Note

On Set Intersections

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Let $L$ be a finite set of nonnegative integers. Let $k$ and $n$ be natural numbers satisfying $n \geq k > \max L$. We call a family $\mathcal{F}$ of $k$-subsets of an $n$-set $X$ an $(n, k, L)$-system if $|F \cap F'| \in L$ for any $F, F' \in \mathcal{F}$, $F \neq F'$. We are interested in the maximum cardinality an $(n, k, L)$-system can have. We denote it by $f(n, k, L)$.

Ryser [4] proved that $f(n, k, \{1\}) \leq n$. This result has been generalized by Ray-Chaudhuri and Wilson [3] to

$$f(n, k, L) \leq \binom{n}{|L|}.$$

Deza, Erdös and Frankl [1] obtained that for $n > n_0(k)$,

$$f(n, k, L) \leq \prod_{l \in L} \frac{n - l}{k - l}.$$

Deza, Erdös and Singhi [2] proved that

$$f(n, k, \{0, 1\}) \leq n \quad \text{whenever} \quad l \neq k.$$

In the present note we are discussing the possible generalizations of this last result.

**Theorem 1.** Suppose that the greatest common divisor of the members of $L$ does not divide $k$. Then $f(n, k, L) \leq n$. 

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Proof. Let $p'$ be a prime power which divides each $i \in L$ but does not divide $k$.

Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be an $(n, k, L)$-system consisting of subsets of $X = \{1, \ldots, n\}$. Let $a_i = (a_{i1}, \ldots, a_{in})$ be the characteristic vector of $F_i$, i.e., $a_{ij} = 1$ if $i \in F_j$ and $a_{ij} = 0$ otherwise. We assert that the vectors $a_1, \ldots, a_m$ are linearly independent over the rationals. This implies the inequality $m \leq n$, hence the theorem.

Assume, to the contrary, that $\sum_{i} \gamma_i a_i = 0$, where we may suppose the $\gamma_i$'s to be integers with g.c.d. $(\gamma_1, \ldots, \gamma_m) = 1$. For $1 \leq q \leq m$, consider the inner product $0 = (a_q, \sum \gamma_i a_i) = \sum \gamma_i(a_q, a_i) - \sum \gamma_i |F_q \cap F_i| = \gamma_q |F_q|$ (mod $p'$). As $|F_q| = k \equiv 0$ (mod $p'$), we infer $p | \gamma_q$. This holds for $q = 1, \ldots, m$, a contradiction.

**Theorem 2.** Let $L = \{l_0, \ldots, l_{s-1}\}$ with $l_0 = 0$. Suppose we can choose $l_1, \ldots, l_t$ not necessarily different members of $L - \{0\}$ such that \( \sum_{q=1}^{t} l_{q} = k \). Then for $n \geq 2k^2$ we have $f(n, k, L) \geq n^2/4k^2$.

**Proof.** Let $p$ be the greatest prime not exceeding $n/k$. Then of course $p \geq n/2k$. Let us choose $k$ pairwise disjoint $p$-subsets $X_r$ of the $n$-set $X$; $X_r = \{x_r^1, \ldots, x_r^n\}$ ($r = 1, \ldots, k$). For $1 \leq i, j \leq p$ set

$$F_{i,j} = \{x_r^{h(r,i,j)}: r = 1, \ldots, k\},$$

where

$$h(r, i, j) = i + (q - 1)j \pmod{p},$$

for

$$\sum_{v=1}^{q} l_{v} < r \leq \sum_{v=1}^{q+1} l_{v}, \quad (0 \leq q < t).$$

Let $\mathcal{F} = \{F_{i,j}: 1 \leq i, j \leq p\}$.

We have $t \leq p$ since $t \leq k \leq n/2k \leq p$. Using this, one readily verifies that $\mathcal{F}$ is an $(n, k, L)$-system. We conclude that

$$f(n, k, L) \geq p^2 \geq n^2/4k^2.$$

**Corollary.** Let $n, k$ be positive integers and $L = \{l_0, l_1, \ldots, l_{s-1}\}$, where $0 = l_0 < l_1 < \cdots < l_{s-1}$, and $s \geq 3$. Assume that $n \geq 2k^2$ and $k \geq (s - 2) l_{s-1}/s - 2$. Then $f(n, k, L) \geq n^2/4k^2$ or $f(n, k, L) \leq n$ according to whether g.c.d.$(l_1, \ldots, l_{s-1})$ divides $k$ or not.

**Proof.** By Theorem 1 we may suppose that g.c.d.$(l_1, \ldots, l_{s-1})$ divides $k$. This implies that $\sum_{i=1}^{s-1} \gamma_i l_i = k$ for some integers $\gamma_1, \ldots, \gamma_s$. Let us choose the $\gamma_i$'s such that the sum of the negative $\gamma_i$'s is maximal. We assert that none of the $\gamma_i$'s is negative.
For, assume \( \gamma_j < 0 \). Then

\[
\sum_{i \neq j} \gamma_i l_i > k \geq (s - 2) l_{s-1} l_{s-2}.
\]

This implies that \( \gamma_q l_q > l_{s-1} l_{s-2} \) for some \( q \neq j \), hence \( (\gamma_q - l_j) l_q > l_{s-1} l_{s-2} - l_l q \geq 0 \). Now, setting \( \delta_j = \gamma_j + l_j \), \( \delta_q = \gamma_q - l_q \) and \( \delta_i = \gamma_i \) for \( i \neq j, q \), we arrive at a contradiction since \( \sum_{i=1}^{s-2} \delta_i l_i = k \) but the sum of the negative \( \gamma_i \)'s is strictly less than the sum of the negative \( \delta_i \)'s.

This proves that \( k \) can be written as a nonnegative integer linear combination of the \( l_i \)'s and \( f(n, k, L) \geq n^a/4k^a \) follows by Theorem 2.

REFERENCES