First countable countably compactifications of first countable spaces

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M-spaces

Definition

A space X is called an **M-space** if there is a countable collection of **open covers** $\{\mathcal{U}_n : n \in \omega\}$ of X, such that:

(i) $\mathcal{U}_{n+1}$ **star-refines** $\mathcal{U}_n$, for all n.

(ii) If $x_n \in St(x, \mathcal{U}_n)$, for all n, then the set $\{x_n : n \in \omega\}$ has an accumulation point.

(the accumulation point in (ii) need not be the fixed x, or we would get a developable space)

M-spaces

Conjecture [Nagata]

Every ***M*-space** is homeomorphic to a **closed subspace** of the **product of a countably compact space and a metric space**.

Morita's reformulation

Definition (Morita)

A space X is **countably-compactifiable** if it has a **countably-compactification**, i.e. there exists a **countably compact** space Y such that

(1) X is a dense subspace of Y,

(2) every **countably compact closed** subset of X is **closed** in Y.

Morita's reformulation

Y is a **countably-compactification** of a dense subspace X iff every **countably compact closed** subset of X is **closed** in Y.

- A countably compact space Y is not necessarily a countable compactification of a dense subspace X: $Y = \omega_1 + 1$ and $X = \omega_1$.

Morita's reformulation

Y is a **countably-compactification** of a dense subspace X iff every **countably compact closed** subset of X is **closed** in Y.

Proposition

A countably compact **first countable** space Y is a countable compactification of a dense subspace X.

Fact

A countably compact subspace A of a first countable space Y is closed.

Morita's reformulation

Theorem (Morita)

*An M-**space** satisfies **Nagata's conjecture** if and only if it is **countably-compactifiable**.*

Reformulated Conjecture

Every M-**space** is countably-compactifi able.

Counterexamples

Theorem (Burke and van Douwen, and A. Kato, independently)

There is a normal, 0-dimensional, locally compact, separable , ***first countable M-space*** *X which does not have a countably-compactification.*

Extensions

Definition

A first-countable space Y is said to be a **maximal first-countable extension** of a space X provided X is a dense subspace of Y and Y is closed in any first countable space $Z \supset Y$.

Theorem (T. Terada and J. Teresawa)

There are first-countable spaces without maximal first-countable extensions.

Extensions

A first-countable space Y is a **maximal first-countable extension** of a space $X \subset Y$ iff $\overline{X}^Z = Y$ for each first countable space $Z \supset Y$.

Proposition

A **countably compact first countable** space Y is a **maximal first countable extension** of any dense subspace X.

Fact

A countably compact subspace Y of a first countable space Z is closed.

An easy example

Proposition

A Ψ-space X does not have a first-countable countably compact extension.

Spaces of cardinality $< \mathfrak{c}$

Theorem

*If $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$ then every **first countable regular space** of cardinality $< \mathfrak{c}$ can be **embedded**, as a dense subspace, into a **first countable countably compact regular space**.*

Corollary

If **Martin's Axiom** holds then Nagata's conjecture holds for every **first countable regular space** X of cardinality $< \mathfrak{c}$, i.e. if X is an M**-space**, then X is homeomorphic to a **closed subspace of the product of a countably compact space and a metric space**. Moreover, X has **maximal first-countable extension**.

M-spaces in generic extensions

Theorem

There is a ***c.c.c. poset*** *of size* 2^ω *such that every* ***first countable regular space*** X ***from the ground model*** *can be* ***embedded****, as a dense subspace, into a* ***first countable countably compact regular space*** Y ***from the generic extension****, and so* X *has a* ***countably-compactification*** *in the generic extension.*

Corollary

There is a c.c.c. poset P of size 2^ω such that for every fi rst countable regular space X from the ground model V **Nagata's conjecture holds for** X **in** V^P, i.e. the following holds in V^P: if X is an M**-space**, then X is homeomorphic to a closed subspace of the **product of a countably compact space and a metric space**.
Moreover, X has **maximal first-countable extension** in V^P .

Extension of 0-dimensional spaces

Lemma

There is a c.c.c poset Q with the following property. If X is a first countable 0-dimensional space ***from the ground model*** *V, and for each $x \in X$ the family $\{U(x,n) : n < \omega\}$ is a clopen neighbourhood base of x*
then ***in the generic extension*** *V^Q there is a first countable 0-dimensional space $Y \supset X$ and for each $y \in Y$ there is a clopen neighbourhood base $\{U'(y,n) : n < \omega\}$ of y*
such that

(i) ***every*** *$A \in [X]^\omega \cap V$* ***has an accumulation point in*** *Y****.***
(ii) *$U'(x,n) \cap X = U(x,n)$ for $x \in X$ and $n < \omega$,*
(iii) *if $U(x,n) \cap U(y,m) = \emptyset$ then $U'(x,n) \cap U'(y,m) = \emptyset$,*
(iv) *if $U(x,n) \subset U(y,m)$ then $U'(x,n) \subset U'(y,m)$,*

Proof of the extension lemma

$V \models \mathcal{U} = \{U(x,n) : n < \omega, x \in X\}$ is a clopen nbhd. base.
$V^Q \models$ Pick $\mathcal{C} \subset [X]^\omega$ and $\forall C \in \mathcal{C}$ add x_C to X s.t. $C \to x_C$ in Y.
Step 1: $V^{Cohen} \models \exists \mathcal{A} \subset [X]^\omega$ A.D. refining $[X]^\omega \cap V$.
$\mathcal{B} = \{A \in \mathcal{A} : A$ is closed discrete in $X\}$.
Step 2: Let $\mathcal{V}$ be a non-trivial ultrafilter on ω, $R_\mathcal{V}$ introduces a pseudo-intersection of $\mathcal{V}$.
$V^{Cohen * R_\mathcal{V}} \models \forall B \in \mathcal{B}\ \exists C_B \in [B]^\omega\ \forall U \in \mathcal{U}\ C_B \cap U$ or $C_B \setminus U$ is finite.
Let $\mathcal{C} = \{C_B : B \in \mathcal{B}\}$
Step 3: $Y = X \cup \{x_C : C \in \mathcal{C}\}$.
$U'(x,n) = U(x,n) \cup \{x_C : C \subset^* U(x,n)\}$.
$V^{Cohen * R_\mathcal{V} * Hechler} \models U'(x_C, n) = \{x_C\} \cup \bigcup\{U'(C(m), d(m)) : m \geq n\}$.
$Q = Cohen * R_\mathcal{V} * Hechler$.
$Q = Cohen * \boxed{R_\mathcal{V}} * Hechler$.

A theorem on almost disjoint refinements

Theorem (Brendle; Balcar and Pazak)

If W is an extension of the ground model V which contains a new real then there is an almost disjoint family $\mathcal{A}$ in W which refines $[\omega]^{\omega} \cap V$.