Cardinal sequences of scattered spaces

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A space X is **scattered** ("hereditarily uncrowded") iff every non-empty subspace $\emptyset \neq Y \subset X$ has an **isolated point**, i.e.

$$I(Y) = \{p \in Y : p \text{ is isolated in } Y\} \neq \emptyset .$$

$\mathcal{P}$: a class of topological spaces

Characterize the elements of $\mathcal{P}$

Cantor-Bendixson Theorem

Every topological space is the disjoint union of a **crowded closed subspace P**, and a **scattered open subspace X**.

Characterize the scattered elements of $\mathcal{P}$

Assign invariants to the scattered elements of $\mathcal{P}$.

$I(Y) = \{p \in Y : p \text{ is isolated in } Y\text{-ben}\}$
X is **scattered** iff $I(Y) \neq \emptyset$ for each nonempty $Y \subset X$.

- $I_0(X)$ is the isolated points of X
- $I_1(X)$ is the isolated points of $X \setminus I_0(X)$

The β^{th} **Cantor-Bendixson level** of X is

$$I_\beta(X) = I(X \setminus \cup\{I_\alpha(X) : \alpha < \beta\})$$

T.F.A.E:

- X is **scattered**
- $X = \cup\{I_\alpha(X) : \alpha \in On\}$.
- X is **right-separated** (there is a **well-ordering** $\preceq$ of X such that $\{y \in X : y \preceq x\}$ is open in X for each $x \in X$.)

The ***height*** of X:

$$\mathrm{ht}(X) = \min\{\beta : I_\beta(X) = \emptyset\}.$$

The **width** of X:

$$wd(X) = \sup\{|I_\alpha(X)| : \alpha < \mathrm{ht}(X)\}$$

The ***reduced height***:

$$\mathrm{ht}^-(X) = \min\{\alpha : I_\alpha(X) \text{ is finite}\}.$$

Clearly, one has

$$\mathrm{ht}^-(X) \le \mathrm{ht}(X) \le \mathrm{ht}^-(X) + 1.$$

The ***cardinal sequence*** of X:

$$SEQ(X) = \langle |I_\alpha(X)| : \alpha < \mathrm{ht}^-(X)\rangle.$$

Main question

What are the cardinal sequences of (locally) compact scattered spaces (or: superatomic boolean algebras)?

$\mathcal{C}(\alpha) = \{SEQ(X) : X$ **locally compact scattered,** $\mathrm{ht}^-(X) = \alpha\}$.

Characterize $\mathcal{C}(\alpha)$!

Decide whether $s = SEQ(X)$ for some LCS space X

space $\equiv$ locally compact scattered space

Absoluteness

- The statement X **is scattered** is absolute.
- The Cantor-Bendixson Hierarchy of a space is absolute.
- The statement $s = SEQ(X)$ is absolute.
- The statement X **is locally compact and scattered** is absolute.

Classical results

$\langle\kappa\rangle_\delta$: constant κ sequence of length δ

Telgarsky, 1968: $\langle\omega\rangle_{\omega_1} \in \mathcal{C}(\omega_1)$**?**

- Rajagopalan, 76: Yes, in ZFC.
- Juhász-Weiss, 78: $\langle\omega\rangle_\alpha \in \mathcal{C}(\alpha)$ for each $\alpha < \omega_2$ in ZFC.
- CH $\Longrightarrow$ $\langle\omega\rangle_{\omega_1} {}^\frown \langle\omega_2\rangle \notin \mathcal{C}(\omega_1 + 1)$ and $\langle\omega\rangle_{\omega_2} \notin \mathcal{C}(\omega_2)$
- Just: $\langle\omega\rangle_{\omega_1} {}^\frown \langle\omega_2\rangle \notin \mathcal{C}(\omega_1 + 1)$ and $\langle\omega\rangle_{\omega_2} \notin \mathcal{C}(\omega_2)$ in the Cohen model
- Roitman: Con(ZFC + $\langle\omega\rangle_{\omega_1} {}^\frown \langle\omega_2\rangle \in \mathcal{C}(\omega_1 + 1)$)
- Baumgartner-Shelah: Con(ZFC + $\langle\omega\rangle_{\omega_2} \in \mathcal{C}(\omega_2)$)
- Martinez: Con(ZFC + $\forall\delta < \omega_3\ \langle\omega\rangle_\delta \in \mathcal{C}(\delta)$.)

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What about $\langle\omega\rangle_{\omega_3} \in \mathcal{C}(\omega_3)$ and $\langle\omega\rangle_{\omega_1} {}^\frown \langle\omega_3\rangle \in \mathcal{C}(\omega_1+1)$?

Cones and the meet function

X is an **LCS space**

$\{I_\alpha(X) : \alpha < \mathrm{ht}(X)\}$

for each $x \in I_\alpha(X)$ fix compact open $U(x) \ni x$ such that
$U(x) \setminus \{x\} \subset I_{<\alpha}(X)$

Investigate the properties of the cone systems $\{U(x) : x \in X\}$!

Simplification:
we assume that the cone system $\{U(x) : x \in X\}$ is **coherent**, i.e.
$x \in U(y)$ implies $U(x) \subset U(y)$

coherent cone system $\equiv$ poset:

$x \preceq y$ iff $x \in U(y)$

$U(x) = \{x' : x' \preceq x\}$.

Notation: for $A \subset X$ write $U[A] = \cup\{U(x) : x \in A\}$

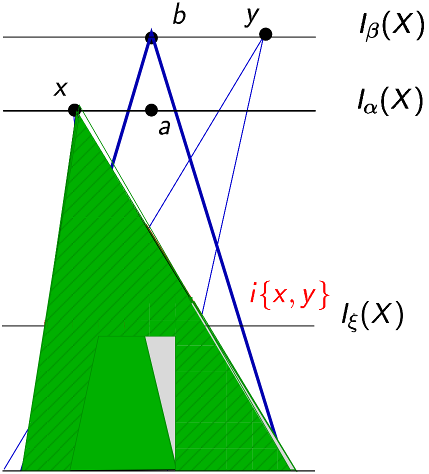
Cones and the meet function

(0) If $a \in I_\alpha(X)$, $b \in I_\beta(X)$, $a \prec b$ then $\alpha < \beta$.

(1): If $x \in I_\alpha(X)$, $y \in I_\beta(X)$, then
$\exists$ **finite** $i\{x,y\}$
$$U(x) \cap U(y) = U[i\{x,y\}].$$

$\forall t \in X$
$t \preceq x \wedge t \preceq y$ iff $(\exists s \in i\{x,y\})\ t \preceq s$

(2): if $F \subset I_{<\alpha}(X)$ finite and $\xi < \alpha$ then
$(U(x) \setminus U[F]) \cap I_\xi(X)$ **is infinite.**



$\{U(x) \setminus U[F] : F \in [U(x) \setminus \{x\}]^{<\omega}\}$ **is base of** x.

Cones and the meet function

Assume $\langle X, \preceq\rangle$ is a partial ordered set, $X = \cup^*\{X_\alpha : \alpha < \kappa\}$, and $i : [X]^2 \to [X]^{<\omega}$ s.t

(0) if $x \in X_\alpha$, $y \in X_\beta$, $x \prec y$ then $\alpha < \beta$.

(1) if x and y are $\preceq$-incomparable elements of X then $t \preceq x \wedge t \preceq y$ iff $\exists s \in i\{x, y\}$ $t \preceq s$.

(2) if $x \in X_\alpha$, $F \in [X_{<\alpha}]^{<\omega}$ and $\xi < \alpha$ then there is $y \in X_\xi$ s.t. $y \preceq x$ but $y \not\preceq F$.

Let $U(x) = \{x' \in X : x' \preceq x\}$. Then

- $\{U(x), X \setminus U(x) : x \in X\}$ is a subbase of an LCS space of $\langle X, \tau\rangle$,
- $U(x)$ is compact open,
- $I_\alpha(X, \tau) = X_\alpha$.

Construction strategy

(0) if $x \in X_\alpha$, $y \in X_\beta$, $x \prec y$ then $\alpha < \beta$.

(1) if x and y are $\preceq$-incomparable elements of X then $t \preceq x \wedge t \preceq y$ iff $\exists s \in i\{x, y\}$ $t \preceq s$.

(2) if $x \in X_\alpha$, $F \in \left[X_{<\alpha}\right]^{<\omega}$ and $\xi < \alpha$ then there is $y \in X_\xi$ s.t. $y \preceq x$ but $y \not\preceq F$.

- Construct an LCS space with cardinal sequence $\langle \kappa_\alpha : \alpha < \mu \rangle$.
- $X_\alpha = \{\alpha\} \times \kappa_\alpha$,
- **Roitman: add generically**
  - **a partial ordering $\preceq$ on X,**
  - **a function $i : \left[X\right]^2 \to \left[X\right]^{<\omega}$**

  satisfying (0)–(1) using finite approximations.
- a typical forcing condition is a triple $\langle a, \leq, i \rangle$, where a is a finite subset of X, $\leq$ is a partial order on a, and i is a function on $\left[a\right]^2$ such that $\langle a, \leq, i \rangle$ satisfies (0) and (1).
- **The poset may not satisfy c.c.c.**

Δ-functions

- Construct an LCS space with cardinal sequence $\langle \kappa_\alpha : \alpha < \mu \rangle$.
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- restrict the possible values of $i\{x, y\}$.
- **Fix $F : [X]^2 \to [X]^\omega$ in V. Demand $i\{x, y\} \in [F(x, y)]^{<\omega}$.**
- Roitman used a construction of Galvin
- Baumgartner-Shelah: Δ-function
- better function F = better space

Δ-functions

- better Δ-function = better space

- Roitman used a construction of Galvin
  - good to get $\langle\omega\rangle_{\omega_1} {}^\frown \langle\omega_2\rangle \in \mathcal{C}(\omega_1+1)$
  - there is no such function to get $\langle\omega\rangle_{\omega_1} {}^\frown \langle\omega_3\rangle \in \mathcal{C}(\omega_1+1)$
- Baumgartner-Shelah:
  - good to get $\langle\omega\rangle_{\omega_2} \in \mathcal{C}(\omega_2)$
  - there is no such function to get $\langle\omega\rangle_{\omega_3} \in \mathcal{C}(\omega_3)$
- **What about $\langle\omega\rangle_{\omega_1} {}^\frown \langle\omega_3\rangle \in \mathcal{C}(\omega_1+1)$?**
- Koszmider constructed "better" Δ-functions.

$F : [\lambda]^2 \longrightarrow \omega_1$ is a **strongly unbounded on λ** iff
for each uncountable family $\mathcal{A} \subset [\lambda]^{<\omega}$ of pairwise disjoint sets

$$\sup\{\min F''[a,b] : a \neq b \in \mathcal{A}\} = \omega_1$$

Theorem (Galvin, Roitman)

It is consistent that there is a strongly unbounded function on ω_2.

Theorem (Koszmider)

If $2^\omega = \omega_1 \leq \lambda$, then there is a strongly unbounded function on λ in some cardinal-preserving generic extension of the ground model.

Theorem (Martinez,S)

It is consistent that $\langle\omega\rangle_{\omega_1} {}^\frown \langle\omega_3\rangle \in \mathcal{C}(\omega_1 + 1)$. If $\delta < \omega_2$ with $cf(\delta) = \omega_1$ then it is consistent that $\langle\omega\rangle_\delta {}^\frown \langle\omega_3\rangle \in \mathcal{C}(\delta + 1)$.

Stepping up in Roitman's theorem

New results

- Baumgartner-Shelah: $\langle\omega\rangle_{\omega_2} \in \mathcal{C}(\omega_2)$
- Bagaria: $2^\omega > \omega_2$, MA_{ω_2} and $\mathcal{C}(\omega_2) \supset {}^{\omega_2}\{\omega, \omega_1, \}$
- S: $2^\omega = \omega_2$ and $\mathcal{C}(\omega_2) \supset {}^{\omega_2}\{\omega, \omega_1, \omega_2\}$

Theorem (Martinez, S)

$2^\omega = \omega_3$ *and* $\mathcal{C}(\omega_2) \supset {}^{\omega_2}\{\omega, \omega_1, \omega_2, \omega_3\}$

- If **there is a strong "Δ-function"** $F : [\omega_2 \times \omega_3]^2 \to [\omega_2]^{<\omega}$ then there is a c.c.c poset R s.t $1_R \Vdash \mathcal{C}(\omega_2) \supset {}^{\omega_2}\{\omega, \omega_1, \omega_2, \omega_3\}$
- If **there is a "strong (ω_1, ω_3)-semimorass"** $\mathcal{F} \subset [\omega_3]^\omega$ then there is a proper poset Q such that $1_Q \Vdash \exists$ **strong "Δ-function"** $F : [\omega_2 \times \omega_3]^2 \to [\omega_2]^{<\omega}$
- If $2^\omega = \omega_1$ then there is a σ-complete, ω_2-c.c. poset P s.t. $V^P \models$ "**there is a "strong (ω_1, ω_3)-semimorass"** $\mathcal{F} \subset [\omega_3]^\omega$"
- $V^{P*Q*R} \models \mathcal{C}(\omega_2) \supset {}^{\omega_2}\{\omega, \omega_1, \omega_2, \omega_3\}$

Problems

Theorem (Just)

In the Cohen model $\langle\omega\rangle_{\omega_2} \notin \mathcal{C}(\omega_2)$.

Theorem (Juhász, Shelah, S, Szentmiklóssy)

$\{\alpha : |I_\alpha(X)| = \omega\} \le \omega_1$ *in the Cohen model.*

Conjecture

$\{\alpha : |I_\alpha(X)| = \omega\} \le \omega_2$.

Shelah: $2^{\aleph_0} < \aleph_\omega \implies (\aleph_\omega)^{\aleph_0} < \aleph_{\omega_4}$

If the Conjecture holds then $2^{\aleph_0} < \aleph_\omega \implies (\aleph_\omega)^{\aleph_0} < \aleph_{\omega_3}$