RESOLVABILITY VS. ALMOST RESOLVABILITY

ISTVÁN JUHÁSZ, SAHARON SHELAH, AND LAJOS SOUKUP

Abstract. A space X is *κ-resolvable* (resp. *almost κ-resolvable*) if it contains κ dense sets that are pairwise disjoint (resp. almost disjoint over the ideal of nowhere dense subsets of X).

Answering a problem raised by Juhász, Soukup, and Szentmiklóssy, and improving a consistency result of Comfort and Hu, we prove, in ZFC, that for every infinite cardinal κ there is an almost 2^κ-resolvable but not ω_1-resolvable space of dispersion character κ.

A space X is said to be *κ-resolvable* if it contains κ dense sets that are pairwise disjoint. X is called *maximally resolvable* iff it is $\Delta(X)$-resolvable, where $\Delta(X) = \min\{|G| : G \neq \emptyset \text{ open}\}$ is the *dispersion character* of X.

V. Malychin, in [4], was the first to suggest studying families of dense sets of a space X that, rather than disjoint, are merely *almost disjoint* with respect to the ideal $\mathcal{N}(X)$, where $\mathcal{N}(X)$ denotes the family of all nowhere dense subsets of the space X. He called a space X *extraresolvable* if it has $\Delta(X)^+$ many dense sets such that any two of them have nowhere dense intersection. This idea was generalized in [3], where the natural notion of *almost κ-resolvability* was introduced: A space X is called *almost κ-resolvable* if it contains κ dense sets that are pairwise almost disjoint over the ideal $\mathcal{N}(X)$ of nowhere dense subsets of X. (Actually, this concept was given a different name in [3], namely: "κ-extraresolvable ", but we think the terminology given here is much better.)

Note that this makes good sense for $\kappa \le \Delta(X)$ as well. But while "almost ω-resolvable" is clearly equivalent to "ω-resolvable", the analogous question for higher cardinals remained open. In particular, the following natural problem was formulated in [3]:

2000 *Mathematics Subject Classification.* 54A25, 03E05.

Key words and phrases. κ-resolvable space, almost κ-resolvable space, extraresolvable space.

The first and third authors were supported by OTKA grant no. 61600.

The second author was supported by the United States – Israel Binational Science Foundation (Grant no. 2002323). Publication 901.

Problem 1. *Let X be an extraresolvable (T_2, T_3, or Tychonov) space with $\Delta(X) \geq \omega_1$. Is X then ω_1- resolvable?*

(The assumption $\Delta(X) \geq \omega_1$ is clearly necessary to make this problem non-trivial.)

Comfort and Hu, see [2, Corollary 3.6], gave a negative answer to this problem, assuming the failure of the continuum hypothesis, CH. More precisely they got the following result:

Theorem . *If κ is an infinite cardinal such that GCH first fails at κ then there is a 0-dimensional T_2 space X with $|X| = \Delta(X) = \kappa^+$ such that X is κ-resolvable, extraresolvable but not κ^+-resolvable, hence not maximally resolvable and if $\kappa = \omega$ then not ω_1- resolvable.*

Our aim in this note is to give the following "final" answer to the above problem, in ZFC.

Theorem 2. *For every cardinal κ there is a 0-dimensional T_2 space of dispersion character κ that is extraresolvable but not ω_1-resolvable.*

We shall actually prove a bit more. Note that no space X can be almost $(2^{\Delta(X)})^+$-resolvable, moreover "almost $2^{\Delta(X)}$-resolvable" can be strictly stronger than "extraresolvable $\equiv$ almost $\Delta(X)^+$-resolvable".

Theorem 3. *For every cardinal κ there is an almost 2^κ-resolvable (and so extraresolvable) but not ω_1-resolvable 0-dimensional T_2 space of cardinality and dispersion character κ. In fact, our example is a κ-dense subspace of the Cantor cube of weight 2^κ.*

To prove this theorem we shall make use of the method of constructing $\mathcal{D}$-forced spaces that was introduced in [3]. Therefore, we first recall some definitions and results from [3].

Let $\mathcal{D}$ be a family of dense subsets of a space X. A subset $M \subset X$ is called a *$\mathcal{D}$-mosaic* iff there is a maximal disjoint family $\mathcal{V}$ of open subsets of X and for each $V \in \mathcal{V}$ there is $D_V \in \mathcal{D}$ such that

$$M = \cup\{V \cap D_V : V \in \mathcal{V}\}.$$

Clearly, every $\mathcal{D}$-mosaic is dense. We say that the space X (or its topology) is *$\mathcal{D}$-forced* iff every dense subset of X includes a $\mathcal{D}$-mosaic.

Let S be any set and $\mathbb{B} = \left\{\left\langle B^0_\zeta, B^1_\zeta\right\rangle : \zeta < \mu\right\}$ be a family of 2-partitions of S. We denote by $\tau_{\mathbb{B}}$ the (obviously zero-dimensional) topology on S generated by the subbase $\{B^i_\zeta : \zeta < \mu,\ i < 2\}$, moreover we set $X_{\mathbb{B}} = \langle S, \tau_{\mathbb{B}}\rangle$.

Given a cardinal κ, we have $\Delta(X_{\mathbb{B}}) \geq \kappa$ iff $\mathbb{B}$ is *κ-independent*, i.e.,

$$\mathbb{B}[\varepsilon] \stackrel{def}{=} \bigcap\{B^{\varepsilon(\zeta)}_\zeta : \zeta \in \operatorname{dom}\varepsilon\}$$

has cardinality at least κ whenever $\varepsilon \in Fn(\mu, 2)$.

Note that $X_{\mathbb{B}}$ is Hausdorff iff $\mathbb{B}$ is *separating*, i.e. for each pair $\{\alpha, \beta\} \in \left[S\right]^2$ there are $\zeta < \mu$ and $i < 2$ such that $\alpha \in B^i_\zeta$ and $\beta \in B^{1-i}_\zeta$.

A set $D \subset X$ is said to be κ*-dense* in the space X iff $|D \cap U| \geq \kappa$ for each nonempty open set $U \subset X$. Thus D is dense iff it is 1-dense. Also, it is obvious that the existence of a κ-dense set in X implies $\Delta(X) \geq \kappa$.

Theorem ([3, Main Theorem 3.3])**.** *Assume that κ is an infinite cardinal and we are given $\mathbb{B} = \left\{\left\langle B^0_\xi, B^1_\xi\right\rangle : \xi < 2^\kappa\right\}$, a κ-independent family of 2-partitions of κ, moreover a non-empty family $\mathcal{D}$ of κ-dense subsets of the space $X_{\mathbb{B}}$. Then there is a separating κ-independent family $\mathbb{C} = \{\langle C^0_\xi, C^1_\xi\rangle : \xi < 2^\kappa\}$ of 2-partitions of κ such that*

(1) every $D \in \mathcal{D}$ is also κ-dense in $X_{\mathbb{C}}$ (and so $\Delta(X_{\mathbb{C}}) = \kappa$),
(2) $X_{\mathbb{C}}$ is $\mathcal{D}$-forced.

Actually, the space $X_{\mathbb{C}}$ has other interesting properties as well but we shall note make use of those here. We are now ready to prove our promised result.

Proof of Theorem 3. Let κ be an arbitrary infinite cardinal. It is well-known, see e. g. [3, Fact 3.2], that we can find two disjoint families $\mathbb{B} = \left\{\langle B^0_i, B^1_i\rangle : i < 2^\kappa\right\}$ and $\mathbb{D} = \left\{\langle D^1_i, D^1_i\rangle : i < 2^\kappa\right\}$ of 2-partitions of κ such that their union $\mathbb{B} \cup \mathbb{D}$ is κ-independent, that is, for any $\eta, \varepsilon \in \mathrm{Fn}(2^\kappa, 2)$ we have

$$\big|\ \mathbb{D}[\eta] \cap \mathbb{B}[\varepsilon]\ \big| = \kappa.$$

In other words, this means that

$$\mathcal{D} = \{\mathbb{D}[\eta] : \eta \in \mathrm{Fn}(2^\kappa, 2)\}$$

is a family of κ-dense subsets of $X_{\mathbb{B}}$, hence we may apply the above cited theorem from [3] to this $\mathbb{B}$ and $\mathcal{D}$, to obtain a family $\mathbb{C}$ of 2^κ many 2-partitions of κ that satisfies conditions (1) and (2) above.

The space that we need will be a further refinement of $X_{\mathbb{C}}$. To obtain that, we next fix a 2-partition $\langle I, J\rangle$ of the index set 2^κ such that $|I| = |J| = 2^\kappa$. For every unordered pair $a \in \left[I\right]^2$ we shall write $a^+ = \max a$ and $a^- = \min a$, so that $a = \{a^-, a^+\}$.

Let $\{j(a, m) : a \in \left[I\right]^2, m < \omega\}$ be pairwise distinct elements of J. For any $a \in \left[I\right]^2$ and $m < \omega$ we then define the sets

$$E^0_{a,m} = D^0_{j(a,m)} \setminus (D^0_{a^-} \cap D^0_{a^+}) \text{ and } E^1_{a,m} = \kappa \setminus E^0_{a,m}.$$

Clearly, then we have

$$E^1_{a,m} = D^1_{j(a,m)} \cup (D^0_{a^-} \cap D^0_{a^+}).$$

In this way we obtained a new family

$$\mathbb{E} = \left\{ \left\langle E^0_{a,m}, E^1_{a,m} \right\rangle : a \in \left[I\right]^2, m < \omega \right\}$$

of 2-partitions of κ. We shall show that the space $X_{\mathbb{C}\cup\mathbb{E}}$ satisfies all the requirements of theorem 3.

Claim 3.1. *For any finite function $\eta \in \operatorname{Fn}(\left[I\right]^2 \times \omega, 2)$ and any ordinal $\alpha \in I$ there is a finite function $\varphi \in \operatorname{Fn}(2^\kappa, 2)$ such that $\alpha \notin \operatorname{dom} \varphi$ and $\mathbb{E}[\eta] \supset \mathbb{D}[\varphi]$.*

Proof of the Claim. For each $a \in \left[I\right]^2$ let us pick $a^* \in a$ with $a^* \neq \alpha$. Then we have

$$\begin{aligned}
\mathbb{E}[\eta] = & \bigcap_{\eta(a,m)=0} E^0_{a,m} \cap \bigcap_{\eta(a,m)=1} E^1_{a,m} \supset \\
& \supset \bigcap_{\eta(a,m)=0} (D^0_{j(a,m)} \setminus (D^0_{a^-} \cap D^0_{a^+}) \cap \bigcap_{\eta(a,m)=1} D^1_{j(a,m)} \supset \\
& \qquad \supset \bigcap_{\eta(a,m)=0} (D^0_{j(a,m)} \cap D^1_{a^*}) \cap \bigcap_{\eta(a,m)=1} D^1_{j(a,m)} = \\
& \qquad\qquad = \bigcap_{\eta(a,m)=0} D^1_{a^*} \cap \bigcap_{\langle a,m\rangle \in \operatorname{dom} \eta} D^{\eta(a,m)}_{j(a,m)}.
\end{aligned}$$

The expression in the last line above is, however, equal to $\mathbb{D}[\varphi]$ for a suitable $\varphi \in \operatorname{Fn}(2^\kappa, 2)$ because j is an injective map of $[I] \times \omega$ into J and $a^* \neq \alpha$ belongs to $I = \kappa \setminus J$ for all $a \in [I]^2$. $\square$

Claim 3.2. $\mathbb{C} \cup \mathbb{E}$ *is κ-independent, hence $\Delta(X_{\mathbb{C}\cup\mathbb{E}}) = \kappa$.*

Proof of the Claim. Let $\varepsilon \in \operatorname{Fn}(2^\kappa, 2)$ and $\eta \in \operatorname{Fn}(\left[I\right]^2 \times \omega, 2)$ be picked arbitrarily. By Claim 3.1 there is $\varphi \in \operatorname{Fn}(2^\kappa, 2)$ such that $\mathbb{E}[\eta] \supset \mathbb{D}[\varphi]$. Since $\mathbb{D}[\varphi] \in \mathcal{D}$ we have $|\mathbb{C}[\varepsilon] \cap \mathbb{D}[\varphi]| = \kappa$ because $\mathbb{C}$ satisfies condition (1). Consequently, we have $|\mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta]| = \kappa$ as well. $\square$

Claim 3.3. *The family $\{D^0_\alpha : \alpha \in I\}$ witnesses that $X_{\mathbb{C}\cup\mathbb{E}}$ is almost 2^κ-resolvable.*

Proof of the Claim. First we show that D^0_α is dense in $X_{\mathbb{C}\cup\mathbb{E}}$ whenever $\alpha \in I$. So fix $\alpha \in I$, moreover let $\varepsilon \in \operatorname{Fn}(2^\kappa, 2)$ and $\eta \in \operatorname{Fn}(\left[I\right]^2 \times \omega, 2)$. By Claim 3.1 there is $\varphi \in \operatorname{Fn}(2^\kappa, 2)$ such that $\alpha \notin \operatorname{dom} \varphi$ and $\mathbb{E}[\eta] \supset \mathbb{D}[\varphi]$. Since $\alpha \notin \operatorname{dom} \varphi$ we have $D^0_\alpha \cap \mathbb{D}[\varphi] \in \mathcal{D}$. Hence, as $\mathbb{C}$ has property (1),

$$\emptyset \neq (D^0_\alpha \cap \mathbb{D}[\varphi]) \cap \mathbb{C}[\varepsilon] \subset D^0_\alpha \cap (\mathbb{E}[\eta] \cap \mathbb{C}[\varepsilon])$$

as well. So D^0_α intersects every basic open subset of $X_{\mathbb{C}\cup\mathbb{E}}$, i. e. D^0_α is dense in $X_{\mathbb{C}\cup\mathbb{E}}$.

Next we show that $D_\alpha \cap D_\beta$ is nowhere dense in the space $X_{\mathbb{C}\cup\mathbb{E}}$ whenever $a = \{\alpha, \beta\} \in \left[I\right]^2$. Indeed, let $\mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta]$ be again a basic open set with $\varepsilon \in \mathrm{Fn}(2^\kappa, 2)$ and $\eta \in \mathrm{Fn}(\left[I\right]^2 \times \omega, 2)$ and let us pick $m < \omega$ such that $\langle a, m\rangle \notin \operatorname{dom}\eta$. Then

$$\eta' = \eta \cup \{\langle\langle a, m\rangle, 0\rangle\} \in \mathrm{Fn}(\left[I\right]^2 \times \omega, 2),$$

hence $\mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta'] \subset \mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta]$ is a (non-empty) basic open set in the space $X_{\mathbb{C}\cup\mathbb{E}}$. Moreover, $E^0_{a,m} = D^0_{j(a,m)} \setminus (D^0_\alpha \cap D^0_\beta)$ implies

$$(D_\alpha \cap D_\beta) \cap \mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta'] \subset (D_\alpha \cap D_\beta) \cap (D^0_{j(a,m)} \setminus (D_\alpha \cap D_\beta)) = \emptyset,$$

consequently, $D_\alpha \cap D_\beta$ is not dense in $\mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta]$. □

Finally, the following simple claim will complete the proof of our theorem.

Claim 3.4. *The space $X_\mathbb{C}$ is ω_1-irresolvable, that is, not ω_1-resolvable.*

Proof of the Claim. Assume that $\{F_\zeta : \zeta < \omega_1\}$ is a family of dense subsets of $X_\mathbb{C}$. By condition (2) the topology of $X_\mathbb{C}$ is $\mathcal{D}$-forced, so every F_ζ includes a $\mathcal{D}$-mosaic in $X_\mathbb{C}$, consequently for all $\zeta < \omega_1$ there are $\varepsilon_\zeta \in \mathrm{Fn}(2^\kappa, 2)$ and $\phi_\zeta \in \mathrm{Fn}(2^\kappa, 2)$ such that $\mathbb{D}[\phi_\zeta] \cap \mathbb{C}[\varepsilon_\zeta] \subset F_\zeta$. By the well-known Δ-system lemma we may then find $\zeta < \xi < \omega_1$ such that $\varepsilon = \varepsilon_\zeta \cup \varepsilon_\xi \in \mathrm{Fn}(2^\kappa, 2)$ and $\phi = \phi_\zeta \cup \phi_\xi \in \mathrm{Fn}(2^\kappa, 2)$. (Actually, much more is true: there is an uncountable set $S \in [\omega_1]^{\omega_1}$ such that the members of both $\{\varepsilon_\zeta : \zeta \in S\}$ and $\{\phi_\zeta : \zeta \in S\}$ are pairwise compatible.) But then we have

$$F_\zeta \cap F_\xi \supset \mathbb{D}[\phi_\zeta] \cap \mathbb{C}[\varepsilon_\zeta] \cap \mathbb{D}[\phi_\xi] \cap \mathbb{C}[\varepsilon_\xi] = \mathbb{D}[\phi] \cap \mathbb{C}[\phi] \neq \emptyset.$$

□

To conclude our proof, it suffices to recall the obvious fact that if a topology on a set is λ-resolvable then so is any coarser topology. Hence the ω_1-irresolvability of $X_\mathbb{C}$ implies that of $X_{\mathbb{C}\cup\mathbb{E}}$. □

Let us point out that as extraresolvability implies almost ω-resolvability that is equivalent to ω-resolvability, any counterexample to problem 1 is automatically an example of an ω-resolvable but not maximally resolvable space, hence it is a solution to the celebrated problem of Ceder and Pearson from [1]. The first Tychonov ZFC examples of such spaces were given in [3] and the spaces constructed in theorem 3 extend the supply of such examples.

References

[1] Ceder, J. I. , Pearson, T. *On products of maximally resolvable spaces.* Pacific J. Math. 22 (1967), 31–45.
[2] Comfort, W.W., Hu, W., *Resolvability properties via independent families*, Top. Appl. 154 (2007), 205-214.
[3] Juhász, István; Soukup, Lajos; Szentmiklóssy, Zoltán *$\mathcal{D}$-forced spaces: a new approach to resolvability.* Topology Appl. 153 (2006), no. 11, 1800–1824.
[4] Malykhin, V. I. *Irresolvability is not descriptively good.* Manuscript.

Alfréd Rényi Institute of Mathematics
E-mail address: juhasz@renyi.hu

Hebrew University
E-mail address: shelah@math.huji.ac.il

Alfréd Rényi Institute of Mathematics
E-mail address: soukup@renyi.hu