All Rationals Occur as Exponents

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For integers \( n \geq k \geq 1 \) and \( L \subseteq \{0, 1, ..., k - 1\} \), \( m(n, k, L) \) denotes the maximum number of \( k \)-subsets of an \( n \)-set so that the size of the intersection of any two among them is in \( L \). It is proven that for every rational number \( r \geq 1 \) there is a choice of \( k \) and \( L \) so that \( cn^r < m(n, k, L) < dn^r \), where \( c, d \) depend on \( k \) and \( L \) but not on \( n \).

1. INTRODUCTION

Suppose \( n \geq k \geq 1 \), \( L \subseteq \{0, 1, ..., k - 1\} \). Let \( X \) be a finite set, \( |X| = n \). A family \( \mathcal{F} \) of subsets of \( X \) is called an \( L \)-system if for any two distinct members \( F, F' \) of \( \mathcal{F} \) one has \( |F \cap F'| \in L \). Define

\[
m(n, L) = \{ \max |\mathcal{F}| : \mathcal{F} \text{ is an } L\text{-system} \};
\]

\[
m(n, k, L) = \{ \max |\mathcal{F}| : \mathcal{F} \text{ is an } L\text{-system and } |F| = k \text{ for all } F \in \mathcal{F} \}.
\]

There is a wide variety of problems related to \( m(n, L) \) and \( m(n, k, L) \). For example, \( m(n, k\{0, 1, ..., t - 1\}) \leq \binom{n}{t} \binom{t}{k} \) with equality holding if and only if a \((n, k, t)\)-Steiner-system exists. This already shows that the determination of these functions is hopeless in general. Let us mention three general upper bounds:

\[
(1) \quad m(n, L) \leq \sum_{i \in L} \binom{n}{i} \quad [13]
\]

\[
(2) \quad m(n, k, L) \leq \binom{n}{|L|} \quad [12]
\]

\[
(3) \quad m(n, k, L) \leq \prod_{l \in L} (n - l)/(k - l) \text{ for } n > n_0(k) \quad [1].
\]

Let us mention some of the recent papers concerning \( m(n, L) \) and \( m(n, k, L) \): [5, 6, 7, 8, 9, 10, 11, 14].

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Let us use the notation \( m(n, k, L) = \Theta(n^r) \) to denote that there exist constants \( c, d \) depending on \( k \) and \( L \) but not on \( n \) so that \( cn^r < m(n, k, L) < dn^r \). It is not known whether such an \( r \) exists for all choices of \( k \) and \( L \). However, if \( r = r(k, L) \) exists then obviously \( r \geq 1 \).

**Theorem 1.1.** For every rational number \( r, r \geq 1 \) there exists \( k \) and \( L \) so that \( m(n, k, L) = \Theta(n^r) \).

The author has examined all cases with \( k \leq 10 \) and proved the existence of \( r(k, L) \). Actually \( r(k, L) \) is an integer for all cases with \( k \leq 9 \) and all but two cases with \( k = 10 \). Its value in the exceptional cases is \( \frac{5}{2} \). In fact Theorem 1.1 follows from the following result.

**Theorem 1.2.** Suppose that \( s, d, a_0, a_1, \ldots, a_d \) are non-negative integers with \( s \geq d \geq 1, \ a_d \geq 1, \text{ and } a_i > \sum_{i=0}^{d} a_i(x_i^{-1}), \text{ define } p(x) = \sum_{i=0}^{d} a_i(x^i). \) Then

\[
m(n, p(s), \{ p(0), \ldots, p(s-1) \}) = \Theta(n^{s/d}).
\]

2. **Some Preparations**

A family \( \mathcal{A} \) of sets is called **closed under intersection** (or shortly **closed**) if \( A, A' \in \mathcal{A} \) implies \( A \cap A' \in \mathcal{A} \). Clearly, to every family \( \mathcal{B} \) there is a smallest closed family \( \overline{\mathcal{B}} \) with \( \mathcal{B} \subseteq \overline{\mathcal{B}} \). \( \overline{\mathcal{B}} \) is called the closure of \( \mathcal{B} \).

For an arbitrary set \( D \), the family \( \overline{\mathcal{B}} \mid D = \{ B \cap D : B \in \mathcal{B} \} \) is closed again.

By a simple averaging argument (cf. [3]) every \( \mathcal{F} \subset \binom{\mathcal{X}}{k} \) contains a subfamily \( \mathcal{F}' \), \( |\mathcal{F}'|/|\mathcal{F}| \geq k!/k^k \) and \( \mathcal{F}' \) being \( k \)-partite, i.e., there exist disjoint sets \( X_1, \ldots, X_k \) satisfying \( |F \cap X_i| = 1 \) for all \( F \in \mathcal{F}' \) and \( i = 1, \ldots, k \).

For a set \( G \) satisfying \( |G \cap X_i| \leq 1 \) define the canonical projection \( \pi(G) \) of \( G \) by \( \pi(G) = \{ i : |G \cap X_i| = 1 \} \). Note that \( |G| = |\pi(G)| \). Also, if \( \mathcal{A} \) is an arbitrary family and \( G \) as above, then the families \( \mathcal{A} \mid G \) and \( \pi(\mathcal{A} \mid G) = \{ \pi(A) : A \in \mathcal{A} \mid G \} \) are isomorphic.

**Theorem 2.1 ([8]).** Suppose \( \mathcal{F} \) is an \((n, k, L)\)-system. Then there exists a positive constant \( c(k, L) \), independent of \( n \), a closed \( L \)-system \( \mathcal{A} \subset 2^{\{1,2,\ldots,k\}} \), and \( \mathcal{F}^* \subset \mathcal{F} \) so that

(i) \( \mathcal{F}^* \) is \( k \)-partite,

(ii) \( |\mathcal{F}^*| \geq c(k, L) |\mathcal{F}| \),

(iii) For every \( F \in \mathcal{F}^* \) one has \( \pi(\mathcal{F}^* \mid F) = \mathcal{A} \).

Note that (iii) implies that \( \mathcal{F}^* \) is an \( L \)-system, i.e., the size of the intersection of any number of members of \( \mathcal{F}^* \) is in \( L \).
Since we are only interested in the order of magnitude of \( m(n, k, L) \), we may suppose \( S = S^* \). To express this fact we say that \( S \) is canonical, we call \( \mathcal{A} \) the intersection pattern of \( S \).

Let us mention without proof the following easy fact.

**Proposition 2.2.** Suppose \( \{ \mathcal{A}_1, \ldots, \mathcal{A}_m \} \) and \( \{ \mathcal{B}_1, \ldots, \mathcal{B}_m \} \) are two families of sets satisfying for all \( j \) and all \( 1 \leq i_1 < \cdots < i_j \leq m \),

\[
|A_{i_1} \cap \cdots \cap A_{i_j}| = |B_{i_1} \cap \cdots \cap B_{i_j}|
\]

Then they are isomorphic.

Suppose \( S \) is a canonical family with intersection pattern \( \mathcal{A} \). For \( A, B \in \mathcal{A} \) satisfying \( A \subseteq B \) and \( G \in S \) with \( \pi(G) = A \) define

\[
\mathcal{J}_S(A, B) = \{ H \in \mathcal{F} : G \supseteq H, \pi(H) = B \}.
\]

We say that \( B \) covers \( A \) if \( A, B \in \mathcal{A} \) and \( A \subseteq B \) but there is no \( C \in \mathcal{A} \) with \( A \subseteq C \subseteq B \).

**Lemma 2.3 (Monotony lemma).** Suppose \( A, B, C, D \in \mathcal{A} \) satisfy \( A \subseteq B \subseteq D \), with \( D \) covering \( B \), \( A \subseteq C \subseteq D \) and \( C \not\subseteq B \). Then for all \( G, H \in \mathcal{A} \) satisfying \( \pi(G) = A \), \( \pi(H) = B \), and \( G \subseteq H \) one has

\[
|\mathcal{J}_S(A, C)| \geq |\mathcal{J}_S(B, D)|.
\]

**Proof.** Suppose \( \mathcal{J}_S(B, D) = \{K_1, \ldots, K_s\} \). Let \( L_i \) be the unique subset of \( K_i \) satisfying \( \pi(L_i) = C \)—such \( L_i \) exists because \( C \subseteq D = \pi(K_i) \). In view of Theorem 2.1 (iii) \( L_i \in S \) holds.

Since \( A \subseteq C \) and \( G \subseteq H \), \( G \subseteq L_i \) holds. To conclude the proof we must show that the \( L_i \)'s are distinct.

Consider \( \pi(K_i \cap K_j) \) for \( i \neq j \). Since \( K_i \neq K_j \), it is a proper subset of \( D \), containing \( B \). As \( D \) covers \( B \), \( \pi(K_i \cap K_j) = B \) follows. Thus \( K_i \cap K_j = H \). Consequently \( L_i \cap L_j \subseteq H \). But \( \pi(L_i) = \pi(L_j) = C \) and \( C \not\subseteq \pi(H) = B \) proving \( L_i \neq L_j \).

3. **The Lower Bound in Theorem 1.2**

The construction we use here was given in [4]. Since we need it in the proof of the upper bound, we repeat it shortly.

Let \( b \) be an integer and \( Z \) a set of cardinality \( a_0 + a_1 b + a_2(b^2) + \cdots + a_d(b^d) \) which we consider as the disjoint union of \( a_i \) copies of \( \{1, \ldots, b\} \), \( i = 0, \ldots, d \). For \( A \subseteq [1, b] \), let \( \varphi(A) \) be the corresponding subset of \( Z \)
with $|\varphi(A)| = \sum_{i=0}^{d} a_i(i_A^d)$. It is very easy to check that for $A, B \subseteq [1, b]$, $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$ holds. Thus for $s$ arbitrary the family $\{\varphi(A) : A \subseteq [1, b], |A| \leq s\}$ is a closed $\{p(0), \ldots, p(s-1)\}$-system showing $m(p(b), p(s), \{p(0), \ldots, p(s-1)\}) \geq \binom{s}{2}$. By choosing $b = \Omega(n^{1/a})$ the desired lower bound follows.

4. Proof of the Upper Bound Part of Theorem 1.2

W.l.o.g. let $\mathcal{F}$ be a canonical closed $L$-system, $L = \{p(0), \ldots, p(s-1)\}$, let $\mathcal{A}$ be the sample family on $[1, p(s)]$. Also, let $\mathcal{P} = \mathcal{P}(s)$ be the sample family from our construction (the point set $Y$ of $\mathcal{P}(s)$ is the disjoint union of $a_0$ copies of the singleton $(1,0)$, $a_1$ copies of $(1,1)$, $\ldots$, $a_d$ copies of $(1,d)$), say

$$Y = \bigcup_{i=0}^{d} \bigcup_{1 \leq j \leq a_i} Y_{ij}.$$

For a subset $B \subseteq [1, s]$ denote by $p(B)$ the subset of size $p(|B|)$ of $Y$ which is the union of the corresponding subsets of $Y_{ij}$. Then

$$\mathcal{P} = \{p(B) : B \subseteq [1, s]\}.$$

Note that as a lattice $\mathcal{P}$ is isomorphic to $A^{[1,s]}$, in particular, all maximal chains have the same size $s$.

We are going to show that $\mathcal{A}$ can be embedded into $\mathcal{P}$, that is, there exists a 1-1 map $\varphi : [1, p(s)] \rightarrow Y$ so that $\varphi(A) \in \mathcal{P}$ holds for all $A \in \mathcal{A}$.

Call a subset $C \subseteq [1, p(s)]$ an atom if $C \cap A \neq \emptyset$ implies $C \subseteq A$ for all $A \in \mathcal{A}$. An element $x \in A \in \mathcal{A}$ is a generic point for $A$ if for all $B \in \mathcal{A}$, $x \in B$ implies $A \subseteq B$.

Note that if $C$ is an atom, $|C| = a_1$, then $\mathcal{A} \cup \{C\}$ will be a closed family. Adding atoms of size $a_1$ successively one obtains finally a family $\mathcal{A}'$ to which one cannot add atoms of size $a_1$. When proving the imbeddability we may assume $\mathcal{A} = \mathcal{A}'$.

Call a set $A \in \mathcal{A}$ with $|A| = p(i)$ filled if it contains $i$ atoms of size $a_1$. For a filled set let $D(A)$ be the union of its atoms, $|D(A)| = ia_1$.

Claim 4.1. All $A \in \mathcal{A}$ are filled.

Proof of Claim 4.1. The claim clearly holds if $|A| = a_1$. Let $A$ be a counterexample of minimal size $|A| = p(i)$. Set $\mathcal{B} = \{B \in \mathcal{A}, B \subseteq A\}$.

Define $M = M(A) = \bigcup_{B \in \mathcal{B}} B$. Since $A - M$ is an atom, $|A - M| < a_1$ holds.
Define also,

\[ K = K(A) = \bigcup_{B \in \mathcal{A}} D(B). \]

Then \( K \subset M \), \( K \) is the union of atoms of size \( a_1 \), thus

\[ |K| = ja_1 \quad \text{holds with some } j < i. \]

For definiteness let \( C_1, \ldots, C_j \) be these atoms. For \( B \in \mathcal{A} \) define

\[ T(B) = \{ v : C_v \subset B \}. \]

Since \( B \) is filled, \( |T(B)| = |B|/a_1 \) holds. If for \( B, B' \in \mathcal{A} \),

\[ |T(B) \cap T(B')| = t \]

then \( ta_1 \leq |B \cap B'| \leq ta_1 + |B - D(B)| < (t + 1)a_1 \) holds. Therefore

\[ |B \cap B'| = p(t). \]

Consequently, the map \( B \to \rho(T(B)) \) defines an embedding of \( \mathcal{A} \) into \( \mathcal{P}^{(j)} \)

(here we used Proposition 2.2). In particular

\[ |M| = \left| \bigcup_{B \in \mathcal{A}} B \right| \leq \left| \bigcup_{P \in \mathcal{P}^{(j)}} P \right| = \rho(j). \quad (1) \]

Thus \( \rho(i) = |A| < \rho(j) + a_1 < \rho(i) \), a contradiction.

Applying the claim to \( [1, \rho(s)] \in \mathcal{A} \), we see that there are \( s \) atoms

\( C_1, \ldots, C_s \) of size \( a_1 \) in it. Define for all \( A \in \mathcal{A} \),

\[ T(A) = \{ v : C_v \subset A \}. \]

Then \( A \to \rho(T(A)) \) gives the desired embedding of \( \mathcal{A} \) into \( \mathcal{P}^{(s)} \).

Note that this implies that every \( A \in \mathcal{A} \) with \( |A| = \rho(d) \) has a generic point (no \( B \in \mathcal{A} \) with \( B \not\subset A \) can contain elements which are mapped on a copy of \( (1, a_1^1) \)). In particular, if \( s = d \), then \( |\mathcal{A}| \leq n \) follows and this will be

the starting case of the induction.

Also, we can add to \( \mathcal{A} \) all subsets of members of \( \mathcal{A} \) which have projection in \( \mathcal{P} \), i.e., the family

\[ \mathcal{H} = \{ H : \pi(H) \in \mathcal{P}, \exists F \in \mathcal{F}, H \subseteq F \} \]

is still closed.

Suppose \( s > d \) and the upper bound is proved for \( s - 1 \). Define

\[ \mathcal{H}_1 = \{ H \in \mathcal{H} : |H| = \rho(s - 1) \}. \]

By induction

\[ |\mathcal{H}_1| \leq \Omega(n^{(s - 1)/d}) \]

holds.

Set \( \mathcal{H}_0^{(0)} = \mathcal{F} \), \( \mathcal{H}_1^{(0)} = \mathcal{H}_1 \). If \( \mathcal{H}_0^{(i)} \) and \( \mathcal{H}_1^{(i)} \) are defined and some member

\( G \in \mathcal{H}_0^{(i)} \) is contained in less than \( n^{1/d} \) members of \( \mathcal{H}_0^{(i)} \) then define

\[ \mathcal{H}_0^{(i+1)} = \mathcal{H}_0^{(i)} - \{ G \}, \mathcal{H}_0^{(i+1)} - \{ H \in \mathcal{H}_0^{(i)} : G \subset H \} \]

and continue. In view of (4) altogether less than \( n^{1/d} \) sets are thrown away. Thus
it will be sufficient to prove the upper bound for the remaining family, which we denote, by abuse of notation, by $\mathcal{F}$. Define
$$\mathcal{H}_i = \{ H \in \mathcal{H}: |H| = p(s-i) \}, \quad 0 \leq i \leq s.$$

**Claim 4.2.** Suppose $G \in \mathcal{H}_i$, $i > 0$, $A, C \in \mathcal{P}$, $\pi(G) = A \subseteq C$, $|C| = p(s-i+1)$. Then $|\mathcal{J}_G(A, C)| \geq n^{1/d}$.

**Proof.** Apply induction on $i$. The case $i = 1$ is fine by the construction. Let $A_0(C_0)$ be the subset of $[1, s]$ satisfying $\varphi(A_0) = A$ ($\varphi(C_0) = C$), respectively. Of course, $|A_0| = |C_0| - 1 = s - i$. Let $j$ be an arbitrary element of $[1, s] - C_0$. Define $B = \varphi(A_0 \cup \{ j \})$, $D = \varphi(C_0 \cup \{ j \})$. Take $G, H \in \mathcal{H}$ with $G \subseteq H$, $\pi(G) = A$, $\pi(H) = B$. By the induction hypothesis and by Lemma 2.3 we have
$$|\mathcal{J}_G(A, C)| \geq |\mathcal{J}_H(B, D)| \geq n^{1/d}.$$

**Claim 4.3.** For $1 \leq i \leq s - 1$ one has $|\mathcal{H}_i| \leq (s/i^{i}) |\mathcal{H}_i| n^{-(i-1)/d}$.

**Proof.** The statement is trivial for $i = 1$. Suppose it has been proved for $i - 1$. Consider the bipartite graph with vertex set $\mathcal{H}_i, \mathcal{H}_{i-1}$ with $(G, H)$ forming an edge if $G \in \mathcal{H}_i$, $H \in \mathcal{H}_{i-1}$ and $G \subseteq H$. Now the degree of $H$ is $s - i + 1$ while the degree of $G$ is at least $n^{1/d}$. This implies
$$|\mathcal{H}_i| \leq \frac{s - i + 1}{i} n^{-1/d} |\mathcal{H}_{i-1}| \leq |\mathcal{H}_i| \frac{(s)}{s} \frac{(i-1)}{i} n^{-(i-1)/d}$$
$$= |\mathcal{H}_i| \frac{(s)}{s} n^{-(i-1)/d}.$$

Now the upper bound is immediate: for an arbitrary $F \in \mathcal{F} = \mathcal{H}_0$ let $A_1(F), \ldots, A_s(F)$ be the $s$ atoms in $F$, i.e., $\pi(A_i(F)) = \varphi(\{i\})$. Then no other member $F'$ of $\mathcal{F}$ contains $A_1(F), \ldots, A_s(F)$ because otherwise $|F \cap F'| \geq sa_i > p(s - 1)$, a contradiction. Consequently,
$$|\mathcal{F}| \leq \binom{|\mathcal{H}_{s-1}|}{s} = O(n^{i/d}).$$

5. **Concluding Remarks**

First of all let us mention an old conjecture of Erdős and Simonovits which has an apparent similarity with Theorem 1.1.
For a class \( \mathcal{C} \) of graphs let \( \text{ex}(n, \mathcal{C}) \) denote the maximum number of edges in a graph with no subgraphs isomorphic to a member of \( \mathcal{C} \).

**Conjecture 5.1** [2]. For every rational number \( r \), \( 1 < r < 2 \), there exists a finite class \( \mathcal{C} \) of bipartite graphs so that \( \text{ex}(n, \mathcal{C}) = \Theta(n^r) \) holds.

Suppose \( p(x) = \sum_{i=0}^{d} a_i x^i \), where \( a_d \geq 1 \), \( a_i \) is integer for \( i = 0, \ldots, d - 1 \). Then there exists a smallest non-negative integer \( t = t(p) \) so that substituting \( y = x - t \) into \( p(x) \) will give a polynomial \( q(y) = p(y + t) = \sum_{i=0}^{d} b_i y^i \) with \( b_d = a_d \) and \( b_i \geq 0 \), integer.

**Conjecture 5.2.** Suppose \( p(x) = \sum_{i=0}^{d} a_i x^i \) and \( t = t(p) \) are as above. Then for \( s \geq s_0(p) \) one has

\[
m(n, p(s), \{ p(j); 0 < j < s \}) = \Theta(n^{(s-t)/d}).
\]

We can prove the above conjecture in several special cases not covered by Theorem 1.2 and can obtain as well a general upper bound of the form \( O(n^{a(p)} + s/d) \), where \( a(p) \) is a constant depending only on the polynomial \( p \).

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