)

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