Covers in Uniform Intersecting Families and a Counterexample to a Conjecture of Lovász

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We discuss the maximum size of uniform intersecting families with covering number at least $k$. Among others, we construct a large $k$-uniform intersecting family with covering number $k$, which provides a counterexample to a conjecture of Lovász. The construction for odd $k$ can be visualized on an annulus, while for even $k$ on a Möbius band.

1. Introduction

Let $X$ be a finite set. $\binom{X}{k}$ denotes the family of all $k$-element subsets of $X$. We always assume that $|X|$ is sufficiently large with respect to $k$. A family $\mathcal{F} \subseteq \binom{X}{k}$ is called $k$-uniform. The vertex set of $\mathcal{F}$ is $X$ and denoted by $V(\mathcal{F})$. An element of $\mathcal{F}$ is called an edge of $\mathcal{F}$. $\mathcal{F} \subseteq \binom{X}{k}$ is called intersecting if $F \cap G \neq \emptyset$ holds for every $F, G \in \mathcal{F}$. A set $C \subseteq X$ is called a cover of $\mathcal{F}$ if it intersects every edge of $\mathcal{F}$, i.e., $C \cap F \neq \emptyset$ holds for all $F \in \mathcal{F}$. A cover $C$ is also called $t$-cover if $|C| = t$. The covering number $\tau(\mathcal{F})$ of $\mathcal{F}$ is the minimum cardinality of the covers of $\mathcal{F}$. The degree of a vertex $x$ is defined by $\deg(x) := |\{F \in \mathcal{F} : x \in F\}|$.  

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For a family $\mathcal{A} \subset 2^X$ and vertices $x, y \in X$, we define $\mathcal{A}(x) := \{ A - \{x\} : x \in A \in \mathcal{A}\}$, $\mathcal{A}(\bar{x}) := \{ A : x \notin A \in \mathcal{A}\}$, etc., and for $Y \subset X$, $\mathcal{A}(Y) := \{ A : Y \subset A \in \mathcal{A}\}$, $\mathcal{A}(\bar{Y}) := \{ A \in \mathcal{A} : Y \cap A = \emptyset\}$. For a family $\mathcal{F} \subset \binom{X}{t}$ and an integer $t \geq 1$, define $\mathcal{C}_t(\mathcal{F}) = \{ C \in \binom{X}{t} : C \cap F \neq \emptyset \text{ holds for all } F \in \mathcal{F}\}$. Note that $\mathcal{C}_t(\mathcal{F}) = \emptyset$ for $t < \tau(\mathcal{F})$. Define

$$p_t(k) = \max\{|\mathcal{C}_t(\mathcal{F})| : \mathcal{F} \subset \binom{X}{k} \text{ is intersecting and } \tau(\mathcal{F}) \geq t\}.$$

Let us first list some useful facts concerning $p_t(k)$. Choosing $|\mathcal{F}| = 1$, one has $p_t(1) = k$.

(1) $p_{t+1}(k) \leq kp_t(k)$.

Proof. Take $\mathcal{F} \subset \binom{X}{t}$, $\mathcal{F}$ intersecting, $\tau(\mathcal{F}) = t + 1$ and $|\mathcal{C}_{t+1}(\mathcal{F})| = p_{t+1}(k)$. Define $\mathcal{C}_t = \mathcal{C}_{t+1}(\mathcal{F})$. Let $F \in \mathcal{F}$ be an arbitrary member of $\mathcal{F}$. By definition, $F \cap C \neq \emptyset$ holds for every $C \in \mathcal{C}_t$. Thus $|\mathcal{C}_t| \leq \sum_{x \in F} |\mathcal{C}_t(x)|$ holds. Therefore, in order to establish (1) it is sufficient to prove $|\mathcal{C}_t(x)| \leq p_t(k)$ for all $x \in F$. Consider $\mathcal{F}(x)$. It is intersecting and $t \leq \tau(\mathcal{F}(x)) \leq \tau(\mathcal{F}) = t + 1$. Moreover, $\mathcal{C}_t(x) \subset \mathcal{C}_t(\mathcal{F}(x))$ is immediate from the definitions. Thus $|\mathcal{C}_t(x)| = 0$ holds if $\tau(\mathcal{F}(x)) = t + 1$ and $|\mathcal{C}_t(x)| \leq p_t(k)$, otherwise.

(2) For $\mathcal{F} \subset \binom{X}{t}$, intersecting, $\tau(\mathcal{F}) = t$ and an arbitrary set $A \in \binom{X}{t}$ with $a < t$, one has $|\mathcal{C}_t(\mathcal{F})(A)| \leq p_{t-a}(k)$.

Proof. This follows from $\mathcal{C}_t(\mathcal{F})(A) = \mathcal{C}_{t-a}(\mathcal{F}(A))$.

(3) $p_2(k) = k^2 - k + 1$.

Using a construction described in the next section, it is not difficult to check that $p_2(k) \geq (k - 1)^3 + 3(k - 1)$ holds for all $k \geq 3$. Actually, this inequality is proved to be an equality if $k \geq 9$ in [4]. (The proof is not simple.) Later we prove $p_2(3) = 14$. The case $4 \leq k \leq 8$ remains open.

(4) For $k \geq k_0$, $p_2(k) = k^4 - 6k^3 + O(k^2)$, $p_3(k) = k^4 - 10k^3 + O(k^2)$.

Let us define

$$r(k) := \max\{|\mathcal{F}| : \mathcal{F} \text{ is } k\text{-uniform and intersecting with } \tau(\mathcal{F}) = k\}.$$

For example, $r(2) = 3$ and the only extremal configuration is a triangle. Note that $\mathcal{C}_t(\mathcal{F}) \subset \mathcal{F}$ for every intersecting $k$-uniform hypergraph, and
equality must hold if $|\mathcal{F}| = r(k)$ holds (together with $\tau(\mathcal{F}) = k$). Recall also, that $r(k) \leq k^4$ was proved by Erdős and Lovász [2]. Clearly, $p_r(k) \geq r(k)$. This inequality is likely to be strict for all $k \geq 3$. E.g. for $k = 3$ consider the family $\mathcal{F} = \{\{1, 2, 3\}, \{3, 4, 5\}, \{5, 6, 1\}, \{2, 4, 5\}, \{4, 6, 1\}, \{6, 2, 3\}\}$. Then $\mathcal{F} \subset \binom{\{1, \ldots, 6\}}{3}$ and $\tau(\mathcal{F}) = 3$ imply $|\mathcal{F}| = \binom{6}{3} - |\mathcal{F}| = 14(G \notin \epsilon(\mathcal{F})$ if $G$ is the complement of some $F \in \mathcal{F}$). On the other hand, $r(3) = 10$ is known. (See Appendix.)

(5) Suppose that $\mathcal{F} \subset \binom{\{1, \ldots, 6\}}{3}$ is an intersecting family with $\tau(\mathcal{F}) = 3$. Then there exists $x \in X$ such that $\deg(x) \geq 3$, and $|\mathcal{F}| \geq 6$.

**Proof.** We can choose $F, F' \in \mathcal{F}$ such that $F = \{1, 2, 3\}, F' = \{1, 4, 5\}$. There exists $G \in \mathcal{F}$ such that $G \cap \{2, 4\} = \emptyset$. If $1 \in G$, then $\deg(1) \geq 3$. Otherwise we may assume $G = \{3, 5, 6\}$. We can choose $G' \in \mathcal{F}$ such that $G' \cap \{3, 4\} = \emptyset$. Since $F' \cap G' \neq \emptyset$, we have $G' \cap \{1, 5\} \neq \emptyset$. This implies $\deg(1) \geq 3$ or $\deg(5) \geq 3$.

Next we prove $|\mathcal{F}| \geq 6$. Assume on the contrary that $|\mathcal{F}| \leq 5$. We choose $x \in X$ such that $\deg(x) \geq 3$. Thus the number of edges which do not contain $x$ is at most $2$. Let $F$ and $F'$ be such edges. Choose $y \in F \cap F'$. Then $\{x, y\}$ is a cover of $\mathcal{F}$, which contradicts $\tau(\mathcal{F}) = 3$. \[\]

(6) $p_3(3) = 14$.

**Proof.** First we consider the case that there exist $F, F' \in \mathcal{F}$ such that $|F \cap F'| = 2$. Let $F = \{1, 2, 3\}, F' = \{1, 2, 4\}$, and $\mathcal{E} = \epsilon(\mathcal{F})$. By (2) and (3), $|\mathcal{E}(1)| \leq 7$ and $|\mathcal{E}(2)| \leq 7$. Thus, since $F, F' \in \mathcal{E}(1) \cap \mathcal{E}(2)$, we have $|\mathcal{E}(1) \cup \mathcal{E}(2)| \leq 7 + 7 - 2 = 12$. Suppose $|\mathcal{E}| \geq 15$. Then $|\mathcal{E}(12)| \geq 3$. Every member of $\binom{\{1, 2\}}{1}$ must meet $F$ at $\{3\}$ and $F'$ at $\{4\}$, and hence $\{3, 4, 5\}, \{3, 4, 6\}, \{3, 4, 7\} \in \mathcal{E}$. Since $\mathcal{F}(34) \neq \emptyset$, we must have $\{5, 6, 7\} \in \mathcal{F}(34)$. But $F \cap \{5, 6, 7\} = \emptyset$, a contradiction.

Now we assume that $|F \cap F'| = 1$ holds for all distinct edges $F, F' \in \mathcal{F}$. Let $\mathcal{E} \in \epsilon(\mathcal{F})$. We may assume that $\deg(1) \geq 3$ (by (5)) and $\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\} \in \mathcal{F}$. Note that if $F \in \mathcal{E}(1)$ then $F \in \binom{\{1, 3\}}{2} \cup \binom{\{1, 4\}}{2} \cup \binom{\{1, 6\}}{2} \cup \binom{\{1, 7\}}{2}$. Consequently, there exist no other edges containing 1, i.e., $\deg(1) = 3$. Hence by (5), we have $\mathcal{F}(1) \geq 3$. Thus, we have $|\mathcal{E}(1)| \leq 2^3 - |\mathcal{F}(1)| \leq 5$. Therefore, $|\mathcal{E}| = |\mathcal{E}(1)| + |\mathcal{E}(1)| \leq 7 + 5 = 12$. \[\]

It is not difficult to check $p_{t+1}(k+1) \geq (k+1)$ and $p_k(k+1) \geq (k+1)$ for $t < k$ and $p_k(k+1) \geq (k+1)$ for $t = k$. Similarly $r(k+1) \geq (k+1)$ for $t < k$. This together with $r(2) = 3$, we obtain

(7) $r(k) \geq \lfloor k!(e-1) \rfloor$.

Actually, (7) was proved by Erdős and Lovász [2].
(8) Let \( k > k_0(\tau), |X| > n_0(k) \). Suppose that \( \mathcal{F} \subset \binom{X}{k} \) is an intersecting family with covering number \( \tau \). Then, \(|\mathcal{F}| \leq p_{\tau-1}(k)(\frac{|X|}{k})^{\tau-1} + O(|X|^{\tau-1})\) holds.

The above claim is proved in [4] for \( \tau = 4 \). One can prove the general case in the same way.

2. A Counterexample to a Conjecture of Lovász

Erdős and Lovász [2] proved that the maximum size of \( k \)-uniform intersecting families with covering number \( k \) is at least \( \lfloor k!(e-1) \rfloor \) and at most \( k^k \). Lovász [10] conjectured that \( \lfloor k!(e-1) \rfloor \) is the exact bound. This conjecture is true for \( k = 2, 3 \). However, for the case \( k \geq 4 \), this conjecture turns out to be false. In this section, we will construct \( k \)-uniform intersecting family with covering number \( k \) whose size is greater than \( ((k+1)/2)^k-1 \).

![Counterexample to a Conjecture of Lovász](image)

The constructions are rather complicated, therefore we first give an outline of them. There is a particular element \( x_0 \) which will have the unique highest degree in general. We construct an intersecting family \( \mathcal{F} \subset \binom{X}{k} \) with \( C(\mathcal{F}) = \{1\} \). (In the Erdős–Lovász case, and \( \tau \leq k \) in general.) Next we define \( \mathcal{B} := \{x_0\} \cup C : C \in \mathcal{F} \setminus \mathcal{F}_0 \). Finally, the family \( \mathcal{F}_0 = \mathcal{F}_0(k, \tau) \) is defined as \( \mathcal{F}_0 := \mathcal{F} \cup \{F \in \binom{X}{k}, \exists B \in \mathcal{B}, B \subset F\} \). Now we give the two examples, according to the parity of \( \tau \).

Example 1 (The Case \( \tau = 2s + 2 \)). Let \( h = k-s \). First we define an infinite \( k \)-uniform family \( \mathcal{G}^* = \mathcal{G}^*(h) \) as follows. Let

\[
V(\mathcal{G}^*) := \{(2i, 2j) : i \in \mathbb{Z}, 0 \leq j < h\} \\
\cup \{(2i+1, 2j+1) : i \in \mathbb{Z}, 0 \leq j < h\}.
\]

We define a broom structure \( \mathcal{G}_i \) as follows. A broom \( \mathcal{G}_i \) has a broomstick

\[
S_i := \{(i, j) : (i, j) \in V(\mathcal{G}^*), |S_i| = h\}
\]

and tails

\[
T_i := \{(i, j_0), (i + 1, j_1), (i + 2, j_2), \ldots, (i + s, j_s)\}:
\]

\[
j_{i+1} - j_i \in \{1, -1\} \text{ for } 0 \leq i \leq s
\]

where

\[
\begin{align*}
j_0 &= \begin{cases} h \quad &\text{if } h + i \text{ is even} \\ h - 1 \quad &\text{if } h + i \text{ is odd} \end{cases}
\end{align*}
\]

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Set $\mathcal{G} := \{ S \cup T : T \in \mathcal{F} \}$. Note that $\mathcal{G}$ is a $k$-uniform family with size $|\mathcal{F}| = 2^k$.

Now define $\mathcal{G}^* := \bigcup_{i \in \mathbb{Z}} \mathcal{G}_i$.

Next we define an equivalence relation $R(s)$ on $V(\mathcal{G}^*)$ induced by

$$(i, j) \equiv (i + 2s + 2h - 1 - j) \text{ for all } i \in \mathbb{Z} \text{ and } 0 \leq j \leq 2h - 1.$$  

Note that this equivalence transforms the infinite tape into a Möbius band. Finally, we define $\mathcal{G}$ as a quotient family of $\mathcal{G}^*$ by $R(s)$, that is, $\mathcal{G} := \mathcal{G}^*/R(s)$. Note that $|V(\mathcal{G})| = (2s + 1)h$.

**Example 2** (The Case $\tau = 2s + 1$). Let $h = k - s$, and

$$V(\mathcal{G}) := \{(2i, j) : i \in \mathbb{Z}_2, 0 \leq i < s, 0 \leq j \leq h\}$$

$$\cup \{(2i + 1, j) : i \in \mathbb{Z}_2, 0 \leq i < S, 0 \leq j \leq h\}$$

$$- \{(2i, 0) : i \in \mathbb{Z}_2, s \leq 2i < 2s\}$$

$$- \{(2i + 1, 2h + 1) : i \in \mathbb{Z}_2, s \leq 2i + 1 < 2s\}$$

Note that $|V(\mathcal{G})| = s(2h + 1)$. We define a broom structure $\mathcal{G}_i$ as follows. A broom $\mathcal{G}_i$ has a broomstick

$${\mathcal{S}}_i := \{(i, j) : (i, j) \in V(\mathcal{G})\},$$

with

$$\lvert {\mathcal{S}}_i \rvert = \cdots = \lvert {\mathcal{S}}_{i - 1} \rvert = h + 1, \quad \lvert {\mathcal{S}}_i \rvert = \cdots = \lvert {\mathcal{S}}_{2s + 1} \rvert = h$$

and tails

$${\mathcal{F}}_i := \{(i, j_0), (i + 1, j_1), (i + 2, j_2), \ldots, (i + u, j_u) : j_{i+1} - j_i \in \{1, -1\} \text{ for } 0 \leq \forall t < u\}$$

where

$$u := \begin{cases} s - 1 & \text{if } i \in \{0, 1, \ldots, s - 1\} \text{ (mod } 2s) \\ s & \text{if } i \in \{s, s + 1, \ldots, 2s - 1\} \text{ (mod } 2s) \end{cases}$$

and

$$j_0 := \begin{cases} h & \text{if } h + i \text{ is even} \\ h + 1 & \text{if } h + i \text{ is odd} \end{cases}$$

Set $\mathcal{G}_i := \{ S \cup T : T \in \mathcal{F}_i \}$, and define $\mathcal{G} := \bigcup_{0 \leq i \leq 2s} \mathcal{G}_i$.

**Remark.** In both examples, any edge of type $\{x_0, x_1, \ldots, x_{\tau - 2}\}$ ($x_i \in S_i$ for all $0 \leq j \leq \tau - 2$) is a cover of $\mathcal{G}$. This implies that $|{\mathcal{C}}_{i-1}(\mathcal{G})| \geq \prod_{j=0}^{i-2} |{\mathcal{S}}_j|$. 


Now we check that the above constructions satisfy the required conditions. It is easy to see that the family $\mathcal{G}$ is intersecting. But $\tau(\mathcal{G}) = \tau - 1$ is not trivial. We only prove the case $\tau = 2s + 2$, because the proof for the case $\tau = 2s + 1$ is very similar.

Let us consider properties of covers of $\mathcal{F}_0$. Define $I_j := \bigcup_{T \in \mathcal{F}_0} (S_j \cap T)$, $J_i := \bigcup_{T \in \mathcal{F}_0} T$, and fix a cover $C \in \mathcal{G}(\mathcal{F}_0)$. A vertex $y_j \in S_j$ is called suspicious (under C) if there exists $T = \{ y_0, y_1, ..., y_s \} \in \mathcal{F}_0$ ($y_j \in S_j$ for all $0 \leq j \leq s$) such that $\{ y_0, y_1, ..., y_s \} \cap C = \emptyset$. Let $L = L(C)$ be the set of all suspicious vertices.

Let us start with a trivial but useful fact.

**Claim 1.** If $C \cap I_{i+1} = \emptyset$ then $|L \cap I_{i+1}| \geq |L \cap I_i| + 1$ and equality holds only if $L \cap I_i$ consists of consecutive vertices on $I_i$.

The following fact is easily proved by induction on $i$.

**Claim 2.** Let $a = |C \cap I_i|$. Suppose that $|C \cap J_l| \leq l$ for all $0 \leq l < i$. Then $|L \cap I_i| \geq i - a + 1$ and equality holds only if $L \cap I_i$ consists of consecutive vertices on $I_i$.

The following is a direct consequence of the above fact.

**Proposition 1.** Suppose that $|C \cap J_l| \leq l$ for all $0 \leq l < i$ and $L \cap I_i = \emptyset$. Then $|C \cap I_i| \geq i + 1$ and equality holds only if $C \cap I_i$ consists of consecutive vertices on $I_i$.

**Proposition 2.** $\tau(\mathcal{G}) = 2s + 1$.

**Proof.** Let $C$ be any cover for $\mathcal{G}$. For each $0 \leq i \leq 2s$, we define the interval $W_i = [i, i + r] \pmod{2s + 1}$ so that $r$ is the minimum non-negative integer satisfying $|C \cap (S_i \cup S_{i+1} \cup ... \cup S_{i+r})| \geq r + 1$. In fact, such an integer $r$ exists by Proposition 1. The following claim can be shown easily.

**Claim 3.** If $W_i$ and $W_j$ have non-empty intersection, then $W_i \subset W_j$ or $W_j \subset W_i$ holds.

Using this, we can choose disjoint intervals from $W_0, W_1, ..., W_s$, whose union is exactly $[0, 2s]$. And so, $|C| \geq 2s + 1$. This completes the proof of $\tau(\mathcal{G}) = 2s + 1$.

Now we know that $\mathcal{F}_0 := \mathcal{G} \cup \{ F \in \binom{S}{\frac{s}{2}} \}$ is intersecting, and $\tau - 1 \leq \tau(\mathcal{F}_0) \leq \tau$. We can check that $\tau(\mathcal{F}_0) = \tau$ using the following easy fact.

**Proposition 3.** Let $\mathcal{G} \subset \binom{S}{\frac{s}{2}}$ be an intersecting family with $\tau(\mathcal{G}) = \tau - 1$. Define $\mathcal{B} := \{ \{ x_0 \} \cup C : C \in \bigcup_{i=-1}^{s-1} S_i(\mathcal{G}) \}$, $\mathcal{F} := \mathcal{G} \cup \{ F \in \binom{S}{\frac{s}{2}} \}$. Then $\tau(\mathcal{F}) = \tau$. The proof of this proposition is similar to the proof of Proposition 2.
\[ \exists B \in \mathcal{B}, B \subseteq F \} \]. Then \( \tau(\mathcal{F}) = \tau \) if and only if for all \( C \in \mathcal{C}_{\tau-1}(\mathcal{G}) \) there exists \( C' \in \mathcal{C}_{\tau-1}(\mathcal{G}) \) such that \( C \subseteq C' \).

Lovász conjectured that \( r(k) = \lceil k!(e-1) \rceil < c^3((k+1)/e)^{k+1} \). Our construction beats this conjecture as follows. Let \( \mathcal{G} \) be a \( k \)-uniform intersecting family defined in Example 1 or Example 2. Then \( \tau(\mathcal{G}) = k \). By Remark 1, we have the following lower bound.

**Theorem 1.**

\[
r(k) > |\mathcal{C}_{k-1}(\mathcal{G})| > \begin{cases} \left( \frac{k+1}{2} \right)^{k-1} & \text{if } k \text{ is even}, \\ \frac{k+3}{2}^{(k-1)/2} \left( \frac{k+1}{2} \right)^{(k-1)/2} & \text{if } k \text{ is odd}. \end{cases}
\]

Thus, our construction is exponentially larger than Erdős–Lovász construction.

3. **Open Problems**

**Problem 1.** Determine the maximum size of 4-uniform intersecting families with covering number four. Does \( r(4) = 42 \) hold?

**Problem 2.** Determine \( p_3(k) \) for \( 4 \leq k \leq 8 \). Does \( p_3(k) = k^3 - 3k^2 + 6k - 4 \) hold in these cases?

**Conjecture 1.** Let \( \mathcal{F} \subseteq \binom{X}{2} \) be an intersecting family with covering number \( \tau \). If \( k > k_0(\tau) \), \( |X| > n_0(k) \), then we have \( |\mathcal{F}| \leq |X|^{k-1} - \left( \frac{k}{2} \right)^{k-1} + c(k, \tau)(|X|^{k-1} - |X|^{k-1}) + O(|X|^{k-2}) \), where \( c(k, \tau) \) is a polynomial of \( k \) and \( \tau \), and the degree of \( k \) is at most \( \tau - 3 \).

Using (8), the above conjecture would follow from the following conjecture by setting \( \tau = t + 1 \).

**Conjecture 2.** Let \( k \geq k_0(t) \). Then \( p_3(k) = k^{t} - \left( \frac{3}{2} \right)k^{t-1} + O(k^{t-2}) \) holds.

This conjecture is true for \( t \leq 5 \) \footnote{5}. It seems that the coefficient of \( k^{t-2} \) in the above conjecture is \( (t/4)(t+1)(t^2-4t+7)/2 \).

For the case \( \tau = k \), we conjecture the following.
Conjecture 3. For some absolute constant \( \frac{1}{2} \leq \mu < 1 \), \( r(k) < (\mu k)^k \) holds.

We close this section with a bold conjecture.

Conjecture 4. Let \( k \geq \tau \geq 4 \) and \( n > n_0(k) \). Let \( \mathcal{F}_0 \) be the family defined in Example 1 or Example 2. Suppose that \( \mathcal{F} \subset \binom{\mathcal{F}_0}{\frac{k}{2}} \) is an intersecting family with covering number \( r \), then \( |\mathcal{F}| \leq |\mathcal{F}| \) holds. Equality holds if and only if \( \mathcal{F} \) is isomorphic to \( \mathcal{F}_0 \).

This conjecture is true if “\( k \geq 4 \) and \( \tau = 2 \)” or “\( k \geq 4 \) and \( \tau = 3 \)” or “\( k \geq 9 \) and \( \tau = 4 \)” (Inequality holds even if “\( k = 3 \) and \( \tau = 2 \)” or “\( k = 3 \) and \( \tau = 3 \)” but the uniqueness of the extremal configuration does not hold in these cases.) Of course, this conjecture is much stronger than Conjecture 1. Note that for \( k = \tau \) this conjecture would give the solution to the problem of Erdős–Lovász, and in particular, it would show that the answer to Problem 1 is 42.

4. Appendix

4.1. Numerical Data

The following is a table of the size of \( k \)-uniform intersecting families with covering number \( k \), i.e., known lower bounds for \( r(k) \).

<table>
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<th>( k )</th>
<th>Erdős–Lovász construction</th>
<th>Example 1</th>
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4.2. $k = \tau = 3$

The maximum size of 3-uniform intersecting families with covering number 3 is 10, i.e., $r(3) = 10$. There are 7 configurations which attain the maximum. The following is the list of these extremal configurations.

\[(\text{#1})\] 123 123 123 123
\[(\text{#2})\] 124 124 124 124
\[(\text{#3})\] 125 125 125 125
\[(\text{#4})\] 345 345 345 345
\[(\text{#5})\] 1 34 1 34 1 34 1 34
\[(\text{#6})\] 1 35 1 46 1 46 1 46
\[(\text{#7})\] 1 45 1 56 1 56 1 56
\[(\text{#7})\] 234 234 234 234
\[(\text{#8})\] 235 235 235 235
\[(\text{#9})\] 2 45 2 45 2 45 2 45
\[(\text{#10})\] 2 46 2 46 2 46 2 46

**References**

