# Math Logic Homework-2.

1. Let $\Sigma = \{X \Rightarrow Y,\ \ Y \Rightarrow Z\}$ and let $\varphi = X \Rightarrow Z$ (all formulas are in propositional logic). Prove that $\Sigma \vdash \varphi$.

2. Suppose $\mathcal{F} \subseteq \mathcal{P}(I)$ is an ultrafilter and let $X \in \mathcal{F}$. Prove that

$$\{X \cap Y : Y \in \mathcal{F}\}$$

is an ultrafilter over $X$.

3. Let $\mathcal{F} \subseteq \mathcal{P}(I)$ be a principal ultrafilter and let $\mathcal{A}$ be a structure. Prove that ${}^{I}\mathcal{A}/\mathcal{F}$ (that is, the ultrapower of $\mathcal{A}$ modulo $\mathcal{F}$) is isomorphic with $\mathcal{A}$.

4. Suppose $\mathcal{F} \subseteq \mathcal{P}(I)$ is an ultrafilter and for each $i \in I$, the structures $\mathcal{A}_i$ and $\mathcal{B}_i$ are elementarily equivalent. Prove that the ultraproducts $\prod_{i \in I} \mathcal{A}_i/\mathcal{F}$ and $\prod_{i \in I} \mathcal{B}_i/\mathcal{F}$ are elementarily equivalent.

5. Prove that the set

$$X = \{2^n : n \in \mathbb{N}\}$$

of natural numbers can be defined by bounded quantifiers in the language of Peano arithmetic.

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