The Zaremba criterion

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1. Subharmonic functions

Let Ω be an open subset of $\mathbb{R}^n$, and let

$$u: \ \Omega \to [-\infty, \infty)$$

be an upper semicontinuous function, which is not identically $-\infty$ on any connected component of Ω.

Recall, that such a function u is said to be **subharmonic** in Ω, if for every relatively compact open subset K of Ω and every function $h \in H(K) \cap C(\overline{K})$ the following implication is true:

$$u \le h \ \text{ on } \partial K \Rightarrow u \le h \quad \text{on } K.$$

The family of all functions, which are subharmonic in Ω we denote by $SH(\Omega)$.

Theorem 1 (The maximum principle) *If Ω is a bounded connected open subset of $\mathbb{R}^n$, and if $g \in SH(\Omega)$, then either g is constant or, for each $x \in \Omega$,*

$$g(x) < \sup_{z \in \partial\Omega} \left\{ \lim \sup_{y \to z, y \in \Omega} g(y) \right\}.$$

Theorem 2 *Let Ω be an open subset of $\mathbb{R}^n$. If $\{g_n\} \subset SH(\Omega)$ is a sequence uniformly convergent in Ω, then $g := \lim_{n\to\infty} g_n$ is subharmonic.*

Theorem 3 *Let Ω be an open subset of $\mathbb{R}^2$, and let $g: \ \Omega \to [-\infty, \infty)$ be upper semicontinuous and not identically $-\infty$ on any connected component of Ω. The function g is subharmonic if and only if in every ball $B(a, r)$ such that $\overline{B}(a, r) \subset \Omega$ satisfies the following condition*

$$g(a) \le \frac{1}{2\pi} \int_0^{2\pi} g(a + re^{i\theta}) d\theta.$$

2. The Zaremba criterion

Let $\Omega \subset \mathbb{R}^2$ be a neighbourhood of the point $z_0 = x_0 + iy_0$ and $g: \ \Omega \to (-\infty, +\infty)$. Put

$$\begin{aligned} \Delta_h(g)(z_0) \ := \ & g(x_0 + h, y_0) + g(x_0 - h, y_0) + \\ & g(x_0, y_0 + h) + g(x_0, y_0 - h) - 4g(x_0, y_0) \end{aligned}$$

We define

$$\overline{Z}g(z_0) = \lim \sup_{h \to 0} \frac{\Delta_h(g)(z_0)}{h^2},$$

$$\underline{Z}g(z_0) = \liminf_{h\to 0} \frac{\Delta_h(g)(z_0)}{h^2},$$

and call these limits **upper** and **lower Zaremba operator**, respectively.

If $\overline{Z}g = \underline{Z}g$, then we say that the **Zaremba operator** exists and write $Zg = \overline{Z}g = \underline{Z}g$.

If the function g has partial derivatives of the second order in (x_0, y_0), then $Zg = \Delta g$. Now we prove the following Zaremba criterion

Theorem 4 *If a function $g(z) = g(x, y)$ is upper semicontinuous in the domain $\Omega \subset \mathbb{R}^2$ and $\overline{Z}g \geq 0$ in every $(x, y) \in \Omega$, then g is subharmonic in Ω.*

Proof. Assume that $\overline{Z}g > 0$. Let $u(z) = u(x, y)$ be harmonic in a ball B, such that $\overline{B} \subset \Omega$ and let the function u be continuous on $\overline{B}$ and satisfy on ∂B the condition

$$g(z) \leq u(z). \tag{1}$$

For

$$G(x, y) = g(x, y) - u(x, y)$$

and every $(x, y) \in B$ we have

$$\overline{Z}G = \overline{Z}g - \overline{Z}u > 0,$$

since $\overline{Z}u = \Delta u = 0$.

To obtain the contradiction, suppose that the condition (1) doesn't hold in the interior of B. Then there exists a point $(x_0, y_0) \in intB$, such that

$$G(x_0, y_0) \geq G(x, y) \quad \text{for } (x, y) \in \overline{B}.$$

Hence

$$\begin{aligned} 4G(x_0, y_0) \;\geq\; & G(x_0 + h, y_0) + G(x_0 - h, y_0) + \\ & G(x_0, y_0 + h) + G(x_0, y_0 - h) \end{aligned}$$

for a sufficiently small h.

Therefore, at the point (x_0, y_0) we get $\overline{Z}G = \overline{Z}g \leq 0$, which contradicts the assumption. Now let $\overline{Z}g \geq 0$. We define the following sequence

$$g_n(x, y) = g(x, y) + \frac{x^2}{n}, \quad n = 1, 2, \ldots \ .$$

Since $\overline{Z}g_n = \overline{Z}g + \frac{2}{n} > 0$, it follows that each g_n is subharmonic. Moreover, g_n is a sequence uniformly convergent to g, so from Theorem 2, g is also subharmonic.

As an immediate consequence of the above theorem we have

Corollary 1 *If $g(z) = g(x, y)$ is upper semicontinuous in the domain Ω and has partial derivatives of the second order and satisfies the condition $\Delta g \geq 0$, then $g \in SH(\Omega)$.*

3. Homogeneous extremal function

Let $\mathcal{P}(\mathbb{C}^n)$ and $\mathcal{H}(\mathbb{C}^n)$ denote the set of polynomials of n complex variables and the set of homogeneous polynomials of n variables, respectively. As usual

$$\mathcal{L}(\mathbb{C}^n) := \{u \in PSH(\mathbb{C}^n) : \ u(z) \leq M_u + \log(1 + \|z\|)\}$$

denotes **Lelong class of plurisubharmonic functions with minimal (logarithmic) growth**.

Recall that, if Ω is an open subset of $\mathbb{C}^n$, then an upper semicontinuous function $u : \Omega \to [-\infty, \infty)$ is said to be **plurisubharmonic** if it is not identically $-\infty$ and for each $a \in \Omega$ and $b \in \mathbb{C}^n$, the function $\lambda \to u(a + \lambda b)$ is subharmonic or identically $-\infty$ on every component of the set $\{\lambda \in \mathbb{C} : a + \lambda b \in \Omega\}$.

Let $E \subset \mathbb{C}^n$ be a compact set, and let $\mathcal{P}_E$ denote the family of all polynomials $p \in \mathcal{P}(\mathbb{C}^n)$ such that $\|p\|_E \leq 1$ and $\deg p \geq 1$. Following Siciak, we define

$$\Phi_E(z) = \sup\{|p(z)|^{1/\deg p} : \ p \in \mathcal{P}_E\}$$

$$\Psi_E(z) = \sup\{|p(z)|^{1/\deg p} : \ p \in \mathcal{P}_E \cap \mathcal{H}(\mathbb{C}^n)\}$$

and call these functions **Siciak's extremal function** (or polynomial extremal function) and **Siciak's homogeneous extremal function**, respectively.

It is well known that

$$\log \Phi_E(z) = V_E(z),$$

where

$$V_E(z) := \sup\{u(z) : \ u \in \mathcal{L}(\mathbb{C}^n), \ u|_E \leq 0\},$$

and

$$\Psi_E(z) = \sup\{u(z) : \ u \in HP(\mathbb{C}^n), \ u|_E \leq 1\}.$$

where $HP(\mathbb{C}^n) := \{u : \ u \text{ is homogeneous psh in } \mathbb{C}^n\}$

Theorem 5 *If E is a circular set, then the following equality holds*

$$\Phi_E(z) = \max(1, \Psi_E(z)).$$

3.1 Cauchy-Poisson transform

Let H_+ and H_- be upper and lower halfplanes, respectively, and let q be a norm in $\mathbb{R}^2$. We put

$$u(t) = \log q(1,t).$$

We denote by $\mathcal{P}u$ **the Cauchy-Poisson transform** of u in H_+ defined as follows

$$\mathcal{P}(\zeta) = \frac{1}{\pi}\int_{-\infty}^{\infty} u(ty+x)\frac{dt}{1+t^2},$$

where $\zeta = x+iy \in H_+$.

Lemma 1 *If $0 < \alpha < 1$ then there exists a constant C_α such that for $x, x' \in \mathbb{R}$ and $y > 0$ we have*

$$|\mathcal{P}(\zeta) - u(x')| \leq C_\alpha(|x-x'|+y)^\alpha, \quad \zeta = x+iy.$$

Corollary 2 *The function $\mathcal{P}u$ extends to a continuous function in $\overline{H}_+$ that is harmonic in H_+. If we set*

$$\mathcal{P}u(\zeta) = \mathcal{P}u(\overline{\zeta}), \quad \zeta \in H_-,$$

we obtain a continuous function in $\mathbb{C}$, symmetric with respect to the real axis and harmonic in $H_+ \cup H_-$. Moreover, for $\zeta = x+iy$, we have

$$\mathcal{P}u(\zeta) = \frac{1}{\pi}\int_{-\infty}^{\infty} u(t|y|+x)\frac{dt}{1+t^2}, \quad \zeta \in \mathbb{C}.$$

Corollary 3 *If $B = \{x \in \mathbb{R}^2 : \; q(x) \leq 1\}$ then*

$$\log \Psi_B(1,\zeta) \leq \mathcal{P}u(\zeta), \quad \zeta \in \mathbb{C}.$$

3.2 Subharmonicity of the Cauchy-Poisson transform

For a fixed $\alpha \in (-1,1)$, define

$$v(\alpha,y) := \frac{1}{2}\log(1+2\alpha y+y^2), \quad y \in \mathbb{R},$$

and set $\beta = \sqrt{1-\alpha^2}$. Note that if $|y| < 1$ then

$$v(-\alpha,y) = -\sum_{k=1}^{\infty}\frac{1}{k}T_k(\alpha)y^k,$$

where $T_k(\alpha) := \cos(k\arccos\alpha)$.

Lemma 2 *For each $y \in \mathbb{R}$,*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} v(\alpha, ty)\frac{dt}{1+t^2} = v(\beta, |y|).$$

Applying the above lemma with

$$\alpha' = \tfrac{\alpha+x}{\sqrt{1+2\alpha x+x^2}} \quad \text{and} \quad y' = \tfrac{|y|}{\sqrt{1+2\alpha x+x^2}}.$$

we obtain

Lemma 3 *If $\zeta = x + iy$ then*

$$\begin{aligned}\mathcal{P}v(\alpha,\zeta) &= \frac{1}{\pi}\int_{-\infty}^{\infty} v(\alpha, t|y|+x)\frac{dt}{1+t^2}\\ &= \frac{1}{2}\log(1+2\alpha x+x^2+2\beta|y|+y^2).\end{aligned}$$

Theorem 6 $\mathcal{P}v(\alpha,\zeta) \in SH(\mathbb{C})$.

Proof. We apply the Zaremba critetion to $\mathcal{P}v(\alpha,\zeta)$.

If $\zeta \in \mathbb{C}\setminus\mathbb{R}$ then

$$\overline{Z}\mathcal{P}v(\alpha,\zeta) = Z\mathcal{P}v(\alpha,\zeta) = 0,$$

since $\mathcal{P}v(\alpha,\zeta)$ is harmonic in $\mathbb{C}\setminus\mathbb{R}$.

If $\zeta \in \mathbb{R}$ then $\mathcal{P}v(\alpha,\zeta) = \frac{1}{2}\log(1+2\alpha x+x^2)$ and

$$\overline{Z}\mathcal{P}v(\alpha,\zeta) = \limsup_{h\to 0}\frac{\Delta_h\mathcal{P}v(\alpha,\zeta)}{h^2} = \infty.$$

As an immediate consequence of the above theorem we have

Corollary 4 *Let $u(t) = \frac{1}{2}\log(at^2+bt+c)$ and $\alpha = b/(2\sqrt{ac})$, where $\Delta = b^2-4ac < 0$, $a, c > 0$. Then $\mathcal{P}u \in SH(\mathbb{C})$.*

Theorem 7 *If q is a norm in $\mathbb{R}^2$ and*

$$u(t) = \log q(1,t),$$

then $\mathcal{P}u \in SH(\mathbb{C})$. This implies that $\mathcal{P}u$ belongs to the Lelong class $\mathcal{L}(\mathbb{C})$.

Theorem 8 *If q is a norm in $\mathbb{R}^2$, $B = \{x \in \mathbb{R}^2 : \ q(x) \le 1\}$ and $u(t) = \log q(1,t)$, $t \in \mathbb{R}$, then*

$$\Psi_B(1,\zeta) = \exp\mathcal{P}u(\zeta), \quad \zeta \in \mathbb{C}.$$

Consequently,

$$\Psi_B(z_1,z_2) = |z_1|\exp\mathcal{P}u(z_2/z_1).$$