On the Border of Finite and Infinite

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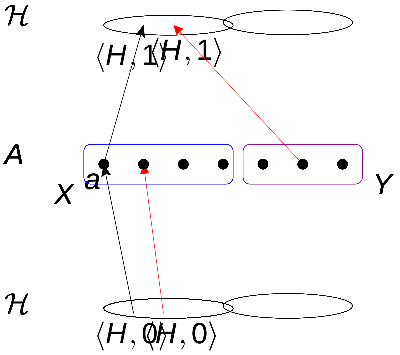
Outline

1. Splitting Antichains (P. L. Erdős, –)
2. Quasi Kernels and Quasi Sinks (P. L. Erdős, A. Hajnal, –)
3. Multiway Cuts

Defi nition

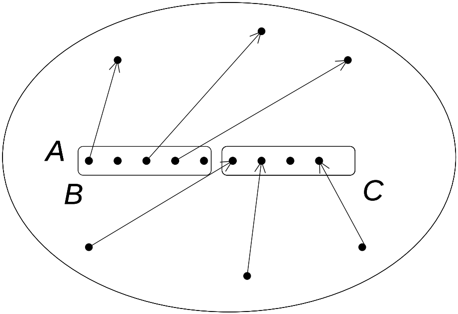
A family $\mathcal{H}$ of subsets of a set A has **property B** iff there is a partition(X, Y) of A such that $H \cap X \neq \emptyset \neq H \cap Y$ for each $H \in \mathcal{H}$.



$Q(\mathcal{H}) = \langle A \cup (\mathcal{H} \times \{0, 1\}), \leq\rangle$.
$\langle 0, H\rangle \leq a \leq \langle 1, H\rangle$ iff $a \in H$.
If $\emptyset \notin \mathcal{H}$ then A is a **maximal antichain**
$A = X \cup^* Y$
(X, Y) **witnesses that** $\mathcal{H}$ **has property B** iff $Q(\mathcal{H}) = X^{\downarrow} \cup Y^{\uparrow}$

Defi nition

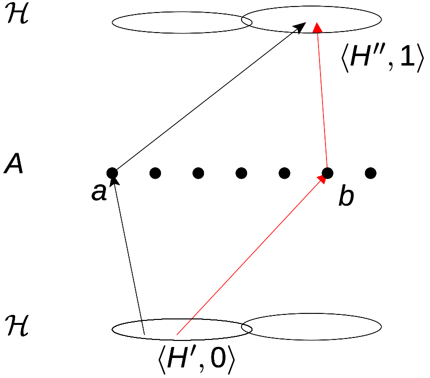
Let $P = \langle P, \leq \rangle$ be a poset, $A \subset P$ be a **maximal antichain**.
$A \subset P$ **splits** iff A has a **partition** $A = B \cup^* C$ such that $P = B^{\uparrow} \cup C^{\downarrow}$.



Property B

Theorem (Lovász, 1979)

*If A is a finite set and $\mathcal{H} \subset [A]^{\geq 2}$ such that $|H' \cap H''| \neq 1$ for each $\{H, H'\} \in [\mathcal{H}]^2$ then $\mathcal{H}$ has **property-B**.*



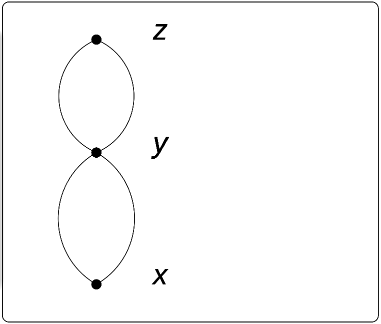
Theorem (Lovász, reformulated)

Let A be a finite set, $\mathcal{H} \subset \mathcal{P}(A)$, $\emptyset \notin \mathcal{H}$, s. t. if $\langle H', 0 \rangle \leq a \leq \langle H'', 1 \rangle$ then there is $b \neq a$ such that $\langle H', 0 \rangle \leq b \leq \langle H'', 1 \rangle$. Then A splits in $Q(\mathcal{H})$.

Property B

Defi nition

Let P be a poset and $y \in P$. y is a **cut point** iff $\exists x, z \in P$ s.t. $x <_P y <_P z$ and $[x, z] = [x, y] \cup [y, z]$.
A subset $A \subset P$ is **cut-free** if it does not contain cut points.



Theorem (Lovász, reformulated)

*Let A be a finite set and $\mathcal{H} \subset \mathcal{P}(A)$, $\emptyset \notin \mathcal{H}$. If A is **cut-free** (in $Q(\mathcal{H})$) then A **splits**.*

Theorem (P. L Erdős - Niall Graham (1993))

In a finite Boolean lattice every max. antichain splits.

Theorem

If P is a finite poset and A is a maximal antichain then the question ***"Does A split?"*** *is NP-complete.*

Theorem (Ahlswede, P. L. Erdős, N. Graham(1995))

In a finite poset every cut-free maximal antichain splits.

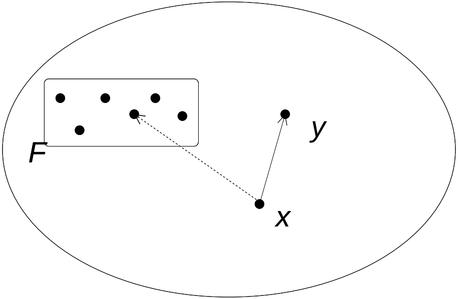
Theorem (Ahlswede, Khachatrian)

There is a maximal, (infinite) non-splitting antichain A in divisor poset of the square-free positive integers.

divisor poset of of the square-free positive integers = $\left\langle [\omega]^{<\omega}, \subset \right\rangle$

Definition

A poset $\mathcal{P}$ is **loose** iff for each $x \in P$ and $F \in [P]^{<\omega}$ if $x \notin F^{\uparrow}$ then there is $y \in x^{\uparrow} \setminus \{x\}$ such that $y \notin F^{\updownarrow}$.



Fact

$\left\langle [\omega]^{<\omega}, \subset \right\rangle$ is loose.

Theorem (P. L. Erdős, –)

A countable, cut-free, loose poset $\mathcal{P} = \langle P, \leq \rangle$ contains a maximal infinite non-splitting antichain A.

Theorem (P. L. Erdős)

In a cut-free poset every finite maximal antichains split.

Theorem

If $\mathcal{P}$ is a poset, A is a cut-free maximal antichain such that $|x^{\updownarrow} \cap A| < \omega$ for all $x \in P$ then A splits.

No proofs with Gödel Compactness Theorem!
If P is cut-free, $Q \subset P$ then Q is not necessarily cut-free

Theorem (P. L. Erdős, –)

Assume that $\mathcal{P}$is a countable poset such that both $\mathcal{P}$ and $\mathcal{P}^{-1}$ are loose. Then $\mathcal{P}$ contains a maximal antichain which splits.

Example

There is a “non-trivial” infi nite cut-free poset s.t. every maximal antichain splits.

Problem

Is there a countable cut-free poset without splitting maximal antichains?

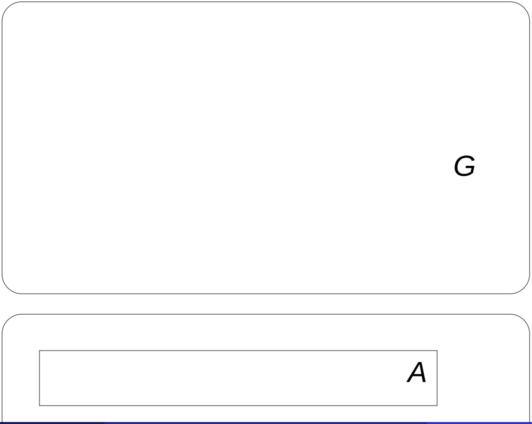
Splitting antichains
Uncountable posets

Deep set-theory, independence, ...

Quasi Kernels and Quasi Sinks

Theorem (Chvatal, Lovász)

Every finite ***digraph*** *(i.e. directed graph) contains an* ***independent set*** *A such that for each point v there is a* ***path of length at most*** *2 from some point of A to v.*



In and Out

Definition

Assume that $G = (V, E)$ is a **digraph**, $A \subset V$ and $n \in \mathbb{N}$. Let

$\mathrm{In}_n(A) = \{v \in V :$ there is a path of length at most n which leads from v to some points of $A\}$, and

$\mathrm{Out}_n(A) = \{v \in V :$ there is a path of length at most n which leads from some points of A to $v\}$.

Definition

Let $G = (V, E)$ be a digraph.

An **independent set** A is a **quasi-kernel** if and only if $V = \mathrm{Out}_2(A)$.

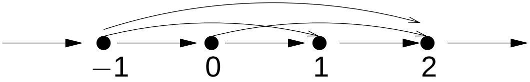
An **independent set** B is a **quasi-sink** if and only if $V = \mathrm{In}_2(B)$.

Theorem (Chvatal, Lovász)

Every finite digraph $G = (V, E)$ contains a ***quasi-kernel (quasi-sink)****.*

Finite $\neq$ Infinite

Plain generalization fails even for **infinite tournaments:**
the tournament $(\mathbb{Z}, <)$ is a counterexample.



$(\mathbb{Z}, <)$ does not have quasi kernel, but $\mathbb{Z} = \mathrm{Out}_1(1) \cup \mathrm{In}_1(0)$.

Problem

Is it true that for each **directed graph** $G = (V, E)$ there are **disjoint, independent subsets** A and B of V such that $V = \mathrm{Out}_2(A) \cup \mathrm{In}_2(B)$.

Hunting for counterexamples

Defi nition

$G = (V, E)$ is a **digraph**, $n, k \in \mathbb{N}$:

$G \in \mathfrak{In}_k \iff \exists$ an **independent set** $A \subset V$ s. t. $V = \mathrm{In}_k(A)$,

$G \in \mathfrak{Out}_n \iff \exists$ an **independent set** $B \subset V$ s.t. $V = \mathrm{Out}_n(B)$.

$G \in \mathfrak{In}_k\text{-}\mathfrak{Out}_n \iff \exists$ **partition** (V_1, V_2) of V s.t $G[V_1] \in \mathfrak{In}_k$ and $G[V_2] \in \mathfrak{Out}_n$.

Theorem (Chvatal, Lovász)

Every finite digraphs is in $\mathfrak{Out}_2$.

Theorem

Every tournament is either in $\mathfrak{Out}_2$ or in $\mathfrak{In}_1$-$\mathfrak{Out}_1$.

Theorem

*If $G = (V, E)$ is a digraph and $\mathrm{In}_1(x)$ **is finite** for each $x \in V$ then $G \in \mathfrak{Out}_2$.*

Theorem

*If the **chromatic number** of G is finite then $G \in \mathfrak{Out}_2$.*

Hunting for counterexamples

Defi nition

If $G = (V, E)$ is a digraph defi ne the ***undirected complement*** of the graph, $\widetilde{G} = (V, \widetilde{E})$ as follows: $\{x, y\} \in \widetilde{E}$ if and only if $(x, y) \notin E$ and $(y, x) \notin E$.

Theorem

Let $G = (V, E)$ be a directed graph. If $K_n \not\subset \widetilde{G}$ for some $n \geq 2$ then $G \in \mathfrak{In}_2\text{-}\mathfrak{Out}_2$. Especially, if the ***chromatic number*** *of $\widetilde{G}$ is finite then $G \in \mathfrak{In}_2\text{-}\mathfrak{Out}_2$.*

Theorem

If $G = (V, E)$ is a digraph such that $\widetilde{G}$ is ***locally finite*** *then $G \in \mathfrak{In}_2\text{-}\mathfrak{Out}_2$.*

A theorem and a new conjecture

Theorem

*For each **directed graph** $G = (V, E)$ there are **disjoint, independent subsets** A and B of V such that $V = \mathrm{Out}_2(A) \cup \mathrm{In}_2(B)$.*

In the positive theorems we obtained $G \in \mathfrak{In}_2\text{-}\mathfrak{Out}_2$!

Conjecture

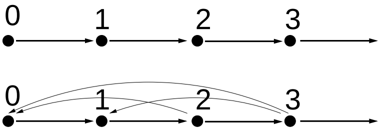
Every directed graph is in $\mathfrak{In}_2\text{-}\mathfrak{Out}_2$.

Problem

Find an infinite digraph G s. t. $G \notin \mathfrak{In}_1\text{-}\mathfrak{Out}_2$.

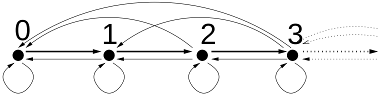
Structure theorems for tournaments

Let $T_\infty = (\mathbb{N}, E)$, where (x, y) is an edge if and only if $y = x + 1$ or $y + 1 < x$.



$T_\infty \notin \mathfrak{Out}_2$, $\quad T_\infty \notin \mathfrak{Out}_n$ for $n \in \mathbb{N}$

Let $\mathbb{G}_\infty = (\mathbb{N}, E)$, as follows: (x, y) is an edge if and only if $x \geq y + 1$.



$\mathbb{G}_\infty$

Theorem

For an infinite tournament $T = (V, E)$ the followings are equivalent:

(i) $T \notin \mathfrak{Out}_3$,

(ii) $T \notin \mathfrak{Out}_n$ *for each* $n \geq 3$,

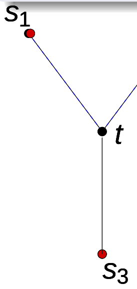
(iii) *there is a surjective homomorphism* $\varphi : T \to \mathbb{G}_\infty$.

Theorem

There is a tournament $T \in \mathfrak{Out}_3 \setminus \mathfrak{Out}_2$.

Multiway Cut Problem

Fix a graph $G = (V, E)$ and a subset S of vertices called **terminals**. A **multiway cut** is a **set of edges** whose removal disconnects each terminal from the others. The **multiway cut problem** is to find the **minimal size** of a multiway cut denoted by $\pi_{G,S}$.



$$S = \{s_1, s_2, s_3\}$$

$$\pi_{G,S} = 2$$

Multiway cuts

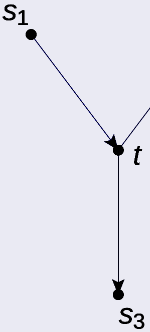
Definition

If $\vec{G} = (V, E)$ is a directed graph and $A, B \subset V$ let $\lambda(\vec{G}, A, B)$ be the maximal number of **edge-disjoint directed paths** from some element of A into some element of B.

Definition

If $G = (V, E)$ is a finite graph, $S \subset V$, and $\vec{G}$ is obtained from G by an orientation of the edges, then let

$$\nu_{\vec{G},S} = \sum_{s \in S} \lambda(\vec{G}, S - s, s)$$



$S = \{s_1, s_2, s_3\}$

$\lambda(\vec{G}, S - s_3, s_3) = 1$, $\lambda(\vec{G}, S - s_2, s_2) = 1$, $\lambda(\vec{G}, S - s_1, s_1) = 0$, $\nu_{\vec{G},S} = 2$

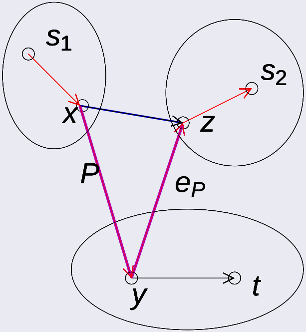
Multiway cuts

Theorem (P. L. Erdős, A. Frank, L. Székely)

If $G = (V, E)$ is a finite graph, $S \subset V$, and $\vec{G}$ is obtained from G by an orientation of the edges, then $\nu_{\vec{G},S} \le \pi_{G,S}$.

Proof.

Fix a multiway cut $F \subset E$ and for $s \in S$ let $\mathcal{P}_s$ be a family of edge-disjoint directed paths from some element of $S - s$ into s.
For each $P \in \mathcal{P}_s$ let e_p be the last element of $P \cap F$ in P.
Then $e_P \neq e_{P'}$ provided $P \neq P'$.



$\square$

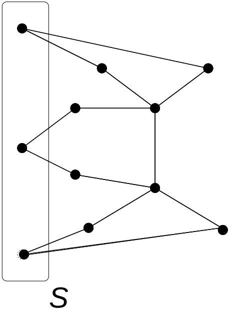
Multiway cuts

Theorem (E. Dahjhaus, D. S. Johson, C. H. Papadimitriou, P.D. Seymout, M Yannakakis)

The multiway cut problem is ***NP-complete***.

Special case:

$G - S$ is a **tree**.



Multiway cuts

Theorem (P. L. Erdős, L. Székely)

If $G = (V, E)$ is a finite graph, $S \subset V$ such that $G - S$ is tree, then

$$\max_{\vec{G}} \nu_{\vec{G},S} = \pi_{G,S}.$$

where the maximum is taken over all orientations $\vec{G}$ of G.

Theorem (P. L. Erdős, A. Frank, L. Székely, reformulated)

*If $G = (V, E)$ is a **finite graph**, $S \subset V$ such that $G - S$ is **tree,** then there is an **orientation** $\vec{G}$ of G, and for each $s \in S$ there is an **edge-disjoint family** $\mathcal{P}_s$ **of** $(S - s, s)$**-paths** in $\vec{G}$ and for each $P \in \mathcal{P}_s$ we can pick an edge $e_P \in P$ such that*

$$\{e_P : P \in \mathcal{P}_s \text{ for some } s \in S\}$$

*is a **multiway cut** (in G for S).*

Multiway cuts

Theorem (–)

If $G = (V, E)$ is a graph, $S \subset V$ such that $G - S$ is tree ***without infinite paths****, then there is an orientation $\vec{G}$ of G, and for each $s \in S$ there is an edge-disjoint family $\mathcal{P}_s$ of $(S - s, s)$-paths in $\vec{G}$ and for each $P \in \mathcal{P}_s$ we can pick an edge $e_P \in P$ such that*

$$\{e_P : P \in \mathcal{P}_s \text{ for some } s \in S\}$$

is a multiway cut (in G for S).

Proposition

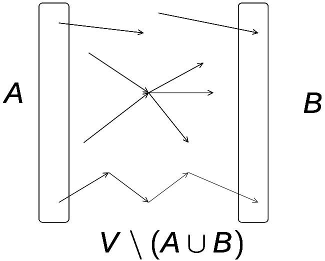
Let $G = (V, E)$ be a finite directed graph, and $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,

(3) $in(x) \le out(x)$ for each $x \in V \setminus (A \cup b)$.

Then there is a family $\mathcal{P}$ of edge-disjoint A-B-paths s .t. $\mathcal{P}$ covers A.



Theorem

Let $G = (V, E)$ be a directed graph which does not contain infinite directed path, and let $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,

(3) $in(x) \le out(x)$ for each $x \in V \setminus (A \cup B)$.

Then there is a family $\mathcal{P}$ of edge-disjoint A-B-paths s .t. $\mathcal{P}$ covers A.

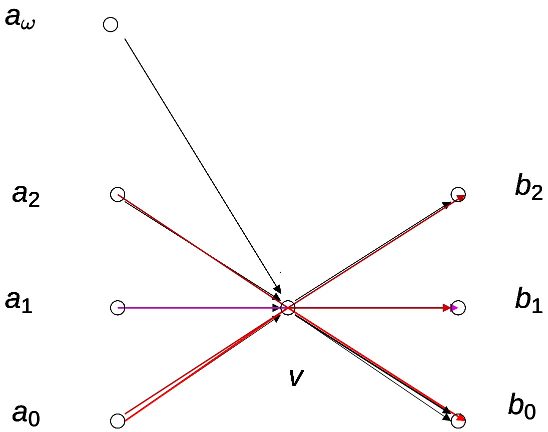
Proof.

G is countable: easy induction: if P is an A-B-path then $G - P$ satisfies (1)–(3)
G is uncountable may got stuck at some point

$\square$

Multiway cuts

Infinite case



Inductive construction, but using the right enumeration
Elementary submodels

Definition

Let $G = (V, E)$ be a graph. A partition (A, B) of V is called **unfriendly** iff every vertex has at least as many neighbor in the other class as in its own.

Observation

Every finite graph has an unfriendly partition.

Theorem (Shelah)

There is an uncountable graph without an unfriendly partition.

Unfriendly Partition Conjecture

Every countable graph has an unfriendly partition.

Fact

Every locally finite graph has an unfriendly partition.

Fact

If $G = (V, E)$ is countable and every $v \in V$ has infinite degree then G has an unfriendly partition.

Question

Let G be a finite graph, and a and b are vertices such that $d_G(a,b) \geq 10^{10^{10}}$. Is there an unfriendly partition of (A,B) of G such that $a \in A$ and $b \in B$?

Answer

No, V. Bonifaci gave counterexample.

Question

Is it true that for each $n \in \mathbb{N}$ there is $f(n) \in \mathbb{N}$ such that for each finite graph G if $deg(x) \leq n$ for each vertices, and a and b are vertices such that $d_G(a,b) \geq f(n)$. then there is an unfriendly partition of (A,B) of G such that $a \in A$ and $b \in B$?

Summary

Many finite problems have infinite counterparts.

Similar, but not the same.

Deep set-theory is not a must.