Rainbow Colourings

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Logic Colloquium 2008

The beginnings

- **Ramsey theory**: find large homogeneous sets
- monochromatic sets
- **anti Ramsey theory**: find large inhomogeneous sets
- polychromatic sets, rainbow sets
- first anti Ramsey theorems : Rado, 1973.
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Let $f : [X]^n \to \lambda$.
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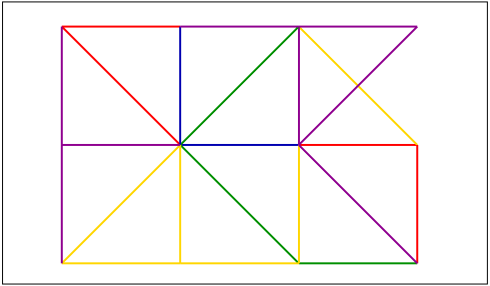

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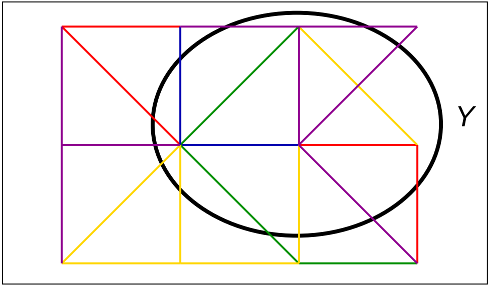

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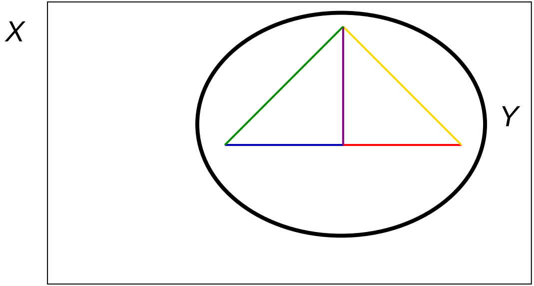

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Assume that $f : [\omega_1]^2 \to 3$ establishes $\omega_1 \not\to [\omega_1]^2_3$.
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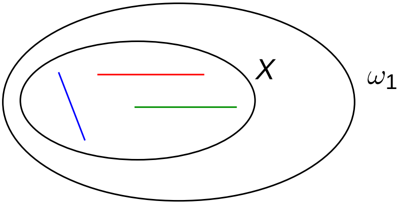

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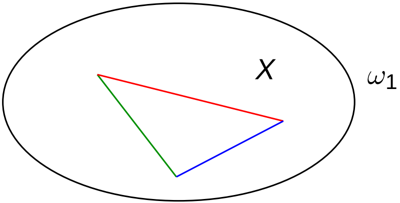

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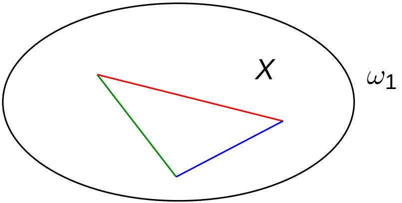

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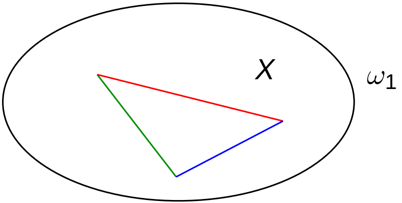

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*If $f \vdash \omega_1 \not\to [(\omega, \omega_1)]^2_3$ then there is a **rainbow triangle**.*

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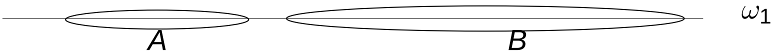

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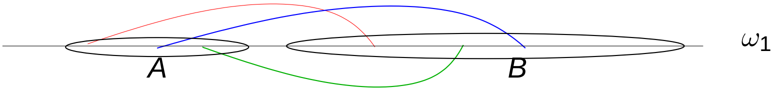

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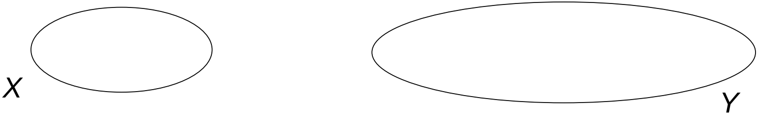

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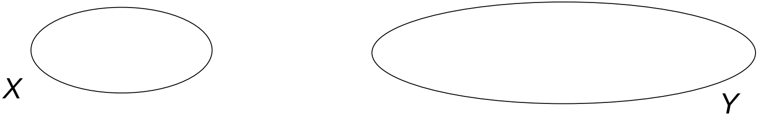

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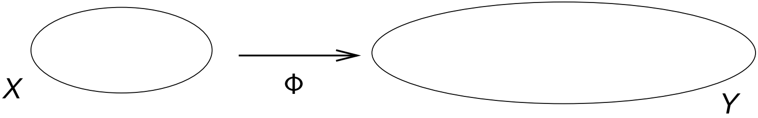

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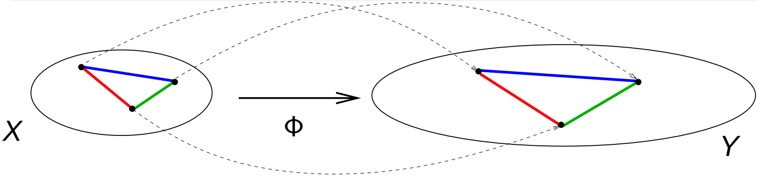

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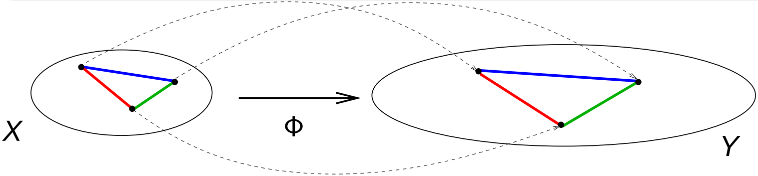

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If $f \vdash \omega_1 \not\to [(\omega, \omega_1)]^2_{\omega_1}$ then f realizes each function $d : [\omega]^2 \to \omega_1$.

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Theorem (Shelah, 1975)

$CH \implies \exists f \vdash \omega_1 \not\to [\omega_1]^2_{\omega}$, *no f-rainbow triangle.*

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- By **transfinite induction** on $\alpha < \omega_1$ define $f(\xi, \alpha)$ for $\xi < \alpha$.
- **First challenge**: $\omega_1 \not\to [\omega_1]^2_\omega$
  - fix a $\clubsuit$-sequence $\langle A_\zeta : \zeta < \omega_1 \rangle \subset [\omega_1]^\omega$
  - if $\zeta < \alpha$ and $A_\zeta \subset \alpha$ then $f''[A_\zeta \times \{\alpha\}] = \omega$
- **Second challenge**: no f-rainbow triangle ???

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- **Second try: Fix the colouring first!** Colour $[2^\omega]^2$ as follows
- $\Delta(x,y) = \min\{n : x(n) \neq y(n)\}$.
- $h : \omega \to \omega$ s.t. $h^{-1}\{k\}$ is infinite for each $k \in \omega$.
- $f(x,y) = h(\Delta(x,y))$.

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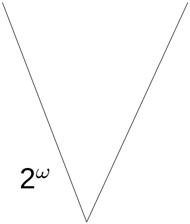

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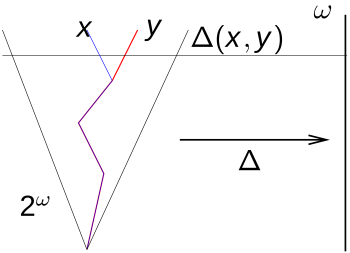

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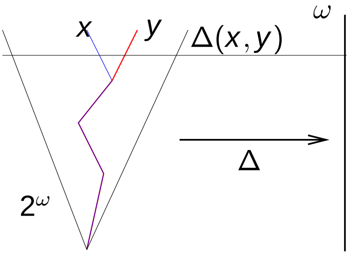

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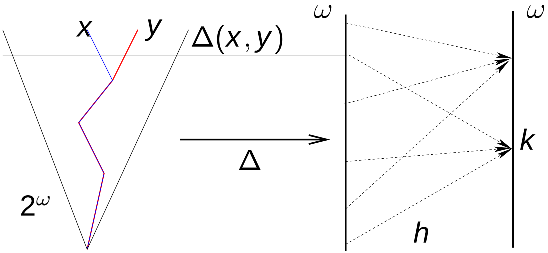

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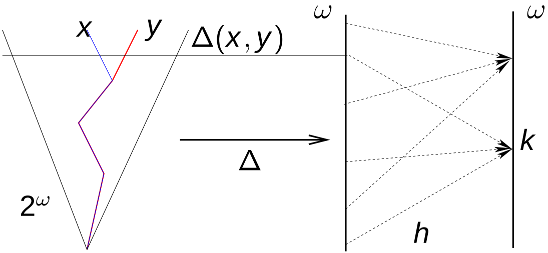

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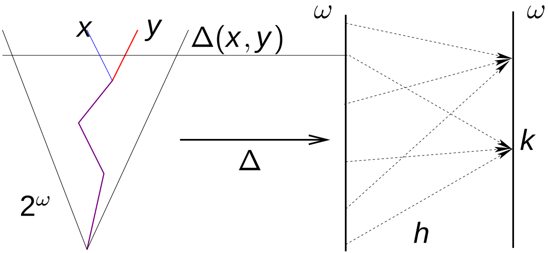

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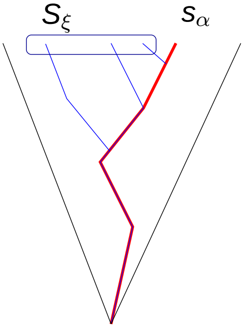

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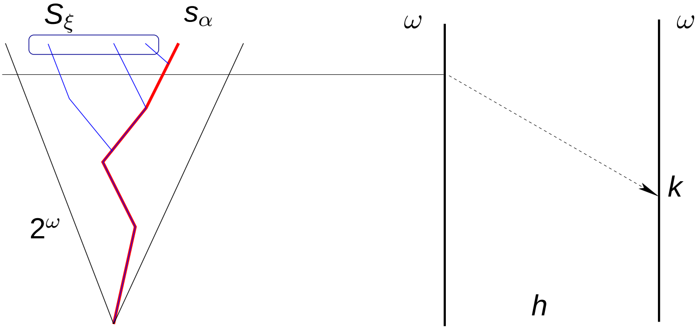

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A weaker partition relation

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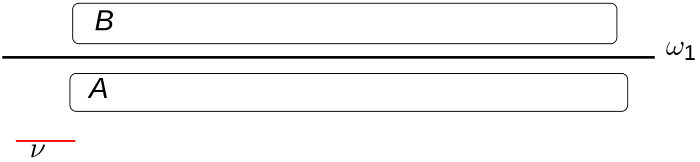

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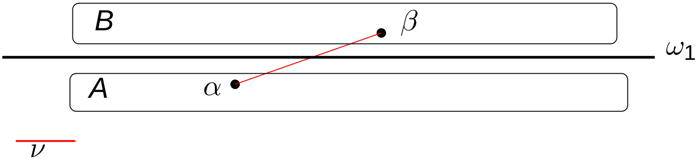

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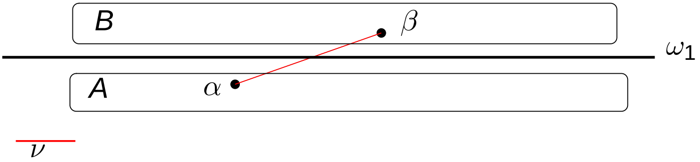

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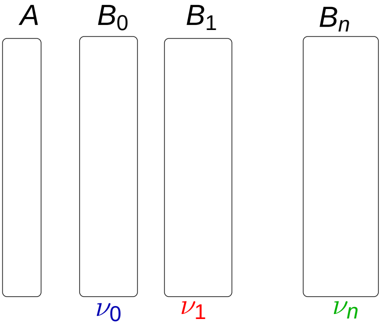

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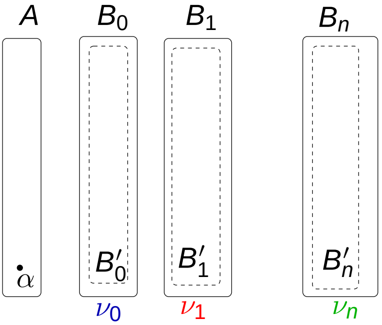

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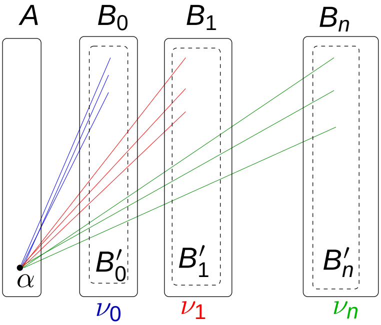

Summary

One more partition relation

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$\forall A, B \in [\omega_1]^{\omega_1}\ \forall \nu < \omega_1\ \exists \alpha \in A\ \exists \beta \in B\ f(\alpha, \beta) = \nu$ and $\alpha < \beta$.

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Assume $f \vdash \omega_1 \not\to [(\omega_1,\omega_1)]^2_{\omega_1}$. Does f realize each function $d : [\omega]^2 \to \omega_1$?

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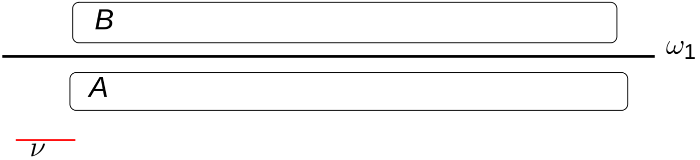

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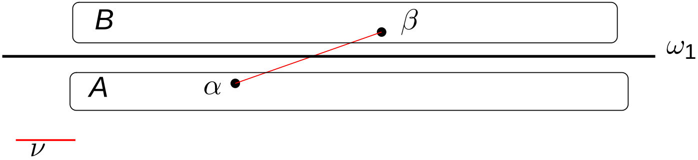

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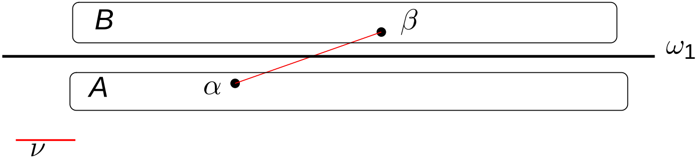

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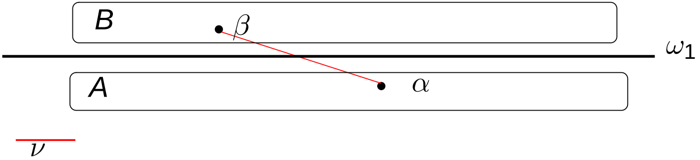

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Theorem (Hajnal)

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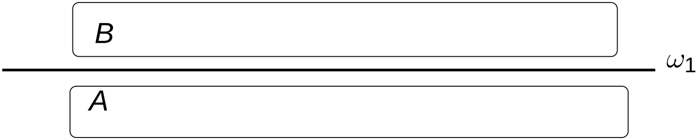

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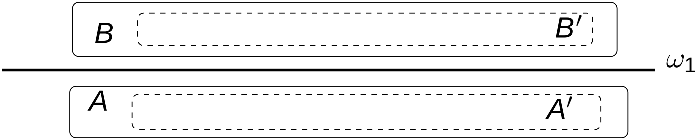

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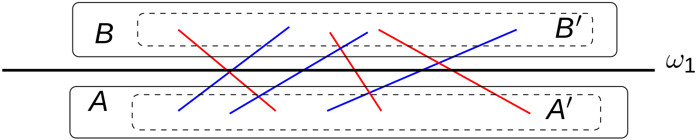

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$\exists f \vdash \omega_1 \not\to [\omega_1]^2_6$ *and* $\exists d : [4]^2 \to 6$ *rainbow s.t.* $d \not\Rightarrow f$.

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Positive theorems

If $f \vdash \omega_1 \not\to [(\omega,\omega_1)]^2_{\omega_1}$ then f realizes each function $d : [\omega]^2 \to \omega_1$.

Assume $f \vdash \omega_2 \not\to [(\omega_1,\omega_2)]^2_2$. Does f realize each function $d : [\omega_1]^2 \to 2$?

Theorem (Shelah, 1975)

(a) If $f \vdash \omega_2 \not\to [(\omega_1;\omega_2)]^2_2$ then $V^{Fn(\omega,2)} \models f \vdash \omega_2 \not\to [(\omega_1;\omega_2)]^2_2$.

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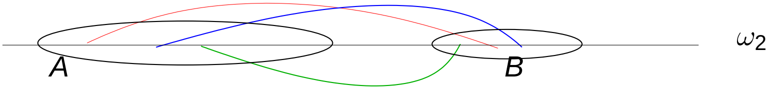

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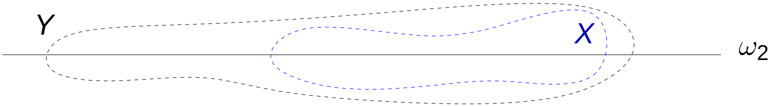

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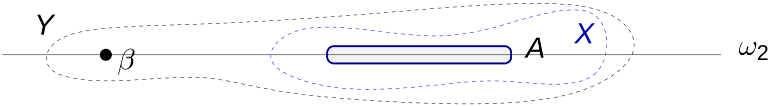

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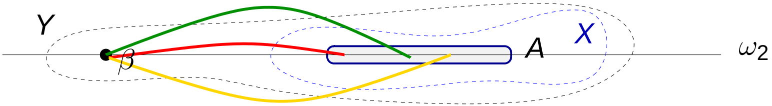

Colourings of ω_2

Problem of Erdős and Hajnal:
Assume GCH and $f \vdash \omega_2 \not\to [(\omega_1;\omega)]^2_{\omega_1}$. Does f realize every $c : [\omega_1]^2 \to 2$?

(Shelah): If $f \vdash \omega_2 \not\to [(\omega_1;\omega_2)]^2_2$ then $V^{Fn(\omega,2)} \models f \vdash \omega_2 \not\to [(\omega_1;\omega_2)]^2_2$.

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Problem (Hajnal)

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It is consistent that GCH holds and $\exists f \vdash \omega_2 \not\to [(\omega_1;\omega)]^2_{\omega_1}$ s.t. there is no uncountable f-rainbow.

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It is consistent that CH holds, 2^{ω_1} is arbitrarily large and $\exists g \vdash 2^{\omega_1} \not\to [(\omega_1,\omega_2)]^2_{\omega_1}$ s. t. there is ***no uncountable*** g***-rainbow*** *subset of 2^{ω_1}.*

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- $f : [X]^n \to C$ is **κ-bounded** iff $|f^{-1}\{c\}| \le \kappa$ for each $c \in C$.
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$\lambda \to (\alpha)^n_k$ implies $\lambda \to^* (\alpha)^n_{k-\mathrm{bdd}}$.

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$\omega_1 \to^* (\alpha)^2_{2-bdd}$ for $\alpha < \omega_1$.

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If $c : [\omega_1]^2 \to \omega_1$ is 2-bounded then T.F.A.E.

- $c \vdash \omega_1 \not\to^* (\omega_1)^2_{2-bdd}$
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Theorem (Abraham, Cummings, Smyth)

*It is consistent that there is a function $c : [\omega_1]^2 \to \omega_1$ which **c.c.c-indestructibly** establishes $\omega_1 \not\to^* (\omega_1)^2_{2-bdd}$.*

If CH holds and there is a Suslin-tree then there is a function $c' : [\omega_1]^2 \to 2$ and there is a c.c.c poset Q such that

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- f is k-bounded,
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Based on some results of Abraham and Todorcevic.

$\mathrm{Fn}_m(\omega_1, K) = \{s : s \text{ is a function, } \mathrm{dom}(s) \in [\omega_1]^m, \mathrm{ran}(s) \subset K\}$

$\langle s_\alpha : \alpha < \omega_1 \rangle \subset \mathrm{Fn}_m(\omega_1, K)$ is ***dom-disjoint*** iff $\mathrm{dom}(s_\alpha) \cap \mathrm{dom}(s_\beta) = \emptyset$

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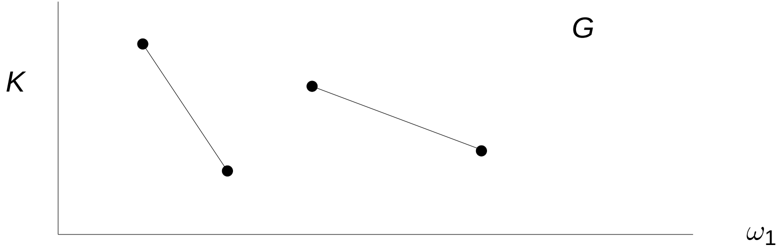

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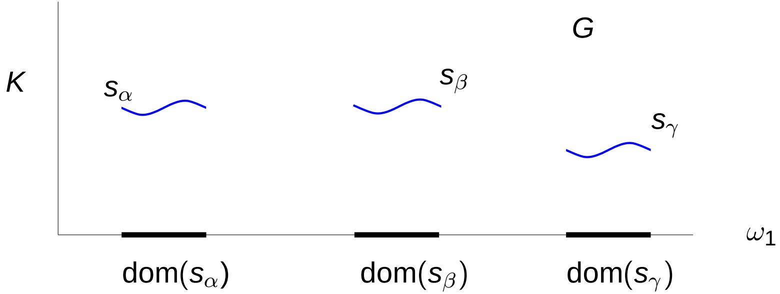

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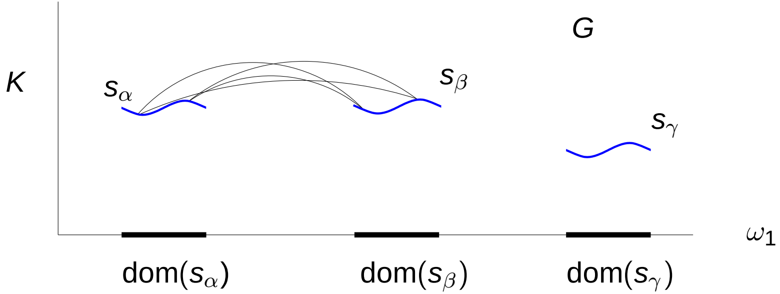

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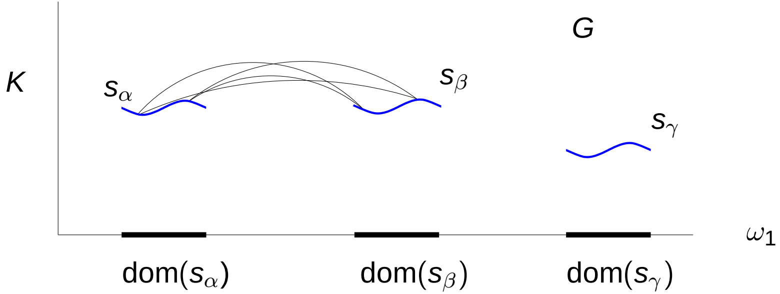

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$$[s_\alpha, s_\beta] \subset G.$$

G is called ***strongly solid*** iff it is m-solid for each $m \in \omega$.

Theorem

*Assume $2^{\omega_1} = \omega_2$. If G is a **strongly solid** graph on $\omega_1 \times K$, where $|K| \le 2^{\omega_1}$, then for each $m \in \omega$ there is a c.c.c poset P of size ω_2 such that*

$$V^P \models \text{“}G \text{ is } \textbf{c.c.c-indestructibly } m\textbf{-solid.”}$$

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ZFC	$\omega_1 \not\to [(\omega,\omega_1)]^2_{\omega_1}$	$\Longrightarrow$	$\forall d : [\omega]^2 \to \omega_1$

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CH	$\omega_1 \not\to [\omega_1]^2_{\omega}$	$\wedge$	**no rainbow** K_3
$\diamondsuit$	$\omega_1 \not\to [\omega_1]^2_{\omega_1}$	$\wedge$	**no rainbow** K_3

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ZFC	$\omega_1 \not\to [(\omega_1, \omega_1)]^2_{\omega_1}$	$\Longrightarrow$	$\forall d : [3]^2 \to \omega_1$
ZFC	$\omega_1 \not\to [(\omega_1, \omega_1)]^2_{10}$	$\wedge$	**not univ for K_5-rainbows**
ZFC	$\omega_1 \not\to [(\omega_1, \omega_1)]^2_{\omega_1}$	$\Longrightarrow$	$\exists$ **infinite rainbow**
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ZFC	$\omega_1 \not\to [\omega_1]^2_{6}$	$\wedge$	**not univ for K_4-rainbows**