LANDAU'S PROBLEMS ON PRIMES

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1. Introduction

In his invited address at the 1912 International Congress of Mathematicians, held in Cambridge, Edmund Landau (1912) gave a survey about developments in the theory of prime numbers and the Riemann zeta-function. Besides this he mentioned (without any further discussion) four specific problems about primes which he considered as "unattackable at the present state of science". The four problems (in the original order) were the following

(1) Does the function $u^2 + 1$ represent infinitely many primes for integer values of u?

(2) Does the equality $m = p + p'$ have for any even $m > 2$ a solution?

(3) Does the equality $2 = p - p'$ have infinitely many solutions in primes?

(4) Does there exist at least one prime between n^2 and $(n + 1)^2$ for any positive integer n?

In the present work we will begin with some historical remarks referring to these problems including the few results known in 1912 about those problems and analyse the connections between the four problems. After this we will give a survey of the most important results of the past nearly 100 years. We will discuss the results in connection with Problems (2)–(4) in more detail and briefly those connected with Problem 1 (Section 19), with special emphasis to recent developments concerning various approximations of the Goldbach and Twin Prime Problems.

2. History of the problems and related results before 1912

Whereas the conjecture that there are infinitely many twin primes may originate from the time of Euclid and Eratosthenes, it seems that it appeared first in print in the work of de Polignac (1849), although in a more general form already. We know much more about the origin of Goldbach's Conjecture, however there are some interesting (and partly not well known) facts to mention concerning its origin. In a letter to Euler, written June 7, 1742, Goldbach formulated his conjecture in two different forms. The first one asserted that

(2.1) if a number N is the sum of two primes, then it can be written as a sum of arbitrarily many primes.

In these formulations we have to keep in mind that in his time the number *one* was considered to be a prime. The second formulation was interestingly found on the *margin* of the same letter. This states that

(2.2) every number greater than 2 can be written as the sum of three primes.

Euler pointed out in his answer of June 30 that the first formulation of Goldbach's Conjecture follows from the conjecture that every even number can be written as the sum of two primes. As Euler remarks in his letter this latter conjecture was communicated to him earlier orally by Goldbach himself. So it is really justified to attribute the Binary Goldbach Conjecture to Goldbach. The correspondence of Euler and Goldbach appeared already in 1843 (cf. Fuss (1843)). While the second formulation (2.2) of Goldbach is clearly equivalent to the usual Binary Goldbach Conjecture, this is not obvious with the first formulation (2.1). However, surprisingly, this is really the case. Let us suppose namely that an even number $2k$ is the sum of two primes. Then $2k$ is also a sum of three primes. One of it has to be even, so $2k-2$ is also a sum of two primes. Continuing the procedure we see that every even integer below $2k$ is the sum of two primes. Since all numbers of the form $2p$ are sums of two primes, the usual Binary Goldbach Conjecture follows from the existence of arbitrarily large primes.

Waring stated Goldbach's Problem in 1770 (Waring (1770)) and added that every odd number is either a prime or is a sum of three primes.

It is much less well known that Descartes formulated a related but not equivalent conjecture much earlier than Goldbach since he died already in 1650. According to this every even number is the sum of at most 3 primes. It is unclear why he formulated this only for even integers, but it is very easy to show that this is equivalent to the following, more natural version (where we do not consider *one* to be prime):

Descartes' Conjecture. *Every integer greater than one can be written as the sum of at most* 3 *primes.*

Let us introduce the following

Definition. An even number is called a Goldbach number (their set will be denoted by $\mathcal{G}$ further on) if it can be written as the sum of two primes.

Then it is easy to see that the Descartes' Conjecture is equivalent to

$$\text{If } N > 2 \text{ is even, then } N \in \mathcal{G} \text{ or } N-2 \in \mathcal{G}. \tag{2.3}$$

It is worth remarking that (as one can easily derive from (2.3)) Descartes' Conjecture is equivalent to a stronger form of it, namely

(2.4) Every integer greater than 1 can be written as a sum of three primes, where the third summand, if it exists, can be chosen as 2, 3 or 5.

Although Descartes' Conjecture is not equivalent to Goldbach's, the question arises: could Euler or Goldbach be aware of Descartes' Conjecture? Theoretically yes, since some copies of his notes and manuscripts circulated in Europe. However, the above two-line long conjecture was not included in his collected works which appeared in 1701 in Amsterdam. It is only contained in the edition of Descartes (1908), under Opuscula Posthuma, Excerpta Mathematica (Vol. 10, p. 298).

Apart from a numerical verification by Desboves (1855) up to 10,000 and Ripert (1903) up to 50,000 actually no result was proved before Landau's lecture. We have to mention however the conjectural asymptotic formula of J. J. Sylvester (1871) for the number $P_2(n)$ of representation of an even n as the sum of two primes:

$$P_2(n) \sim 4e^{-\gamma} C_0 \frac{n}{\log^2 n} \prod_{\substack{p \mid n \\ p > 2}} \left(1 + \frac{1}{p-2}\right), \tag{2.5}$$

where C_0 is the so-called twin prime constant,

$$C_0 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = 0.66016\dots \ . \tag{2.6}$$

Here and later p (as further on p', p'', p_i) will always denote primes, $\mathcal{P}$ will denote the set of all primes.

It was proved later by Hardy and Littlewood (1923) that this formula is definitely not correct. They expressed the same conjecture with $4e^{-\gamma}C_0$ replaced by $2C_0$. By now, we know that the analogue of (2.5) with $2C_0$ is true for almost all even numbers.

The only area where some non-trivial results existed before Landau's lecture was Problem 4 about large gaps between primes. Bertrand (1845) stated the assertion – called *Bertrand's Postulate* – that there is always a prime between n and $2n$. The same assertion – also without any proof – appeared about 100 years earlier in one of the unpublished manuscripts of Euler (see Narkiewicz (2000), p. 104). Bertrand's Postulate was proven already 5 years later by Čebyšev (1850). He used elementary tools to show

$$0.92129 \frac{x}{\log x} < \pi(x) < 1.10555 \frac{x}{\log x} \quad \text{for } x > x_0, \tag{2.7}$$

where $\pi(x)$ denotes the number of primes not exceeding x. Further on we will use the notation

$$d_n = p_{n+1} - p_n. \tag{2.8}$$

Čebyšev's proof implies

$$d_n < \left(\frac{6}{5} + \varepsilon\right) p_n \quad \text{for } n > n_0(\varepsilon). \tag{2.9}$$

The next step

$$d_n = o(p_n) \tag{2.10}$$

was a consequence of the Prime Number Theorem (PNT)

$$\pi(x) \sim \frac{x}{\log x} \sim \operatorname{li} x := \int_0^x \frac{dt}{\log t}, \tag{2.11}$$

shown simultaneously using different arguments by J. Hadamard (1896) and de la Vallée Poussin (1896).

The last step before 1912, the inequality

$$d_n < p_n \exp\left(-c\sqrt{\log p_n}\right) \tag{2.12}$$

was a consequence of the Prime Number Theorem with remainder term

$$\pi(x) = \operatorname{li} x + O\left(x \exp\left(-c\sqrt{\log x}\right)\right), \tag{2.13}$$

proved by de la Vallée Poussin (1899).

Finally we mention that H. Brocard (1897) gave an incorrect proof of the closely related conjecture that there exists a prime between any two consecutive triangular numbers. This shows that Problem 4 of Landau was examined before 1912, although in a slightly different form.

As we mentioned already, the Twin Prime Conjecture appeared in print already the first time in a more general form, due to de Polignac (1849):

(2.14) Every even number can be written in an infinitude of ways as the difference of two consecutive primes.

Kronecker (1901) mentioned the same conjecture (with reference to an unnamed writer) in a weaker form as

(2.15) Every even number can be expressed in an infinitude of ways as the difference of two primes.

Maillet (1905) commented on de Polignac's conjecture that

(2.16) Every even number is the difference of two primes.

When the even number is 2 or 4 then (2.14) and (2.15) are equivalent, otherwise (2.15) is weaker than (2.14), while (2.16) is weaker than (2.15). The form (2.16) is trivial for every concrete small even number and today we know its truth for almost all even numbers. In strong contrast to this we do not know whether there is any number for which (2.15) or (2.14) is true.

The Goldbach and Twin Prime Conjecture were mentioned in the celebrated address of Hilbert at the International Congress of Mathematicians in Paris, 1900 (see Hilbert (1935)). In his Problem No. 8 he mentioned them together with the Riemann Hypothesis, using the following words:

"After a comprehensive discussion of Riemann's prime number formula we might be some day in the position to give a rigorous answer on Goldbach's Problem, whether every even number can be expressed as the sum of two primes, further on the problem whether there exist infinitely many primes with difference 2 or on the more general problem whether the diophantine equation

$$ax + by + c = 0 \tag{2.17}$$

is always solvable in primes x, y if the coefficients a, b, c are given pairwise relatively prime integers."

There are close ties between Landau's problems. These connections depend strongly upon which formulation of the Conjectures (2.14)–(2.16) we consider. The first two are really generalizations of the Twin Prime Conjecture, the third one, (2.16) is obviously trivial if the difference is two. As the cited lines of Hilbert's lecture also indicate, both Goldbach's Conjecture and the Twin Prime Conjecture are special cases of linear equations of type (2.17) for primes. Using the formulation of (2.16) there is really a very strong similarity between the equations $p + p' = N$ and $p - p' = N$ for even values of N. In fact, most of the results for Goldbach's Conjecture are transferable to the other equation, too.

On the other hand, the Twin Prime Conjecture is also connected with Problem 4. The former one refers to the smallest possible gaps between consecutive primes, the latter one for the largest possible gaps.

Finally, the Twin Prime Conjecture and Problem 1 admit a common generalization, formulated first by A. Schinzel (Schinzel, Sierpiński 1958): if $f_1, \dots, f_k$ are irreducible polynomials in $\mathbb{Z}[X]$ and their product does not have a fixed factor, then for infinitely many integers n all values $f_i(n)$ are prime. Bateman and Horn (1962) formulated a quantitative form of it. The special case $f_i(x) = x + h_i$, $h_i \in \mathbb{Z}$ of Schinzel's conjecture was formulated by L. E. Dickson (1904) more than a hundred years ago, while the quantitative version of it is due to Hardy and Littlewood (1923).

In the simplest case $k = 2$, Dickson's conjecture is clearly equivalent to (2.15). On the other hand, if $k = 1$ and $f(x) = ax + h$, then this is Dirichlet's theorem (see Dirichlet (1837). Landau's Problem No. 1 is the simplest case of Schinzel's conjecture if $k = 1$ and $\deg f > 1$. There is no single non-linear polynomial for which we would know the answer for Schinzel's conjecture, even for $k = 1$. However, if primes are substituted by almost primes, then Schinzel's conjecture is true in case of $k = 1$ for an arbitrary polynomial f (see Section 19 for the case when f is an irreducible polynomial).

According to the above connections between Landau's problems we will organize the material into four areas as follows (the first three discussed in detail, the fourth one briefly):

(i) Large gaps between primes

(ii) Small gaps between primes and the prime k-tuple conjecture of Dickson, Hardy and Littlewood

(iii) Goldbach's Conjecture and numbers of the form $N = p_1 - p_2$

(iv) Approximations to Problem No. 1.

3. Upper bounds for large gaps between consecutive primes

As mentioned in Section 2, the only area where significant results existed before 1912, was the upper estimation of the differences $d_n = p_{n+1} - p_n$. These estimations were trivial consequences of the deep results (2.7), (2.11) and (2.13) concerning estimation and asymptotics of $\pi(x)$. However, this approach has its natural limits. The Riemann–Von Mangoldt Prime Number Formula (cf. Davenport (1980), Chapter 17)

$$\Delta(x) := \psi(x) - x := \sum_{n \le x} \Lambda(n) - x = -\sum_{|\gamma| \le T} \frac{x^\varrho}{\varrho} + O\left(\frac{x \log^2 x}{T}\right) \tag{3.1}$$

(where $\varrho = \beta + i\gamma$ denotes the zeros of Riemann's Zeta-function, $T \le x$ and $\Lambda(n) = \log p$ if $n = p^m$, $\Lambda(n) = 0$ otherwise) tells us that any zero ϱ itself implies an expected oscillation of size $x^\beta/|\varrho|$ for the remainder term $\Delta(x)$. Answering a question of Littlewood, this was proved rigorously in an effective way first by Turán (1950), later in an improved form by Pintz (1980a) and in the sharpest (in some sense optimal) form

$$\sup_x \frac{|\Delta(x)|}{x^\beta/|\varrho_0|} \ge \frac{\pi}{2}, \tag{3.2}$$

by Révész (1988).

The crucial observation, which helps to reach improvements of the estimate (2.12) is, that subtracting the two formulas of type (3.1) for $x + y$ and x we obtain

$$\psi(x+y) - \psi(x) = y - \sum_{|\gamma| \le T} \frac{(x+y)^\varrho - x^\varrho}{\varrho} + O\left(\frac{x \log^2 x}{T}\right) \tag{3.3}$$

and in (3.3) any single zero ϱ has only an effect of size

$$\frac{(x+y)^\varrho - x^\varrho}{\varrho} \le \min\left(2\frac{(x+y)^\beta}{|\varrho|}, yx^{\beta-1}\right), \tag{3.4}$$

which is alone always inferior to y. So, unlike in the problem of estimating $\Delta(x)$ as in (3.2), one single zero itself can never destroy everything. It is the number of zeros with large real part β and not too large imaginary parts $|\gamma|$, which influences the estimation of d_n. The earliest of such results, called today as density theorems, was reached by Carlson (1920):

$$N(\alpha, T) = \sum_{\substack{\zeta(\varrho)=0 \\ \beta \geq \alpha, |\gamma| \leq T}} 1 \ll T^{4\alpha(1-\alpha)+\varepsilon}. \tag{3.5}$$

Based on this, Hoheisel (1930) could reach the first result of type

$$d_n \ll p_n^{\vartheta_1} \qquad (\vartheta_1 < 1) \tag{3.6}$$

with the value $\vartheta_1 = 1 - 1/33\ 000$.

His result was improved later significantly as

$\vartheta_1 = 3/4 + \varepsilon$	Čudakov (1936),
$\vartheta_1 = 5/8 + \varepsilon$	Ingham (1937),
$\vartheta_1 = 3/5 + \varepsilon$	Montgomery (1969),
$\vartheta_1 = 7/12 + \varepsilon$	Huxley (1972).

The result of Ingham shows that we have always primes between n^3 and $(n+1)^3$ if n is sufficiently large. These results all showed beyond (3.6) that the PNT is valid in intervals of length x^{ϑ_1}.

$$\pi(x+y) - \pi(x) \sim \frac{y}{\log x} \qquad \bigl(y = x^{\vartheta_1}\bigr). \tag{3.7}$$

Riemann's Hypothesis (RH) implies (cf. v. Koch (1901))

$$\Delta(x) \ll \sqrt{x} \log^2 x, \tag{3.8}$$

and thereby

$$d_n \ll \sqrt{x} \log^2 x. \tag{3.9}$$

This result was improved under RH by Cramér (1920) to

$$d_n \ll \sqrt{x} \log x, \tag{3.10}$$

which still falls short of answering Landau's question No. 4 positively, even supposing RH.

In the case of the unconditional estimates the exponent 7/12 is still the best known one for which (3.7) holds. However, concerning (3.6), a breakthrough happened when Iwaniec and Jutila (1979) obtained by an ingenious combination of analytic and sieve methods the result

$$\vartheta_1 = 13/23. \tag{3.11}$$

A theoretical importance of the results of Iwaniec and Jutila was to overcome the 'parity obstacle' (to be discussed later in Sections 10–11) which in general prevents sieve methods to reveal the existence of primes in a suitable set (cf. Greaves (2001) p. 171).

The later developments all used both analytic and sieve methods and showed similarly to Iwaniec and Jutila (1979) an inequality weaker than (3.7) but stronger than (3.6), namely,

$$\pi(x+y) - \pi(x) \gg \frac{y}{\log x} \quad \bigl(y = x^{\vartheta_1}\bigr). \tag{3.12}$$

$$
\begin{array}{lll}
& \vartheta_1 = 11/20 & \text{Heath-Brown, Iwaniec (1979),} \\
& \vartheta_1 = 17/31 & \text{Pintz (1981, 1984),} \\
& \vartheta_1 = 23/42 & \text{Iwaniec–Pintz (1984),} \\
(3.13) & \vartheta_1 = 11/20 - 1/384 & \text{Mozzochi (1986),} \\
& \vartheta_1 = 6/11 & \text{Lou–Yao (1992, 1993),} \\
& \vartheta_1 = 7/13 & \text{Lou–Yao (1992, 1993),} \\
& \vartheta_1 = 107/200 & \text{Baker–Harman (1996).}
\end{array}
$$

Finally the best known result is

Theorem (Baker–Harman–Pintz (2001))**.** $d_n \ll p_n^{21/40}$.

If we are contented with results which guarantee the existence of primes in almost all short intervals of type

$$[x, x+y], \quad y = x^{\vartheta_2}, \tag{3.14}$$

then the method of Huxley (1972) yields this with (3.7) for

$$\vartheta_2 = 1/6 + \varepsilon. \tag{3.15}$$

Further, the combination of analytic and sieve methods lead to (3.12) in almost all short intervals with

$$
\begin{array}{lll}
& \vartheta_2 = 1/10 + \varepsilon & \text{Harman (1982)} \\
(3.16) & \vartheta_2 = 1/14 + \varepsilon & \text{Ch. Jia (1995a), Watt (1995)} \\
& \vartheta_2 = 1/15 + \varepsilon & \text{H. Z. Li (1997)} \\
& \vartheta_2 = 1/20 + \varepsilon & \text{Ch. Jia (1996a).}
\end{array}
$$

These results will have later also significance in the examination of gaps between consecutive Goldbach numbers in Section 14.

Landau's Problem No. 4 can be approximated in other ways if primes are substituted by almost primes. We call an integer a P_r number if it has at most r prime factors (counted with multiplicity). Already Viggo Brun (1920) showed that there is a P_{11} number in any interval of type $(x, x+\sqrt{x})$ for $x > x_0$ (consequently between neighboring squares, if $n > n_0$). After various improvements, J. R. Chen (1975) showed this for P_2 numbers, too.

Another approach is to show that we have a number n in every interval of type $(x, x+\sqrt{x})$ such that the greatest prime factor of it

$$P(n) > n^{c_1} \tag{3.17}$$

with a $c_1 < 1$, possibly near to 1. The first and the last results of this type are

$$
\begin{array}{lll}
(3.18) & c_1 = 15/26 = 0.5769\ldots & \text{Ramachandra (1969),} \\
& c_1 = 0.738 & \text{H. Q. Liu, J. Wu (1999).}
\end{array}
$$

It is interesting to observe that if we consider the slightly larger interval $[x, x+x^{1/2+\varepsilon}]$ then we have already numbers with much larger prime factors, namely $P(n) > n^{c_2}$, where again the first and the latest results are the following

$$
\begin{array}{lll}
(3.19) & c_2 = 2/3 - \varepsilon & \text{Jutila (1973),} \\
& c_2 = 25/26 - \varepsilon & \text{Ch. Jia, M. Ch. Liu (2000).}
\end{array}
$$

We mention that the methods leading to the strong results about $P(n)$, use, similarly to the work of Iwaniec and Jutila, a combination of analytic and sieve methods including the linear sieve with Iwaniec's bilinear expression of the error term.

4. The expected size of large gaps. Cramér's probabilistic model

Empirical data suggest that the largest gaps between primes are much smaller than the size $d_n < 2\sqrt{p_n}$ which would imply Landau's conjecture (and is of about the same strength). It was Cramér who first used a probabilistic approach to predict the size of the largest possible gaps between consecutive primes. His probabilistic model (Cramér (1935, 1936)) is a good starting point to formulate conjectures about the asymptotic behaviour of primes. Based on the Prime Number Theorem (2.11) he defined the *independent* random variables $\xi(n)$ for $n \geq 3$ by

$$\mathbb{P}(\xi_n = 1) = \frac{1}{\log n}, \quad \mathbb{P}(\xi_n = 0) = 1 - \frac{1}{\log n}. \tag{4.1}$$

On the basis of his model he conjectured

$$\limsup_{n\to\infty} \frac{d_n}{\log^2 p_n} = 1, \tag{4.2}$$

which would be true with probability 1 in his model.

This model would predict the truth of all four conjectures of Landau and seemed to scope with our knowledge about primes when used for *appropriate* problems. The Cramér model (CM) predicts namely asymptotically the same number of even and odd primes below a given bound, which is clearly not true. That made no obstacle as long as the mathematical community believed to know which are the *appropriate* problems. Cramér's model *predicted* the truth of PNT in short intervals of size $(\log x)^\lambda$ for any $\lambda > 2$, for example, that is, the relation (cf. (3.7))

$$\pi(x+y) - \pi(x) \sim \frac{y}{\log x}, \quad y = (\log x)^\lambda, \quad \lambda > 2. \tag{4.3}$$

It is naturally quite difficult to check numerically conjectures like (4.3) for all short intervals for really large values of x. However, the general belief was that this is an *appropriate* problem and CM can be applied to predict relations as (4.3) despite the obvious deficiencies of the model. It was therefore a great surprise when Maier (1985) showed that taking an arbitrarily large fix λ, the relation (4.3) will be always false for suitable values $x_n, y_n = (\log x_n)^\lambda \to \infty$.

As explained in Granville (1994, 1995) the reason why CM makes a wrong prediction for (4.3) is the same as mentioned already, the model does not contain the trivial information that *primes have no small divisors.* If CM is corrected in the way that all numbers having a divisor below a given parameter $z = z(x)$ $\big($with $\frac{\log z(x)}{\log\log x} \to \infty\big)$ are a priori excluded from the set of possible primes (and the remaining numbers are chosen with a probability proportional to $1/\log n$), then the contradiction discovered by Maier disappears. The corrected CM (CCM) will predict falsity of (4.3) for a suitable (rare) set of short intervals. On the basis of this corrected model, Granville (1993) conjectures that (4.2) holds with 1 replaced by $2e^{-\gamma}$.

However, the present author has shown that any type of modification preserving the independence of the variables ξ_n will still be in a 'non-trivial' contradiction with the true distribution of primes. We have, namely, still a contradiction with the global result

$$\frac{1}{X}\int_0^X \Bigg(\pi(x) - \sum_{2<n\leq x} \frac{1}{\log n}\Bigg)^2 dx \ll \frac{X}{\log^2 X}, \tag{4.4}$$

valid on RH. If RH is not true, then we have a much more significant contradiction with CM-type probabilistic models, since then we have much larger oscillation than $\sqrt{X}$ as shown by the result (3.2) of Révész. We remark that in case of existence of zeros with $\beta_0 > 1/2$, also the average size of the error is larger, as shown first by Knapowski (1959) and later in a stronger form by Pintz (1980b, c).

What makes the contradiction between (4.4) and Cramér's model more peculiar is the fact that the result (4.4) was proved 15 years before the discovery of Cramér's model and the mathematician who showed (4.4) was Harald Cramér (1920) himself.

The theorem below shows that in order to avoid conflict with reality, our set from which we choose our 'possible primes' (which was the set of numbers without prime divisors of size $O(\log^\lambda x)$ for any λ earlier) has to coincide nearly exactly with the set of primes. Our freedom is just to add a thin set of composite numbers to the primes whose cardinality is less than that of the primes by a factor at least $c \log x$. This means that any reasonable new model has to give up the simple condition of independence.

Theorem 1 (J. Pintz (200?)). *Let x be a large even number, $I = (x/2, x] \cap \mathbb{Z}$. Let S_x^* be arbitrary with*

$$\mathcal{P}_x^* := \mathcal{P} \cap I \subseteq S_x^* \subseteq I, \quad A = \frac{|I|}{|S_x^*|}. \tag{4.5}$$

Let us define independent random variables η_n for all $n \in I$ as

$$\eta_n = 0 \quad \text{if } n \notin S_x^*; \tag{4.6}$$

while for $n \in S_x^*$ let

$$\mathrm{P}(\eta_n = 1) = \frac{A}{\log n}, \quad \mathrm{P}(\eta_n = 0) = 1 - \frac{A}{\log n}. \tag{4.7}$$

Then the truth of the relation

$$\mathrm{D}^2\left(\sum_{n\in I} \eta_n\right) \ll \frac{x}{\log^2 x} \tag{4.8}$$

implies

$$|S_x^* \setminus \mathcal{P}_x^*| \ll \frac{x}{\log^2 x}. \tag{4.9}$$

5. Lower bounds for large gaps between primes. The Erdős–Rankin Problem

The Prime Number Theorem (2.11) obviously implies

$$\lambda := \limsup_{n\to\infty} \frac{d_n}{\log n} \geq 1. \tag{5.1}$$

This was improved to $\lambda \geq 2$ by Backlund (1929) and $\lambda \geq 4$ by Brauer, Zeitz (1930). Soon after this, further improvements were made. Westzynthius (1931) proved just one year later that $\lambda = \infty$, by showing

$$\limsup_{n\to\infty} \frac{d_n}{\log p_n \log_3 p_n / \log_4 p_n} \geq 2e^\gamma, \tag{5.2}$$

where $\log_\nu x$ denotes the ν-fold iterated logarithmic function. Erdős (1935) succeeded to improve $\log\log\log p_n$ to $\log\log p_n$. More precisely, he proved

$$\limsup_{n\to\infty} \frac{d_n}{\log p_n \log_2 p_n/(\log_3^2 p_n)} > 0. \tag{5.3}$$

Rankin (1938) could add a further factor $\log_4 p_n$ to it:

$$\limsup_{n\to\infty} \frac{d_n}{\log p_n \log_2 p_n \log_4 p_n/(\log_3^2 p_n)} \ge c_0 \text{ with } c_0 = 1/3. \tag{5.4}$$

The value of $c_0 = \frac{1}{3}$ was increased to e^γ by Ricci (1952) and Rankin (1962/63).

In 1979 Erdős offered a price of USD 10,000 for the proof that (5.4) is true with $c_0 = \infty$, the highest price ever offered by Erdős for any mathematical problem. Two improvements of the constant c_0 were reached in the past 28 years. Maier and Pomerance (1990) showed this with $c_0 = 1.31\ldots e^\gamma$, while the best known result is the following

Theorem (Pintz (1997))**.** (5.4) *is true with* $c_0 = 2e^\gamma$.

The usual way to find lower estimation for d_n is by showing a lower estimate for the function $J(x) = \max_{n\le x} j(n)$, where $j(n)$ stands for the maximal gap between consecutive integers prime to n (Jacobstahl's function).

The results before 1970 used Brun's sieve and estimates of de Bruijn for the number of integers below a given x composed of primes less than a suitably chosen $y = y(x)$. The work of Maier and Pomerance relied on deep analytic results about the distribution of generalized twin primes in arithmetic progressions. Finally, the work of the author needed beyond the tools of Maier and Pomerance a new result about colorings of graphs, which was shown in Pintz (1997) by probabilistic methods.

6. Small gaps between primes. Earlier results

Contrary to the uncertainty concerning the size of possible large gaps between primes, the smallest possible gaps d_n occurring infinitely often between consecutive primes are generally believed to be 2, as predicted by the Twin Prime Conjecture. Hence, we try to give upper estimates for the size of the small gaps in terms of p_n. Since the average value of d_n is $\log p_n$ by the Prime Number Theorem, analogously to (5.1) we try to give upper bounds for the corresponding quantity

$$\Delta_1 := \liminf_{n\to\infty} \frac{d_n}{\log p_n} \le 1. \tag{6.1}$$

The advance in case of the analogous problem of lower estimation of λ was rather quick. One year after the first non-trivial estimate of Backlund (1929) the bound $\lambda \ge 4$, two years after it, $\lambda = \infty$ was reached. This was not the case with this problem (cf. (6.7)). The first non-trivial result was reached 80 years ago: Hardy and Littlewood (1926) showed

$$\Delta_1 \le 2/3 \ \text{ on GRH} \tag{6.2}$$

by the circle method, where GRH stands for the Generalized Riemann Hypothesis. It was 14 years later that Rankin (1940) improved (6.2) to $\Delta_1 \le 3/5$, also assuming

GRH. In the same year Erdős (1940) succeeded to obtain the first unconditional estimate

$$\Delta_1 < 1 - c \tag{6.3}$$

with an unspecified but explicitly calculable small positive constant c. He could namely show that values of d_n cannot accumulate too strongly around the mean value $\log p_n$, since every even value $2k$ appears as a difference of two primes p_1, p_2 at most

$$C\mathfrak{S}(2k)\frac{N}{\log^2 N} \qquad \left(\mathfrak{S}(2k) := \prod_{p\mid k,\, p>2}\left(1 + \frac{1}{p-2}\right)\right) \tag{6.4}$$

times for $p_1 = p_2 + 2k \le N$.

This result was improved by Ricci (1954) to $\Delta_1 \le 15/16$, later by Wang, Xie, Yu (1965) to $\Delta_1 \le 29/32$.

A breakthrough came when Bombieri and Davenport (1966) refined and made unconditional the method of Hardy and Littlewood by substituting the Bombieri–Vinogradov theorem for the GRH and obtained $\Delta_1 \le 1/2$. They also combined their method with that of Erdős to obtain

$$\Delta_1 \le \frac{2+\sqrt{3}}{8} = 0.4665\ldots\ . \tag{6.5}$$

Their result was further improved to

(6.6)

$0.4571\ldots$	Pilt'ai (1972),
$0.4542\ldots$	Uchiyama (1975),
$0.4463\ldots$	Huxley (1973),
$0.4425\ldots$	Huxley (1977),
$0.4393\ldots$	Huxley (1984),
0.4342	Fouvry, Grupp (1986).

Finally, Maier (1988) succeeded to apply his celebrated matrix method to improve Huxley's estimate by a factor $e^{-\gamma}$, where γ is Euler's constant. He obtained

$$\Delta_1 \le e^{-\gamma} \cdot 0.4425\cdots = 0.2484\ldots, \tag{6.7}$$

which was the best result until 2005.

The method of Bombieri and Davenport (1966) was also suitable to give an estimate for chains of consecutive primes. They showed

$$\Delta_\nu = \liminf_{n\to\infty} \frac{p_{n+\nu} - p_n}{\log p_n} \le \nu - \frac{1}{2}, \tag{6.8}$$

which was improved by Huxley (1968/69, 1977) to

$$\Delta_\nu \le \nu - \frac{5}{8} + O\left(\frac{1}{\nu}\right). \tag{6.9}$$

Finally, similarly to the case $\nu = 1$, Maier (1988) obtained an improvement by a factor $e^{-\gamma}$:

$$\Delta_\nu \le e^{-\gamma}\left(\nu - \frac{5}{8} + O\left(\frac{1}{\nu}\right)\right). \tag{6.10}$$

The method of Huxley (1968/69) also yielded an extension of the result (6.9) to small gaps between consecutive primes in an arithmetic progression of a fixed difference q.

Finally we have to mention an important conditional result of Heath–Brown (1983). He proved that the existence of Siegel zeros implies the Twin Prime Conjecture, and more generally that every even number can be expressed in infinitely many ways as the difference of two primes. Naturally most mathematicians believe that there are no Siegel zeros. (The truth of GRH trivially implies this, for example.) So there is not much hope to prove the Twin Prime Conjecture via Heath-Brown's result. However, this result means that if we try to prove the Twin Prime Conjecture, or any weaker version of it as the Small Gap Conjecture or Bounded Gap Conjecture (see Section 7), for example, then we are entitled to assume that there are no Siegel zeros. In the light of the results of the next section it is also interesting to note that both

(i) the existence of Siegel zeros, that is, extreme irregularities in the distribution of primes in some arithmetic progressions (AP), and

(ii) improvements of the Bombieri–Vinogradov theorem, that is, a very regular distribution of primes in most AP's

imply the Bounded Gap Conjecture.

7. Small gaps between primes. Recent results

In the present section, extending the discussion of Section 2, we will formulate in more detail several conjectures related to the Twin Prime Conjecture and describe some recent results about them. All these results were reached in collaboration with D. A. Goldston and C. Y. Yıldırım. The results of Section 6 raised the goal to prove the

Small Gap Conjecture. $\Delta_1 = 0$,

as an approximation to the Twin Prime Conjecture. A much better approximation would be to show the

Bounded Gap Conjecture. $\liminf_{n\to\infty}(p_{n+1} - p_n) < \infty$.

It turned out (as it often happens in mathematics) that in order to approach the above weaker form of the Twin Prime Conjecture it is worth to examine the much stronger generalizations of it, formulated in a qualitative form by Dickson (1904), and in a quantitative form by Hardy and Littlewood (1923). Let $\mathcal{H} = \{h_i\}_{i=1}^k$ be a set composed of k distinct non-negative integers, and let us examine whether we have infinitely many natural numbers n such that all $n + h_i$ are simultaneously primes, that is

$$\{n + h_i\}_{i=1}^k \in \mathcal{P}^k \quad \text{i.o.,} \tag{7.1}$$

where i.o. stands for infinitely often.

Dickson (1904) formulated the conjecture that if a trivial necessary condition is true for $\mathcal{H}$, then (7.1) really happens for infinitely many values n. The condition is that the number $\nu_p(\mathcal{H})$ of residue classes covered by $\mathcal{H}$ mod p should satisfy

$$\nu_p(\mathcal{H}) < p \quad \text{for every prime } p. \tag{7.2}$$

Such sets $\mathcal{H}$ are called *admissible*. Hardy and Littlewood (1923) examined also the frequency of values n for which (7.1) is expected to be true. They arrived through an analytic way at the conjecture that

$$\sum_{\substack{n\le N\\ \{n+h_i\}\in\mathcal{P}^k}} 1 \sim \mathfrak{S}(\mathcal{H})\frac{N}{\log^k N}, \tag{7.3}$$

where $\mathfrak{S}(\mathcal{H})$ is the so-called singular series, the convergent non-negative product defined by

$$\mathfrak{S}(\mathcal{H}) := \prod_p \left(1 - \frac{\nu_p(\mathcal{H})}{p}\right)\left(1 - \frac{1}{p}\right)^{-k}. \tag{7.4}$$

It is easy to show that

$$\mathfrak{S}(\mathcal{H}) > 0 \Longleftrightarrow \mathcal{H} \text{ is admissible.} \tag{7.5}$$

A heuristic reasoning was provided by Pólya (1959) for the validity of (7.3) on a probabilistic basis (at least for $k = 2$). If the probabilities

$$n + h_i \in \mathcal{P}, \quad n + h_j \in \mathcal{P} \tag{7.6}$$

would be pairwise independent, we would obtain (7.3) without the extra factor $\mathfrak{S}(\mathcal{H})$. However, for a fixed p we have

$$\mathbb{P}\big(p \nmid (n + h_i),\ i = 1, 2, \dots, k\big) = 1 - \frac{\nu_p(\mathcal{H})}{p} \tag{7.7}$$

in contrast to $(1 - 1/p)^{-k}$, which would be the probability if the events $p \mid n + h_i$, $p \mid n + h_j$ would be pairwise independent. Hence we have to multiply the naive probability $(\log N)^{-k}$ with the product of all the correction factors $\bmod p$: this is exactly the quantity $\mathfrak{S}(\mathcal{H})$ in (7.4).

Since the conjecture about the infinitude of prime k-tuples is usually associated by the names of Hardy and Littlewood and they were the first who examined it in greater detail, we will define the qualitative form of it as

Hardy–Littlewood–Dickson (HLD) Conjecture. *If $\mathcal{H} = \{h_i\}_{i=1}^k$ is admissible, then all components $n+h_i$ are simultaneously primes for infinitely many natural numbers n.*

Since this conjecture is extremely deep, we will formulate an easier version of it as

HLD(k, ν) Conjecture. *If $\mathcal{H} = \{h_i\}_{i=1}^k$ is admissible, then there are at least ν primes among $\{n + h_i\}$ for infinitely many values of n.*

In order to see the depth of this we may remark that if there is any $k, \nu \ge 2$ and any single $\mathcal{H} = \{h_i\}_{i=1}^k$ for which the above conjecture is true, then the Bounded Gap Conjecture is obviously also true.

In the next section we will sketch an almost successful attempt to prove HLD$(k, 2)$ for sufficiently large values of k, which will, however, yield the truth of the Short Gap Conjecture. The method will also yield HLD$(k, 2)$ for sufficiently large values $k > k(\delta)$ and thus the Bounded Gap Conjecture if the Bombieri–Vinogradov Theorem can be improved as to include arithmetic progressions with differences up to $X^{1/2+\delta}$ with a fixed $\delta > 0$. We will introduce the

Definition. We say that ϑ is an admissible level of the distribution of primes if for any $A > 0$, $\varepsilon > 0$ we have

$$\sum_{q \le X^{\vartheta-\varepsilon}} \max_{\substack{a \\ (a,q)=1}} \left| \sum_{p \equiv a(q)} \log p - \frac{X}{\varphi(q)} \right| \ll_{\varepsilon,A} \frac{X}{(\log X)^A}. \tag{7.8}$$

By the Bombieri–Vinogradov theorem (Bombieri (1965), A. I. Vinogradov (1965)) the number $\vartheta = 1/2$ is an admissible level. The Elliott–Halberstam Conjecture (EH) asserts that $\vartheta = 1$ is an admissible level (see Elliott–Halberstam (1968/69)).

The following results are proved in Goldston–Pintz–Yıldırım (200?):

Theorem 2 (Goldston–Pintz–Yıldırım (200?))**.** *If the primes have an admissible level $\vartheta > 1/2$ of distribution, then for $k > C(\vartheta)$ any admissible k-tuple contains at least two primes infinitely often. If $\vartheta > 0.971$, then this is true for $k \ge 6$.*

Since the 6-tuple $(n, n+4, n+6, n+10, n+12, n+16)$ is admissible, the Elliott–Halberstam (EH) Conjecture implies

$$\liminf_{n\to\infty} d_n \le 16, \tag{7.9}$$

that is, $p_{n+1} - p_n \le 16$ for infinitely many n. Unconditionally we are able to show the truth of the Short Gap Conjecture.

Theorem 3 (Goldston–Pintz–Yıldırım (200?))**.** $\Delta_1 = \liminf\limits_{n\to\infty}(d_n/\log p_n) = 0.$

(For a simplified but self-contained proof of this assertion see Goldston–Motohashi–Pintz–Yıldırım (2006).)

Remark. We have to mention here that the HLD$(k,2)$ Conjecture is for any given value of k in some sense much stronger than its immediate consequence, the Bounded Gap Conjecture. Let us return, namely, for the three different analogues of the twin prime conjecture, given in (2.14)–(2.16), more specially to the formulation (2.15) of Kronecker. Let us denote by $\mathcal{K}$ the set of all even integers which can be expressed in an infinitude of ways as the difference of two primes. Although we believe that every even integer belongs to $\mathcal{K}$ (as formulated in (2.15)), even the assertion that $\mathcal{K} \ne \emptyset$ is very deep. The following assertion is trivial:

Proposition 1. *$\mathcal{K} \ne \emptyset$ is equivalent to the Bounded Gap Conjecture.*

On the other hand it is easy to show

Proposition 2. *If the* HLD$(k,2)$ *Conjecture is true for any given k then the lower asymptotic density of $\mathcal{K}$, $\underline{d}(\mathcal{K}) > c(k)$, with an explicitly calculable positive constant $c(k)$, depending only on k.*

The same approach yields more generally for blocks of consecutive primes the following

Theorem 4 (Goldston–Pintz–Yıldırım (200?))**.** *If the primes have an admissible level ϑ of distribution, then for $\nu \ge 2$ we have*

$$\Delta_\nu \le \left(\sqrt{\nu} - \sqrt{2\vartheta}\right)^2, \tag{7.10}$$

in particular we have on EH

$$\Delta_2 = \liminf_{n\to\infty} \frac{p_{n+2} - p_n}{\log p_n} = 0. \tag{7.11}$$

It is also possible to combine Maier's method with the method of Goldston–Pintz–Yıldırım (200?) to obtain an improved form of the result (6.10) of Maier, generalized for arithmetic progressions, where, beyond the results of Huxley (1968/69) we may allow q to tend (slowly) to infinity with N.

Theorem 5 (Goldston–Pintz–Yıldırım (2006))**.** *Let ν be an arbitrary fixed positive integer. Let ε and A be arbitrary fixed positive numbers. Let q and N be arbitrary, sufficiently large integers, satisfying*

$$q_0(A,\varepsilon,\nu) < q < (\log\log N)^A, \quad N > N_0(A,\varepsilon,\nu), \tag{7.12}$$

and let a be arbitrary with $(a,q)=1$. Let $p_1', p_2', \dots$ denote the consecutive primes $\equiv a (\operatorname{mod} q)$. Then there exists a block of $\nu+1$ primes $p_n', \dots, p_{n+\nu}'$ such that

$$\frac{p_{n+\nu}' - p_n'}{\varphi(q)\log p_n'} < e^{-\gamma}\left(\sqrt{\nu}-1\right)^2 + \varepsilon, \quad p_n' \in [N/3, N]. \tag{7.13}$$

Consequently,

$$\Delta_\nu(q,a) := \liminf_{n\to\infty} \frac{p_{n+\nu}' - p_n'}{\varphi(q)\log p_n'} \le e^{-\gamma}\left(\sqrt{\nu}-1\right)^2, \tag{7.14}$$

and in particular

$$\Delta_1(q,a) = 0. \tag{7.15}$$

The above results left open the quantitative question: how can we estimate d_n as a function of p_n from above, beyond the relation $d_n = o(\log p_n)$ infinitely often $\iff \Delta_1 = 0$. We mentioned that we are not able to show $d_n \le C$ infinitely often, for example. However, we were able to substantially refine the methods of proof of $\Delta_1 = 0$ as to yield the following result.

Theorem 6 (Goldston–Pintz–Yıldırım (200??))**.**

$$\liminf_{n\to\infty} \frac{d_n}{(\log p_n)^{1/2}(\log\log p_n)^2} < \infty. \tag{7.16}$$

We may remark that although the result $\Delta_1 = 0$ was proved 75 years later than the analogous $\lambda = \infty$ (cf. (5.2)), our present understanding (7.16) of small gaps is much better than of large gaps (cf. (5.4)).

8. The ideas of the proof on the small gaps

The idea is based on a variant of Selberg's sieve, which appears in connection with almost primes in Selberg (1991) in the special case $k=2$, and for general k in Heath-Brown (1997).

We can sketch the ideas leading to Theorems 2 and 3 as follows.

Step 1. Instead of the special problem of twin primes we consider the general problem of Dickson: we try to show that for any admissible set $\mathcal{H} = \{h_i\}_{i=1}^k$ we have infinitely many k-tuples of primes in $\{n+h_i\}_{i=1}^k$, that is

$$\{n+h_i\}_{i=1}^k \in \mathcal{P}^k \quad \text{i.o.} \tag{8.1}$$

The simultaneous primality of all components $n + h_i$ is essentially equivalent to

$$\Lambda(n;\mathcal{H}) := \sum_{d \mid P_{\mathcal{H}}(n)} \mu(d)\left(\log\frac{n}{d}\right)^k \ne 0 \quad \text{i.o.}, \tag{8.2}$$

where

$$P_{\mathcal{H}}(n) = \prod_{i=1}^{k}(n+h_i), \tag{8.3}$$

since the generalized von Mangoldt function $\Lambda(n;\mathcal{H})$ detects numbers $P_{\mathcal{H}}(n)$ with at most k different prime factors. If we would be able to evaluate for any fixed tuple $\mathcal{H}$ the average of $\Lambda(n;\mathcal{H})$ for $n \in (N, 2N]$, that is, $n \sim N$, this could answer our question. Unfortunately this is not the case, especially due to the large divisors d of $P_{\mathcal{H}}(n)$.

Step 2. As usual in sieve theory we try to approximate the detector function $\Lambda(n;\mathcal{H})$ of the prime k-tuples by the truncated divisor sum

$$\Lambda_R(n;\mathcal{H}) := \sum_{\substack{d \mid P_{\mathcal{H}}(n) \\ d \le R}} \mu(d)\left(\log \frac{R}{d}\right)^k. \tag{8.4}$$

In this case we can evaluate the average of $\Lambda_R(n;\mathcal{H})$ for $n \sim N$ if $R \le N(\log N)^{-B_0(k)}$. However, we do not obtain any direct arithmetic information from this. The reason for this is that, although we believe that $\Lambda_R(n;\mathcal{H})$ and $\Lambda(n;\mathcal{H})$ are on average close to each other, we have no means to prove this. Further, the sum (8.4) might take negative values too, which is a handicap for us.

Step 3. More generally we look for non-negative weights (depending on the given set $\mathcal{H}$, $|\mathcal{H}| = k$)

$$a(n) \ge 0 \ \text{ for } \ n \sim N, \quad A := \sum_{n \sim N} a(n) > 0, \quad w(n) = \frac{a(n)}{A}, \tag{8.5}$$

such that the average number of primes of the form $n + h_i$,

$$E(N;\mathcal{H}) = \sum_{i=1}^{k} \sum_{n \sim N} w(n)\chi_{\mathcal{P}}(n+h_i) \tag{8.6}$$

should be as large as possible, where $\chi_{\mathcal{P}}(n)$ is the characteristic function of primes,

$$\chi_{\mathcal{P}}(n) = 1 \ \text{ if } \ n \in \mathcal{P}, \quad \chi_{\mathcal{P}}(n) = 0 \ \text{ otherwise.} \tag{8.7}$$

If we obtain

$$E(N;\mathcal{H}) > 1 \ \text{ for } \ N > N_0 \tag{8.8}$$

(or at least for a sequence $N = N_\nu \to \infty$), we showed that there are at least two primes in the given k-tuple. If we obtained

$$E(N;\mathcal{H}) > k-1 \ \text{ for } \ N = N_\nu \to \infty, \tag{8.9}$$

then we would prove the HLD prime k-tuple conjecture for the given k-tuple. Some candidates for $a(n)$ are the following:

$$\begin{aligned} &a_1(n) = 1 && \Longrightarrow \quad E(N,\mathcal{H}) \sim \frac{k}{\log N} \ (N \to \infty) \\ &a_2(n) = \chi_{\mathcal{P}}(n+h_j) \text{ with a given } j && \Longrightarrow \quad E(N;\mathcal{H}) \ge 1 \\ &a_3(n) = 1 \text{ if } \{n+h_i\}_{i=1}^k \in \mathcal{P}^k && ? \\ &a_3'(n) = \Lambda(n,\mathcal{H}) && ? \end{aligned} \tag{8.10}$$

The first choice of the uniform weights $a_1(n)$ shows the difficulty of (8.8). Although $a_2(n)$ yields nearly (8.8), we have no idea how to proceed further. The other two, nearly equivalent choices $a_3(n)$ and $a_3'(n)$ could lead to (8.9) (in fact in case of $a_3(n)$ we would obtain $E(N, \mathcal{H}) = k$) if we had

$$A_3(N) = \sum_{n \sim N} a_3(n) > 0, \quad \text{resp.} \quad \sum_{n \sim N} a_3'(n) > 0. \tag{8.11}$$

However, $A_3(N) > 0$ for $N = N_\nu \to \infty$ is trivially equivalent to the HLD conjecture. So all these obvious choices look clearly dead-ends. Nevertheless, our final choice will still originate from $a_3'(n)$. As we have seen in Step 2, the truncated version $\Lambda_R(n, \mathcal{H})$ of $a_3'(n)$ can be evaluated on average and we believe that it is close to $a_3'(n)$ in some sense. Since it does not fulfill the non-negativity condition, we can try to square it and examine

$$a_4(n) = \Lambda_R^2(n; \mathcal{H}). \tag{8.12}$$

We can still evaluate their sum $A_4(N)$ asymptotically, but, due to the squaring, only for

$$R \le N^{1/2} (\log N)^{-B_1} \qquad B_1 = B_1(k). \tag{8.13}$$

If we restrict R further to

$$R = N^{\vartheta/2 - \varepsilon} \tag{8.14}$$

where ϑ is an admissible level for the distribution of primes (cf. (7.8)), we can also evaluate

$$E_i(N, \mathcal{H}) := \sum_{n \sim N} w_4(n) \chi_{\mathcal{P}}(n + h_i) = \frac{\vartheta - \varepsilon_0(k)}{k}. \tag{8.15}$$

This yields

$$E(N, \mathcal{H}) = \vartheta - \varepsilon_0(k), \quad \lim_{k \to \infty} \varepsilon_0(k) = 0, \tag{8.16}$$

which is much better than the result $k / \log N$ at the uniform measure, but still less than 1, even supposing EH, that is, $\vartheta = 1$. (However, as we will see later, this approach could already yield $\Delta_1 = 0$ on EH with some additional ideas.)

Step 4. Since we would be very happy to find at least two primes in the k-tuple $\{n + h_i\}_{i=1}^k$ for infinitely many n, there is no compelling (heuristic) reason to restrict our attention for the approximation of the detector function $\Lambda(n, \mathcal{H})$ (cf. (8.2)) of prime k-tuples. We can try also to approximate the detector function of those values n for which

$$\omega\big(P_H(n)\big) = \omega\left(\prod_{i=1}^k (n + h_i)\right) \le k + \ell, \quad 0 \le \ell < k - 2, \tag{8.17}$$

where ℓ is a free parameter, to be chosen later. This leads to the weight

$$a_5(n) = \Lambda_R^2(n; \mathcal{H}, \ell) := \left(\sum_{\substack{d \mid P_{\mathcal{H}}(n) \\ d \le R}} \mu(d) \left(\log \frac{R}{d} \right)^{k+\ell} \right)^2. \tag{8.18}$$

This approximation is twice as good as $a_4(n)$ in the sense that under the mild restriction

$$\ell = \ell(k) \to \infty, \quad \ell = o(k), \tag{8.19}$$

we obtain on average twice as many primes as in Step 4. Namely, supposing (8.19) with these weights we obtain in place of (8.16)

$$E(N,\mathcal{H}) = 2\vartheta - \varepsilon_1(k,\ell), \quad \lim_{k\to\infty} \varepsilon_1(k,\ell) = 0. \tag{8.20}$$

If $\vartheta > 1/2$ we immediately obtain (8.8), which implies Theorem 2. However, unconditionally we must take $\vartheta = 1/2$, so we have only

$$E(N,\mathcal{H}) = 1 - \varepsilon_1(k,\ell), \tag{8.21}$$

which is still weaker than the result by the trivial choice $a_2(n)$.

Step 5. We missed by a hairbreadth an unconditional proof of the existence of at least two primes in any k-tuple which implies the Bounded Gap Conjecture, but what about the Small Gap Conjecture, $\Delta_1 = 0$? The fact that we obtained *on average* already $1 - \varepsilon_1(k,\ell)$ primes is of crucial importance. We missed the proof of the Bounded Gap Conjecture but the primes we found during our trial are still there. If we can collect more than $\varepsilon_1(k,\ell)$ primes on average among

$$n + h, \quad 1 \le h \le H := \eta \log N, \quad h \ne h_i, \tag{8.22}$$

where $\eta > 0$ is an arbitrary fixed parameter, then we obtain $\Delta_1 = 0$. Since the weights $a_5(n)$ are not specially sensible for the primality (or prime divisors) of $n+h$ for $h \ne h_i$, we can expect, similarly to the uniform distribution $a_1(n)$, to obtain on average

$$\sum_{n\sim N} w_5(n)\chi_{\mathcal{P}}(n+h) \sim \frac{1}{\log N} \tag{8.23}$$

primes for any $h \ne h_i$. This would yield in total on average

$$\sim \frac{H-k}{\log N} \sim \frac{H}{\log N} = \eta > \varepsilon_1(k,\ell) \ \text{ if } \ k > k_0(\eta) \tag{8.24}$$

new primes among $n + h$ for $h \in [1,H]$, $h \ne h_i$.

This heuristic works in practice too, with a slight change. Although (8.23) is not true in the exact form given above, we can show the similar relation $(h \ne h_i)$

$$\sum_{n\sim N} w_5(n)\chi_{\mathcal{P}}(n+h) \sim \frac{\mathfrak{S}(\mathcal{H}\cup\{h\})}{\mathfrak{S}(\mathcal{H})} \cdot \frac{1}{\log N} \quad (N \to \infty). \tag{8.25}$$

After this, in order to show $\Delta_1 = 0$, it is sufficient to show

$$\frac{1}{X}\sum_{h=1}^{X} \frac{\mathfrak{S}(\mathcal{H}\cup\{h\})}{\mathfrak{S}(\mathcal{H})} \ge C(\mathcal{H}) \qquad X > X_0(\mathcal{H}) \tag{8.26}$$

for at least one choice of an admissible $\mathcal{H}$, $|\mathcal{H}| = k$ for any k (or for a series $k_\nu \to \infty$). Let us choose $\mathcal{H}$ as

$$P = \prod_{p\le 3k} p, \quad h_i = iP, \quad i = 1,2,\dots,k. \tag{8.27}$$

Then we have for any given even h

$$\nu'(p) := \nu_p(\mathcal{H}\cup\{h\}) \le \nu_p(\mathcal{H}) + 1 := \nu(p) + 1 \le k+1, \tag{8.28}$$

$$\begin{aligned}\frac{\mathfrak{S}(\mathcal{H}\cup\{h\})}{\mathfrak{S}(\mathcal{H})} &\geq \prod_{2<p\leq 3k}\frac{1-\frac{2}{p}}{\left(1-\frac{1}{p}\right)^2}\prod_{p>3k}\frac{1-\frac{\nu(p)+1}{p}}{\left(1-\frac{\nu(p)}{p}\right)\left(1-\frac{1}{p}\right)}\\ &\geq \prod_{2<p\leq 3k}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p>3k}\frac{1}{1+\frac{\nu(p)}{p(p-1-\nu(p))}}\\ &\geq \big(2C_0+o_k(1)\big)\exp\Bigg(-\sum_{p>3k}\frac{k}{p^2/2}\Bigg)\geq 2C_0+o_k(1),\end{aligned}\tag{8.29}$$

which clearly proves (8.26) with $C(\mathcal{H}) = C_0 = 0.66\ldots$ (cf. (2.6)). We remark that it is also easy to show (8.26) for any fixed admissible $\mathcal{H}$ and with the better lower estimate $1 + o_k(1)$. However, the exact analogue of (8.29) is not true for all admissible tuples $\mathcal{H}$, although some similar estimate can be given if $\mathcal{H}\cup\{h\}$ is admissible. To have an idea about (8.26) for general $\mathcal{H}$ we may note that the contribution of primes with $p > 3k$ is for any single h at least $1+o_k(1)$, as in (8.29). On the other hand, the contribution of the primes $p \leq 3k$ to the left-hand side of (8.29) depends only on h modulo

$$P_0 = P_0(3k) = \prod_{p\leq 3k} p,\tag{8.30}$$

and for a full period we have the average (with $\nu_p(\mathcal{H}) = \nu(p)$)

$$\frac{1}{P_0}\sum_{p=1}^{P_0}\prod_{p\leq 3k}\frac{1-\frac{\nu_p(\mathcal{H}\cup\{h\})}{p}}{\left(1-\frac{\nu(p)}{p}\right)\left(1-\frac{1}{p}\right)} = \prod_{p\leq 3k}\frac{\left(1-\frac{\nu(p)}{p}\right)\left(1-\frac{\nu(p)+1}{p}\right)+\frac{\nu(p)}{p}\left(1-\frac{\nu(p)}{p}\right)}{\left(1-\frac{\nu(p)}{p}\right)\left(1-\frac{1}{p}\right)} = 1.\tag{8.31}$$

We remark that changing the above argument slightly even the dependence of $o_k(1)$ on k can be omitted and we can prove for any given $\mathcal{H}$ and k

$$\liminf_{X\to\infty}\frac{1}{X}\sum_{h=1}^{X}\frac{\mathfrak{S}(\mathcal{H}\cup\{h\})}{\mathfrak{S}(\mathcal{H})}\geq 1.\tag{8.32}$$

9. Linear equations with almost primes

In Section 3 we mentioned the work of Brun (1920), according to which Problem 4 is true if primes are substituted by almost primes of the form P_{11} (that is, numbers with at most 11 prime factors). In the same work Brun showed the first significant results concerning the Goldbach and Twin Prime Conjectures. Using his sieve method he was able to prove that

$$P_9 = P_9' + 2 \quad \text{infinitely often}\tag{9.1}$$

and that all even numbers $N > 2$ can be expressed as

$$N = P_9 + P_9'.\tag{9.2}$$

Brun's sieve was used later to show several results of similar type, where by $\{a, b\}$ we abbreviate the assertion that every large even integer can be written as a sum of type $P_a + P_b$. The same method leads to results of type $P_a = P_b + 2$ or

$P_a = P_b + 2d$ for every integer d

$$
\text{(9.3)} \quad
\begin{array}{ll}
\{7,7\} & \text{H. Rademacher (1924),} \\
\{6,6\} & \text{T. Estermann (1932),} \\
\{5,7\},\ \{4,9\},\ \{3,15\}\ \{2,366\} & \text{G. Ricci (1936, 1937),} \\
\{5,5\} & \text{A. A. Buhštab (1938),} \\
\{4,4\} & \text{W. Tartakowski (1939a, 1939b),} \\
 & \text{A. A. Buhštab (1940).}
\end{array}
$$

These results were further improved by the weighted sieve of P. Kuhn (1941) who was able to show in that and later works (cf. Kuhn (1953, 1954)) that every even number can be written as a sum of two numbers having altogether at most 6 prime factors.

The first result where at least one of the summands could be taken as a prime $\{1, K\}$ was proved by A. Rényi (1947, 1948) using Linnik's large sieve. (Here K was a large unspecified constant.) The main novelty of the method was that he (essentially) showed that primes have a positive distribution level (cf. Section 7).

A few years later Selberg (1950) noted that his sieve can yield $\{2, 3\}$ without working out the details. In fact, the results below were reached by Selberg's sieve:

$$
\text{(9.4)} \quad
\begin{array}{ll}
\{3,4\} & \text{Wang Yuan (1956),} \\
\{3,3\} & \text{A. I. Vinogradov (1957),} \\
\{2,3\} & \text{Wang Yuan (1958).}
\end{array}
$$

For the detailed proof of Selberg see Selberg (1991).

The next developments were based on the method introduced by Rényi. The distribution level of primes was proved to be at least $\vartheta = 1/3$, later $3/8$ by Pan Cheng Dong and Barban. These results led to the strong approximation of the Goldbach and Twin Prime Conjectures as

$$
\text{(9.5)} \quad
\begin{array}{ll}
\{1,4\} & \text{Pan Cheng Dong (1962, 63), M. B. Barban (1963),} \\
\{1,3\} & \text{A. A. Buhštab (1965).}
\end{array}
$$

The celebrated Bombieri–Vinogradov theorem, the level $\vartheta = 1/2$ (cf. (7.8)) enabled a simpler proof of Buhštab's result $\{1, 3\}$, but did not yield $K = 2$.

The presently best known results were reached by Jing Rum Chen (1966, 1973). He used a form of Kuhn's weighted sieve, the switching principle and the newly invented Bombieri–Vinogradov theorem to show

Chen's Theorem. *Every sufficiently large even integer can be written as a sum of a prime and a P_2 number. Further, every even number can be written in infinitely many ways as the difference of a prime and a P_2 number.*

10. Small gaps between products of two primes. The Hardy–Littlewood–Dickson Conjecture for almost primes

Chen's theorem (see Section 9) showed that at least one of the equations ($p, p', p_i \in \mathcal{P}$)

$$p + 2 = p', \tag{10.1}$$

$$p + 2 = p_1 p_2 \tag{10.2}$$

has infinitely many solutions. The phenomenon that we cannot specify which one of the two equations (10.1) and (10.2) has infinitely many solutions (in reality most probably both) is the most significant particular case of the parity problem. This is a heuristic principle (observed and formulated by Selberg in the 1950's, see p. 204 of Selberg (1991) for example) stating that sieve methods cannot differentiate between integers with an even and odd number of prime factors.

Whereas Chen's theorem is relatively close to the Twin Prime Conjecture, it might be surprising to note that the seemingly much easier assertion that for infinitely many primes p, $p+2$ has an odd (or even) number of prime factors, is still open. The reason for it is the parity obstacle (cf. Hildebrand (2002)).

According to this, until very recently, problems involving numbers with *exactly* two distinct prime factors seemed to be as difficult as problems involving primes. We will introduce the

Definition. We call a natural number an E_2-number if it is the product of two distinct primes.

Denoting the consecutive E_2-numbers by $q_1 < q_2 < \dots$ we may remark that the analogue of $\Delta_1 = 0$ (cf. (6.1)), the relation

$$\liminf_{n\to\infty} \frac{q_{n+1} - q_n}{\log q_n / \log\log q_n} = 0 \tag{10.3}$$

was not known (the function in the denominator corresponds to the average distance between E_2-numbers as the function $\log p$ in case of primes).

In collaboration with D. Goldston S. W. Graham, C. Y. Yıldırım (Goldston, Graham, Pintz, Yıldırım 200?, 200?? to be abbreviated later by GGPY 200? and GGPY 200??) we examined various problems. The results obtained (cf. the present and the next section) showed that the method of Section 8 is suitable to discuss these problems as well. We are not only able to prove analogous results for E_2 numbers, but we obtain much stronger ones. For the difference of E_2 numbers we obtained the analogue of the Bounded Gap Conjecture in the following strong form.

Theorem 7 (GGPY 200?)**.** $\liminf_{n\to\infty}(q_{n+1} - q_n) \le 6.$

Let us consider more generally the appearance of almost primes in admissible k-tuples, the qualitative analogue of the Hardy–Littlewood–Dickson (HLD) conjecture for almost primes. (As we will see, Theorem 7 is the consequence of such a result – Theorem 9 – as well.) Chen's theorem (Section 9) gives a complete answer (for the qualitative case) for any integer d, we have

$$p + 2d \in \mathcal{P}_2 \quad \text{i.o.} \tag{10.4}$$

This trivially implies that we have infinitely often at least two P_2 numbers in any admissible k-tuple. In other words, the HLD$(k,2)$ Conjecture, formulated in Section 7 is true for P_2 numbers for any $k \ge 2$.

We will examine the problem, whether we can guarantee for every ν the existence of ν P_2-numbers (or at least ν P_r-numbers with a given fixed r, independent of ν) in any admissible k-tuple if k is sufficiently large, that is, $k \ge C_0(\nu)$.

Such a result seems to be unknown for any fixed value of r. The strongest result in this direction is due to Heath-Brown (1997). He showed for any admissible k-tuple $\{h_i\}_{i=1}^k$

$$\max_{1\le i\le k} \omega(n + h_i) < C \log k, \tag{10.5}$$

where $\omega(n)$ stands for the number of distinct prime divisors of n.

Our method enables us to prove that the HLD(k,ν) Conjecture is true for $k > C(\nu)$ if primes are replaced by E_2-numbers (or E_r-numbers, for any fixed r). This will imply the existence of infinitely many blocks of ν consecutive E_2-numbers with a bounded diameter (depending on ν) for any given ν. We remark that in case of primes we cannot prove $\Delta_\nu = 0$ if $\nu > 2$ (cf. (6.8)) and Theorem 5). In case of $\nu = 2$ we were able to show $\Delta_2 = 0$ (but not bounded gaps $p_{n+2} - p_n$ i.o.) only under the very deep Elliott–Halberstam Conjecture (cf. (7.10)–(7.11)).

We can prove the result in a more general form. Let

$$L_i(x) = a_i x + b_i \quad (1 \le i \le k) \quad a_i, b_i \in \mathbb{Z}, \quad a_i > 0 \tag{10.6}$$

be an admissible k-tuple of distinct linear forms, that is, we suppose that $\prod L_i(x)$ has no fixed prime divisor.

Theorem 8 (GGPY 200?). *Let D be any constant, $L_i(x)$ as above. Then there are $\nu + 1$ forms among them, which take simultaneously E_2-numbers with both prime factors above D if*

$$k \ge C_1(\nu) = (e^{-\gamma} + o(1))e^\nu. \tag{10.7}$$

Corollary. $\liminf_{n\to\infty}(q_{n+\nu} - q_n) \le C_2(\nu) = (e^{-\gamma} + o(1))\nu e^\nu.$

Theorem 8 does not specify how many forms we need, to find among them $\nu+1$ E_2-numbers for given small values of ν. The most important particular case is the following, which clearly implies Theorem 7.

Theorem 9 (GGPY2). *Let $L_i(x)$ be an admissible triplet of linear forms. Among these there exist two forms L_i, L_j such that $L_i(n)$ and $L_j(n)$ are both E_2-numbers infinitely often.*

11. Small gaps between almost primes and some conjectures of Erdős on consecutive integers

Erdős had many favourite problems on consecutive integers (see the work of Hildebrand (2002)). Let us denote by $\Omega(n)$ and $\omega(n)$, resp., the number of prime factors of n with and without multiplicity. Let $d(n)$ stand for the divisor-function.

The celebrated Erdős–Mirsky (1952) Conjecture refers to the divisor function, the others to consecutive values of Ω and ω.

Conjecture 1 (Erdős–Mirsky). $d(x) = d(x+1)$ *infinitely often.*

Conjecture 2 (Erdős (1983)). $\Omega(x) = \Omega(x+1)$ *infinitely often.*

Conjecture 3 (Erdős (1983)). $\omega(x) = \omega(x+1)$ *infinitely often.*

These conjectures would follow from the Twin Prime Conjecture with $x+1$ replaced by $x+2$ in the following sharp form:

$$d(x) = d(x+2) = 2, \quad \Omega(x) = \Omega(x+2) = \omega(x) = \omega(x+2) = 1 \text{ i.o.} \tag{11.1}$$

In their original forms Conjectures 1–3 would follow from an analogue of the conjecture of Sophie Germain. Sophie Germain conjectured that $2p+1 \in \mathcal{P}$ for infinitely many primes p. It is possible to show by Chen's method that, similarly to (10.1)–(10.2), either

$$2p + 1 \in \mathcal{P} \tag{11.2}$$

or

$$2p + 1 = p_1 p_2 \quad p_i \in \mathcal{P} \tag{11.3}$$

is true for infinitely many primes p. Now if (11.3) itself holds infinitely often, then C1–C3 hold, namely,

$$d(2p) = d(2p+1) = 4, \quad \omega(2p) = \Omega(2p) = \Omega(2p+1) = \omega(2p+1) = 2. \tag{11.4}$$

Due to this connection also Conjectures C1–C3 were considered extremely difficult, if not hopeless. (Problem 11.3 is believed to be of the same depth as the Twin Prime Conjecture, as remarked by Hildebrand (2002)).

It was a great surprise when C. Spiro (1981) could prove

$$d(n) = d(n + 5040) \quad \text{i.o.} \tag{11.5}$$

Independently of (11.5) Heath-Brown (1982) found a conditional proof of C2 under EH. Finally he succeeded (Heath-Brown (1984)) to show C1 unconditionally using the ideas of Spiro in combination with significant new ideas of himself.

His method led also to C2, but C3 remained open. C3 was proved just recently by J.-C. Schlage-Puchta (2003/2005). His method involved both theoretical and computational methods.

An important feature of all these results were, as pointed out by Heath-Brown (1982) in connection with the conditional solution of C2: 'It should be noted at this point that in solving $\Omega(n) = \Omega(n+1)$ we shall not have specified $\Omega(n)$, or even the parity of $\Omega(n)$. Thus we avoid the parity problem, rather than solve it.'

Our Theorem 9 yields in a rather quick way a new solution of Conjectures C1–C3 with the additional advantage that we can solve them even if the common consecutive value of $f(n) = f(n+1)$ ($f = d, \Omega$ or ω) is specified.

More precisely we can prove

Theorem 10 (GGPY 200??). *For any $A \geq 3$ we have $\omega(n) = \omega(n+1) = A$, i.o.*

Theorem 11 (GGPY 200??). *For any $A \geq 4$ we have $\Omega(n) = \Omega(n+1) = A$, i.o.*

Theorem 12 (GGPY 200??). *For any $24 \mid A$ we have $d(n) = d(n+1) = A$, i.o.*

Conjectures C1–C3 are also interesting if the shift 1 is replaced by 2 or by a general shift $b \in \mathbb{Z}^+$, that is, the problem, whether

$$f(n) = f(n + b) \tag{11.6}$$

holds infinitely often if $f = d, \Omega$ or ω.

These results were proved by C. Pinner (1997) for every value b using an ingenious extension of Heath-Brown's method for $f = d$ and Ω. Y. Buttkewitz (2003) extended Puchta's result for $f = \omega$ and for infinitely many integer shifts b.

Our methods yield a full extension of the results for $f = \omega$ and Ω with specified common values of $f(n) = f(n+b) = A$ and a partial extension for $d(n) = d(n+b) = A$.

Theorem 13 (GGPY 200??). *If $b \in \mathbb{Z}^+$, $A \geq 4$, then $\Omega(n) = \Omega(n+b) = A$, i.o.*

Theorem 14 (GGPY 200??). *If $b \in \mathbb{Z}^+$, $A \geq A(b)$, then $\omega(n) = \omega(n+b) = A$, i.o.*

Theorem 15 (GGPY 200??). *If $b \in \mathbb{Z}^+$, $b \not\equiv 15 \pmod{30}$, $48 \mid A$, then $d(n) = d(n+b) = A$, i.o.*

We will show below the simple deduction of Theorem 10 from Theorem 9 in the most important case $A = 3$. We consider the admissible system

$$18m + 1,\ 24m + 1,\ 27m + 1 \tag{11.7}$$

and the relations

$$\begin{aligned} 3(18m + 1) &= 2(27m + 1) + 1, \\ 4(18m + 1) &= 3(24m + 1) + 1, \\ 9(24m + 1) &= 8(27m + 1) + 1. \end{aligned} \tag{11.8}$$

Since by Theorem 9 at least two of the forms in (11.7) will be simultaneously E_2-numbers not divisible by 2 and 3 for infinitely many values m, we obtain by (11.8) a sequence $x_i \to \infty$ with

$$\omega(x_i) = \omega(x_i + 1) = 2 + 1 = 3. \tag{11.9}$$

The case of a general $A > 3$ can be deduced in a similar way from Theorem 9 with some additional ideas.

12. The exceptional set in Goldbach's Problem

Hardy and Littlewood (1924) examined the problem whether one can bound from above the number $E(X)$ of Goldbach exceptional even numbers below X, which cannot be expressed as a sum of two primes, i.e.

$$E(X) = |\mathcal{E}| = \big|\{n \le X,\ 2 \mid n,\ n \ne p + p'\}\big|. \tag{12.1}$$

They attacked the problem with the celebrated circle method, invented by Hardy, Littlewood and Ramanujan. They could not prove any result unconditionally. However, they showed

$$E(X) \ll X^{1/2+\varepsilon} \tag{12.2}$$

under the assumption of GRH. This result is even today the best conditional one, apart from the improvement of Goldston (1989/1992) who substituted X^ε by $\log^3 X$.

The first unconditional estimate of type $E(X) = o(X)$ was made possible more than a decade later by the method of I. M. Vinogradov (1937) which yielded the celebrated Vinogradov's three prime theorem. This theorem states that every sufficiently large odd number can be written as a sum of three primes.

Thus, using Vinogradov's method, van der Corput (1937), Estermann (1938) and Čudakov (1938) simultaneously and independently proved the unconditional estimate

$$E(X) \ll_A X(\log X)^{-A} \quad \text{for any} \quad A > 0. \tag{12.3}$$

This shows that Goldbach's Conjecture is true in the statistical sense that almost all even numbers are Goldbach numbers. The above result was the best known for 35 years when Vaughan (1972) improved it to

$$E(X) \ll X \exp\big(-c\sqrt{\log X}\big). \tag{12.4}$$

Just three years later Montgomery and Vaughan (1975) succeeded to show the very deep estimate

$$E(X) \le X^{1-\delta} \quad \text{for} \quad X > X_0(\delta), \tag{12.5}$$

with an unspecified but explicitly calculable, $\delta > 0$. Several attempts were made to show (12.5) with a reasonable (not too small) value of δ. These investigations led to the values

$$\begin{aligned} &\delta = 0.05 \quad \text{(Chen, Liu (1989))},\\ &\delta = 0.079 \ \text{(H. Z. Li (1999))},\\ &\delta = 0.086 \ \text{(H. Z. Li (2000a))}. \end{aligned} \tag{12.6}$$

Finally, the author could show unconditionally (see Section 18 for some details)

Theorem 16 (J. P.). *There exists a fix $\vartheta < 2/3$ such that $E(X) \ll X^{\vartheta}$.*

If we consider the analogous problem (cf. (12.6)) for the difference of primes, then the mentioned results are all transferable to this case as well, thereby furnishing an approximation to the Generalized Twin Prime Conjecture. Thus, defining analogously to $E(X)$ in (12.1)

$$E'(X) = \{n \le X, 2 \mid n, n \ne p - p'\}, \tag{12.7}$$

we can show that all but $X^{2/3}$ even integers below X can be written as the difference of two primes.

Theorem 17 (J. P.). *$E'(X) \ll X^{\vartheta}$ with a fixed constant $\vartheta < 2/3$.*

Naturally, we are not able to prove the stronger generalization of the twin prime conjecture (2.15) for any single even number N since this would imply the Bounded Gap Conjecture.

13. The Ternary Goldbach Conjecture and Descartes' Conjecture. Primes as a basis

The name Ternary Goldbach Conjecture refers to the conjecture that every odd integer larger than 5 can be written as a sum of three primes. This conjecture appeared first actually at Waring (1770). The first, albeit conditional result concerning this was achieved in the mentioned pioneering work of Hardy and Littlewood (1923). They showed that if there is a $\theta < 3/4$ such that no Dirichlet L-function vanishes in the halfplane $\operatorname{Re} s > \theta$ (a weaker form of GRH), then every sufficiently large odd number can be written as a sum of three primes. As an approximation to the Goldbach Conjecture we may consider the problem whether the set $\mathcal{P}$ of primes (extended with the element 0) forms a basis or an asymptotic basis of finite order. The existence of a number S^* such that every integer larger than 1 can be written as the sum of at most S^* primes was first proved by Schnirelman (1930, 1933). Let us denote by S the smallest number S^* with the above property. Similarly let S_1 denote the smallest number with the property that all sufficiently large numbers can be expressed as the sum of S_1 primes. As we mentioned in Section 2, the conjecture formulated by Descartes prior to Goldbach is equivalent to $S = 3$. So we may restate it as

Descartes' Conjecture. $S = S_1 = 3$.

Approaching the Goldbach Conjecture from this direction we can try to give upper bounds for S and S_1. The original work of Schnirelman relied on two basic results proved by Brun's sieve and combinatorial methods, respectively.

(i) The number $G(x)$ of Goldbach numbers below x satisfies $G(x) \ge cx$ for $x \ge 4$ with an absolute constant $c > 0$.

(ii) Every set $\mathcal{A} \subset \mathbb{Z}^+$ with positive lower asymptotic density forms an asymptotic basis of finite order.

This combinatorial approach led to the estimates

$$
(13.1)\qquad \begin{array}{ll} S_1 \le 2208 & \text{Romanov (1935)}, \\ S_1 \le 71 & \text{Heilbronn–Landau–Scherk (1936)}, \\ S_1 \le 67 & \text{Ricci (1936, 1937)}. \end{array}
$$

The breakthrough came in 1937 when I. M. Vinogradov (1937) invented his method to estimate trigonometric sums over primes and proved his celebrated theorem, according to which every sufficiently large odd number can be written as the sum of three primes, which implies

Theorem (I. M. Vinogradov)**.** $S_1 \le 4$.

Although his result is nearly optimal, it gave no clue for a good estimate of S. The estimates for S were first reached by the elementary method of Schnirelman. Later results used a combination of elementary and analytic methods involving in many cases heavy computations as well. Subsequent improvements for S were as follows:

$$
\begin{array}{ll} S < 2 \cdot 10^{10} & \text{Šanin (1964)}, \\ S \le 610^9 & \text{Klimov (1969)}, \\ S \le 159 & \text{Deshouillers (1972/73)}, \\ S \le 115 & \text{Klimov, Pilt'ai, Šeptickaja (1972)}, \\ S \le 61 & \text{Klimov (1978)}, \\ S \le 55 & \text{Klimov (1975)}, \\ S \le 27 & \text{Vaughan (1977)}, \\ S \le 26 & \text{Deshouillers (1975/76)}, \\ S \le 24 & \text{Zhang, Ding (1983)}, \\ S \le 19 & \text{Riesel, Vaughan (1983)}, \\ S \le 7 & \text{Ramaré (1995)}. \end{array}
$$

The best known conditional results are the following

Theorem (Kaniecki (1995))**.** *RH implies* $S \le 6$.

Theorem (Deshouillers, Effinger, te Riele, Zinoview (1997), Saouter (1998))**.** *The assumption of GRH implies the validity of the Ternary Goldbach Conjecture for every odd integer* $N > 5$*, and consequently the estimate* $S \le 4$.

It is unclear yet whether it is easier to deal with the Descartes Conjecture than with the (Binary) Goldbach Conjecture. Earlier methods for estimation of the exceptional set would yield the same estimates for the exceptional sets of the two different problems. However, the method leading to Theorem 16 (for a brief discussion see Section 18, for more details see Pintz (2006)) yields a better bound for this case. We can prove, namely,

Theorem 18. *All but* $O(X^{3/5} \log^{10} X)$ *integers below* X *can be written as the sum of at most three primes, where the last prime (if it exists) can be chosen as* 2, 3 *or* 5.

We remark that while the methods of proving Theorems 16 and 18 are similar, neither of one implies the other (however, the proof of Theorem 18 is easier, comparable to an estimate of type $E(X) \ll X^{4/5} \log^c X$ (cf. (12.1)) for Goldbach's Problem).

14. Gaps between consecutive Goldbach numbers

Denoting the consecutive Goldbach numbers by $4 = g_1 < g_2 < \dots$ we may try to give upper bounds for the occurring maximal gaps

$$A(X) = \max_{g_k \le X} (g_{k+1} - g_k) \tag{14.1}$$

in this sequence. Goldbach's Conjecture is naturally equivalent to $A(X) = 2$ for $X \ge 4$. Thus, the problem of upper estimation of $A(X)$ represents a new approximation to Goldbach's Conjecture. In other words we may ask: for which functions $f(X)$ can we guarantee at least one Goldbach number in an interval of type $(X, X + f(X))$. This problem differs from the other approximations in the following aspect. The sharpest results for the problems of Sections 12–13 and 15–18 all use the circle method, which is specifically designed to treat additive problems. In contrast to this, the best estimates for $A(X)$ can be derived from information concerning the distribution of primes.

The following proposition is contained in the special case $\vartheta_1 = 7/12 + \varepsilon$, $\vartheta_2 = 1/6 + \varepsilon$ in the work of Montgomery, Vaugham (1975)

Proposition. *Let us suppose we have four positive constants $\vartheta_1, \vartheta_2, c_1$ and $c_2 < c_1\vartheta_1$ with the following properties:*

(a) *every interval of type $[X - Y, X]$ with $X^{\vartheta_1} < Y < X/2$ contains at least $c_1 Y/\log X$ primes for any $X > X_0$,*

(b) *for all but $c_2 X/\log X$ integer values $n \in [X, 2X]$ the interval $[n - X^{\vartheta_2}, n]$ contains a prime for any $X > X_0$. Then*

$$A(X) \ll X^{\vartheta_1\vartheta_2}. \tag{14.2}$$

In such a way, the combination of any pair of estimates from (3.13) and (3.16) implies a bound for $A(x)$. Combining the result $\vartheta_1 = 21/40$ of Baker–Harman–Pintz (2001) with the estimate $\vartheta_2 = 1/20 + \varepsilon$ of Ch. Jia (and taking into account that the first mentioned work gives actually some exponent ϑ_1, slightly less than 21/40) we obtain

Theorem (Baker, Harman, Jia, Pintz)**.** *All intervals of type $[X, X + X^{21/800}]$ contain at least one Goldbach number, that is,*

$$g_{n+1} - g_n \ll g_n^{21/800} \Leftrightarrow A(X) \ll X^{21/800}. \tag{14.3}$$

The first conditional estimate,

$$g_{n+1} - g_n \ll \log^3 g_n \Leftrightarrow A(X) \ll \log^3 X \quad \text{on RH} \tag{14.4}$$

was proved by Linnik (1952), while the best is the following:

Theorem (Kátai (1967))**.** *RH implies $g_{n+1} - g_n \ll \log^2 g_n \Leftrightarrow A(X) \ll \log^2 X$.*

15. Goldbach exceptional numbers in short intervals

The results of Section 12 raise the problem whether we can prove the analogue of $E(X) = o(X)$ for short intervals, that is,

$$E(X, Y) := E(X + Y) - E(X) = o(Y), \tag{15.1}$$

for some function $Y = Y(X)$. The above relation means that almost all even integers are Goldbach numbers in a short interval of type $[X, X + Y(X)]$.

The first result of such type was proved by Ramachandra (1973) with an interval of type

$$Y = Y(X) = X^{\vartheta_3}, \quad \vartheta_3 = \frac{3}{5} + \varepsilon. \tag{15.2}$$

The result of Ramachandra was improved to $\vartheta_3 = 1/2 + \varepsilon$ in Perelli, Pintz (1992). Soon afterwards, simultaneously and independently Perelli–Pintz (1993) and Mikawa (1992) proved the significantly stronger estimates $\vartheta_3 = 7/36 + \varepsilon$ and $\vartheta_3 = 7/48 + \varepsilon$, respectively. A new feature of Mikawa's result was that, similarly to the work of Iwaniec and Jutila (1979), it was based on a combination of analytic and sieve methods. Further refinements of this method yielded the sharper results

$$\begin{aligned} &\vartheta_3 = 7/78 + \varepsilon && \text{Ch. Jia (1995b, 1995c),} \\ &\vartheta_3 = 7/81 + \varepsilon && \text{H. Z. Li (1995),} \\ &\vartheta_3 = 11/160 + \varepsilon && \text{Baker, Harman, Pintz (1995/97).} \end{aligned} \tag{15.3}$$

Finally, the best known estimates are the following.

Theorem (Ch. Jia 1996b). *Almost all even integers are Goldbach numbers in every interval of type* $[X, X + X^{\vartheta_3}]$, *for* $\vartheta_3 = 7/108 + \varepsilon$. *More precisely we have*

$$E(X, X^{\vartheta_3}) \ll_A X^{\vartheta_3} \log^{-A} X \quad \text{for any } A > 0. \tag{15.4}$$

Theorem (Kaczorowski–Perelli–Pintz (1993)). *Under the GRH we have*

$$E(X, \log^{6+\varepsilon} X) = o(\log^{6+\varepsilon} X) \quad \text{for any } \varepsilon > 0. \tag{15.5}$$

We conclude this section with a further problem, which is a combination of the approaches of Section 12 and the present section. We can namely ask for the shortest interval $Y = Y(X)$ for which we can guarantee beyond (15.1) an estimate of type

$$E(X, Y) \ll Y^{1-\delta}, \tag{15.6}$$

with a given absolute constant $\delta > 0$. The strongest known result of this kind was reached recently by A. Languasco (2004).

Theorem (Languasco). *The estimate* (15.6) *is true for* $Y = X^{7/24+\varepsilon}$ *if* $\varepsilon > 0$ *is arbitrary, with a suitably chosen* $\delta = \delta(\varepsilon) > 0$.

16. The Goldbach–Linnik Problem

As another approximation to Goldbach's Problem Linnik (1951, 1953) proved that every even integer can be expressed as a sum of two primes and K powers of two. In his original work K was an unspecified large number. One can try to show Linnik's theorem with explicitly given values of K, at least for even numbers $N > N_0$. The best possible estimate $K = 0$ is clearly equivalent to Goldbach's Conjecture for $N > N_0$.

Linnik's proof was significantly simplified by Gallagher (1975). Later explicit estimates for K were based on Gallagher's work:

$$\begin{aligned} &K = 54\,000 && \text{Liu–Liu–Wang (1998b),} \\ &K = 25\,000 && \text{H. Z. Li (2000b),} \\ &K = 2\,250 && \text{T. Z. Wang (1999),} \\ &K = 1\,906 && \text{H. Z. Li (2001).} \end{aligned} \tag{16.1}$$

The conditional estimates, proved under GRH, were the following:

$$\begin{aligned} &K = 770 && \text{J. Y. Liu, M. C. Liu, T. Z. Wang (1998a),}\\ &K = 200 && \text{J. Y. Liu, M. C. Liu, T. Z. Wang (1999),}\\ &K = 160 && \text{Tianze Wang (1999).} \end{aligned} \tag{16.2}$$

These results were improved by D. R. Heath-Brown, J.-C. Puchta (2002) to $K = 13$ (and $K = 7$ on GRH) and simultaneously and independently by I. Ruzsa and the author to

Theorem 19 (Pintz–Ruzsa (2003, 200?))**.** *All sufficiently large even integers can be expressed as a sum of two primes and at most eight powers of two. Under GRH the same result is true with at most seven powers of two.*

The proof of Theorem 19 relies on the Structural Theorem of Section 18 and on a much more effective treatment of the exponential sum $\sum\limits_{\nu=1}^{L} \exp(2^\nu \alpha)$ than those applied in (16.1)–(16.2).

As a natural refinement of the result of Linnik we may ask for an asymptotic formula for the number $R'_k(N)$ of representation of an even integere as the sum of two primes and k powers of two. This problem is open for evey value of k. Recently, however, in collaboration with A. Languasco and A. Zaccagnini, we were able to show the following

Theorem 20 (Languasco–Pintz–Zaccagnini (200?))**.** *Let k be any positive integer, X sufficiently large. Then, after the eventual deletion of at most $C(k)X^{3/5}(\log X)^{(10)}$ even integers below X, we can give an asymptotic formula for $R'_k(N)$ for the remaining even values of $N < X$.*

17. Further approximations to Goldbach's Conjecture

H. Mikawa (1993) studied the moments of the differences of Goldbach numbers $\mathcal{G} = \{g_k\}_{k=1}^{\infty}$,

$$M_\alpha(X) = \sum_{g_n \le X} (g^*_{n+1} - g_n)^\alpha, \quad g^*_k = \min(g_k, X), \quad \alpha \ge 0,\ 0^0 = 1. \tag{17.1}$$

He proved $M_3(X) \ll X \log^{300} X$ and

$$M_\alpha(X) = 2^{\alpha-1} X + o(X) \quad \text{for } \ 0 < \alpha < 3. \tag{17.2}$$

The above moments are sensible both for the total size $E(X)$ of the exceptional set (cf. (12.1)) and for the concentration of Goldbach exceptional numbers. As a sharper form of (17.2) we may formulate the

Conjecture A. $M_\alpha(X) = 2^{\alpha-1}X + O(X^{1-\delta(\alpha)})$ *holds for any* $\alpha \ge 0$ *and corresponding* $\delta(\alpha) > 0$.

To see the difficulty of Conjecture A we remark that

(i) for $\alpha = 0$ the assertion is equivalent to the deep theorem $E(X) \ll X^{1-\delta}$ of Montgomery and Vaughan (cf. (12.5)).

(ii) The truth of Conjecture A is equivalent to

Conjecture B. $g_{n+1} - g_n \ll_\varepsilon g_n^\varepsilon$ *for any* $\varepsilon > 0$.

Using the Structural Theorem of Section 18 and the theorem of Baker–Harman–Jia–Pintz (cf. (14.3)) about the gaps between consecutive Goldbach numbers we can deduce

Theorem 21 (J. P.)**.** *Conjecture A is true for any* $\alpha < \frac{341}{21} = 16.238\dots$

The following result refers to the concentration of Goldbach exceptional numbers.

Theorem 22 (J. P.)**.** *One can discard a set* $\mathcal{E}'$ *of at most* $O(X^{3/5}\log^{10} X)$ *Goldbach exceptional numbers* $m \in [X/2, X]$ *so that the remaining set would contain at most* C *Goldbach exceptional numbers in any interval of type* $[Y, Y + Y^{1/3}] \subseteq [X/2, X]$*, where* C *is an explicitly calculable absolute constant.*

The above result clearly implies $E(X) \ll X^{2/3}$.

In Section 16 we have seen that starting from any even number and subtracting from it a number of the form

$$\sum_{i=1}^{K} 2^{\nu_i} \quad K \le 8 \tag{17.3}$$

(an empty sum means 0) we arrive at a Goldbach number. The numbers of type (17.3) form a very thin set, having at most $(\log X)^K$ elements below X. One can prove the analogue of this result, too, that starting from any even number and adding to it a number of type (17.3) we can reach a Goldbach number. The result below shows that this can be achieved for any $N \le X$ with a set having just two suitably chosen elements below X.

Theorem 23. *Let* $X > X_0$*. Then we have integers* $a, b \le X$ *such that for every* $N \le X$ *at least one of* N*,* $N + a$*,* $N + b$ *is a Goldbach number*

From the above result we can further deduce the existence of an arbitrarily thin universal set $\mathcal{S}$ with

$$\mathbb{Z}^+ = \mathcal{G} - \mathcal{S}. \tag{17.4}$$

Theorem 24. *Let* $f(x)$ *be an arbitrary increasing function with* $\lim_{x\to\infty} f(x) = \infty$*. Then we can find a set* $\mathcal{S}$ *such that*

$$|n \le X; n \in \mathcal{S}| \le f(X) \text{ for } X > X_0, \quad \mathcal{G} - \mathcal{S} = \mathbb{Z}^+, \tag{17.5}$$

that is, every integer can be written as the difference of a Goldbach number and an element of $\mathcal{S}$*.*

18. Explicit formulas in the additive theory of primes

In this section we sketch one of the basic ideas behind the new approximation of Goldbach's Conjecture. In Section 2 we cited that Hilbert expressed his hope that the Riemann–von Mangoldt Prime Number Formula (3.1) might help in the solution of the Goldbach, Twin Prime and Generalized Twin Prime Conjectures. We will sketch below a two-dimensional analogue of (3.1), which plays a basic role in the proof of all Theorems 16–24 (in some cases directly, in some cases through Theorem 16, for example).

Let X be any large number, $P \le \sqrt{X}$ a parameter to be chosen later. We will apply the circle method with major and minor arcs defined by

$$\mathfrak{M} = \bigcup_{q \le p} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \left[\frac{a}{q} - \frac{P}{qX}, \frac{a}{q} + \frac{P}{qX} \right], \qquad \mathfrak{m} = [0,1] \setminus \mathfrak{M}. \tag{18.1}$$

We consider as usual

(18.2)

$$R(m) = \sum_{p_1+p_2=m} \log p_1 \cdot \log p_2 = \int_0^1 S^2(\alpha) e(-m\alpha) d\alpha = \int_{\mathfrak{M}} + \int_{\mathfrak{m}} = R_1(m) + R_2(m),$$

where

$$S(\alpha) = \sum_{X_1 < p \le X} \log p e(p\alpha), \qquad X_1 = X^{1-\varepsilon_0}, \quad L = \log X. \tag{18.3}$$

The contribution of the minor arcs can be estimated well on average by Parseval's identity and Vinogradov's method or Vaughan's method. This yields for $P \le X^{2/5}$

$$|\mathcal{E}_2| = \left| \left\{ m; 2 \mid m, m \in \left[\frac{X}{2}, X \right], R_2(m) > \frac{X}{\sqrt{L}} \right\} \right| \ll \frac{L^{10} X}{\sqrt{P}}. \tag{18.4}$$

In such a way many problems are reduced to the behaviour of $R_1(m)$. In the classical treatment $\mathfrak{M}$, that is, P is chosen in such a way that an asymptotic evaluation of $R_1(m)$ as

$$R_1(m) \sim \mathfrak{S}(m) := C_0 \prod_{p \mid m} \left(1 + \frac{1}{p-1} \right) \tag{18.5}$$

would be possible, due to the uniform distribution of primes in the arithmetic progressions with moduli $q \le P$. This requires by Siegel's theorem (cf. Siegel (1936)) the strong upper bound

$$P \ll \log^A X \tag{18.6}$$

for any arbitrarily large but fixed $A > 0$. This yields a relatively weak bound in (18.4) for $|\mathcal{E}_2|$.

The idea of Montgomery and Vaughan (1975) was to evaluate exactly the effect of the possible (essentially unique) Siegel zero and to show some weaker positive lower bound for $R_1(m)$ in place of (18.5) by using a deep theorem of Gallagher (1970) about the statistically good distribution of primes in arithmetic progressions modulo

$$q \le P := X^{\delta} \tag{18.7}$$

with a small but fixed positive value of δ.

As a generalization of this idea we evaluate exactly the effect of all 'generalized exceptional zeros' $\varrho = \beta + i\gamma$ of all L-functions modulo

$$q \le P := X^{2/5} \tag{18.8}$$

with

$$\beta \ge 1 - \frac{H}{\log X}, \qquad |\gamma| \le T, \tag{18.9}$$

where H and T are large parameters. We denote the set of the above zeros by $\mathcal{E}(H, T)$. In practice we may choose H and T as large absolute constants.

The evaluation gives rise to 'generalized singular series' $\mathfrak{S}(\chi_i, \chi_j, m)$ satisfying

$$|\mathfrak{S}(\chi_i, \chi_j, m)| \le \mathfrak{S}(\chi_0, \chi_0, m) = \mathfrak{S}(m). \tag{18.10}$$

The evaluation of $R_1(m)$ depends on the generalized exceptional zeros, whose number is bounded by Ce^{4H} by a density theorem of Jutila (1977). The arising explicit formula forms a basis for many further results.

Theorem 25 (J. P.)**.** *Under the above conditions we have for even $n \in [X/2, X]$ the explicit formula:*

$$\begin{aligned} R_1(m) = \mathfrak{S}(m) + \sum_{\varrho_i, \varrho_j \in \mathcal{E}(H,T)}\!\!\sum \mathfrak{S}(\chi_i, \chi_j, m) \frac{\Gamma(\varrho_i)\Gamma(\varrho_j)}{\Gamma(\varrho_i + \varrho_j)} m^{\varrho_i + \varrho_j - 1} \\ + O(Xe^{-cH}) + O(X^{-1/3H}) + O\left(\frac{X}{\sqrt{T}}\right), \end{aligned} \tag{18.11}$$

where the generalized singular series $\mathfrak{S}(\chi_i, \chi_j, m)$ satisfy

$$|\mathfrak{S}(\chi_i, \chi_j, m)| \le \varepsilon \tag{18.12}$$

unless the following three conditions all hold ($\operatorname{cond}\chi$ *is the conductor of χ, $C(\varepsilon)$ a constant depending on ε)*

$$\operatorname{cond}\chi_i \mid C(\varepsilon)m, \ \operatorname{cond}\chi_j \mid C(\varepsilon)m, \ \operatorname{cond}\chi_i\overline{\chi_j} < \varepsilon^{-3}. \tag{18.13}$$

Introducing the notation

$$\mathcal{E}_0(X) = \left\{ m; 2 \mid m, \ m \in \left[\frac{X}{2}, X\right], \ m \notin \mathcal{G} \right\}, \tag{18.14}$$

one can deduce from Theorem 24 results about the 'structure' of a set $\mathcal{E}_1(X)$ containing the even m's with $R_1(m) < \frac{X}{\sqrt{L}}$.

Theorem 26 (J. P., Weak Structural Theorem)**.** *There are positive absolute constants c_0, K and a set $\mathcal{E}_1(X)$ with the properties:*

(18.15)

$$\mathcal{E}_0(X) \subseteq \mathcal{E}_1(X) \cup \mathcal{E}_2(X), \ |\mathcal{E}_2(X)| \ll L^{10}X^{3/5}, \ \mathcal{E}_1(X) \subseteq \bigcup_{\nu=1}^{K} \mathcal{A}_{d_\nu}, \quad d_\nu > X^{c_0},$$

where $\mathcal{A}_d$ denotes the multiples of an integer d.

Theorem 26 clearly implies $E(X) \ll X^{1-\delta}$. However, in a much more sophisticated way we can show the much stronger

Theorem 27 (J. P., Strong Structural Theorem)**.** *Theorem 26 is true with a $c_0 > 1/3$.*

This result immediately yields Theorem 16 and plays a crucial role in the proofs of some other results among Theorems 16–24.

Finally we just briefly mention that an analogous explicit formula and results analogous to Theorems 19–27 can be proved for the representation of even integers as *differences* of two primes.

19. Approximations to Landau's first problem

In this last section we will briefly summarize the most important results in connection with the problem whether the polynomial n^2+1 represents infinitely many primes.

Let f be a polynomial with integer coefficients irreducible over the rationals and without a fixed prime divisor. Let $p(f)$ be the minimal number such that f represents infinitely often integers with at most $p(f)$ prime factors and let $\deg f$ be the degree of f. The first result,

$$p(f) \le 4\deg f - 1, \tag{19.1}$$

was reached more than 80 years ago by H. Rademacher (1924). Later results were

$$\begin{array}{ll} p(f) \le 3\deg f - 1 & \text{Ricci (1936),} \\ p(f) \le \deg f + c\log(\deg f) & \text{Kuhn (1953, 1954),} \\ p(f) \le \deg f + 1 & \text{Buhštab (1967).} \end{array} \tag{19.2}$$

If $\deg f = 2$, then first Kuhn (1953) proved $p(f) \le 3$, while the best result is at present

Theorem (Iwaniec (1978)). *If $\deg f = 2$ and $f(0)$ is odd, then $p(f) \le 2$.*

Corollary. *$n^2+1 = P_2$ infinitely often.*

Another approach is to ask Ω-type estimates about the largest prime divisor $P(f(n))$ of $f(n)$. For the special case of $f(n) = n^2+1$, such estimates were reached by Hooley (1967)

$$P(n^2+1) > n^{1.1} \quad \text{i.o.}; \tag{19.3}$$

finally the sharpest known estimate is

Theorem (Deshouillers, Iwaniec (1982)). *$P(n^2+1) > n^{1.202468\ldots}$ i.o.*

It is easy to see that Landau's first conjecture would follow if we could show

$$\{\sqrt{p}\} < \frac{c}{\sqrt{p}} \quad \text{i.o.,} \tag{19.4}$$

with a suitable constant c.

This was shown in the weaker form $\{\sqrt{p}\} \le cp^{-\alpha}$ with

$$\begin{array}{ll} \alpha = \dfrac{1}{15} - \varepsilon & \text{I. M. Vinogradov (1940),} \\ \alpha = \dfrac{1}{10} - \varepsilon & \text{I. M. Vinogradov (1976, Ch. 4),} \\ \alpha = 0.163\ldots & \text{Kaufman (1979).} \end{array} \tag{19.5}$$

The best known result is

Theorem (Balog (1983), Harman (1983)). *$\{\sqrt{p}\} \ll_\varepsilon p^{-\frac{1}{4}+\varepsilon}$ i.o.*

Finally we mention that Hardy and Littlewood (1923) expressed a number of conjectures in their landmark paper about additive problems involving primes. Some of them deal with prime values of polynomials; one of them is exactly the quantitative form of the 1st conjecture of Landau.

Conjecture (Hardy–Littlewood (1923))**.** *The number of primes $n^2 + 1 \le x$ is asymptotically equal to*

$$\prod_{p>2} \left(1 - \frac{1}{p-1}\left(\frac{-1}{p}\right)\right) \frac{\sqrt{X}}{\log X}. \tag{19.6}$$

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