Notes

On the number of nonnegative sums

Peter Frankl 1

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Hungary

A R T I C L E  I N F O

Article history:
Received 26 June 2012
Available online 22 August 2013

Keywords:
Subset sums
Hypergraphs

A B S T R A C T

A short proof is presented for the following statement. If \( X \) is a set of \( n \) real numbers summing up to 0 and \( n \geq (3/2)k^3 \) then at least \( \binom{n-1}{k-1} \) of the subset sums involving \( k \) numbers are nonnegative.

© 2013 Elsevier Inc. All rights reserved.

1. Introduction

Let \( X := \{x_1, x_2, \ldots, x_n\} \) be a set of not necessarily distinct real numbers listed in decreasing order and satisfying \( x_1 + \cdots + x_n = 0 \). Let \([n] := \{1, 2, \ldots, n\}\). For a set \( S \subset [n] \) we define

\[
x(S) := \sum_{i \in S} x_i.
\]

For \( 1 \leq k \leq n \), define the family of nonnegative \( k \)-sums

\[
P(X, k) := \{S \subset [n] : |S| = k, x(S) \geq 0\}.
\]

A longstanding conjecture of Manickam, Miklós and Singhi is as follows.

Conjecture 1.1. (Cf. [4,5].)

\[
|P(X, k)| \geq \binom{n-1}{k-1}
\]

holds for all \( X \) and \( n \geq 4k \).

References

E-mail address: peter.frankl@gmail.com.

1 Mailing address: Peter Frankl Office, Shibuya-ku, Shibuya 3-12-25, Tokyo, Japan.

0095-8956/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jctb.2013.07.002
Manickam and Singhi [5] proved (1) for all $n$ that are divisible by $k$. However, the general case proved unexpectedly difficult and only limited progress was made (cf. [1] for detailed reference). In a recent paper Alon, Huang and Sudakov [1] made a breakthrough by establishing the validity of the conjecture for $n \geq 33k^2$ thus significantly improving the previous superexponential lower bound. The aim of the present note is to provide a short proof of a somewhat weaker result still giving a polynomial lower bound for $n$.

**Theorem 1.2.** Let $n \geq (3/2)k^3$. Then one of the following must hold.

(i) All $k$-subsets of $[n]$ containing 1 are in $\mathcal{P}(X, k)$, or

(ii) $|\mathcal{P}(X, k)| \geq 2 \binom{n-k^2}{k-1} > \left(\frac{n-1}{k-1}\right)$.

2. Proof of the theorem

Let us define the following $k$ pairwise disjoint $(k-1)$-element sets $S_i := (n-i(k-1)+1, \ldots, n-((i-1)(k-1))$ for $i = 1, \ldots, k$. By monotonicity of the $x_j$’s the sum $x(S_1)$ is the smallest among all sums involving $k-1$ elements of $X$. Consequently, if $x_1 + x(S_1) > 0$ holds then the case (i) follows.

Suppose that $x_1 + x(S_1) < 0$ holds for some $k \geq r > 1$. Define

$$R := [n] - ([r] \cup S_1 \cup \cdots \cup S_{r-1}).$$

By the monotonicity of the $x_i$’s, for all $(k-1)$-element sets $Q \subset R$ and all $1 \leq j \leq r$ the sum $x_j + x(Q)$ is nonnegative. Thus

$$|\mathcal{P}(X, k)| \geq |\mathcal{P}([r] \cup R, k)| \geq r \binom{n-r-(r-1)(k-1)}{k-1} \geq 2 \binom{n-k^2}{k-1},$$

yielding case (ii).

From now on, we can assume that $x_1 + x(S_1) < 0$ holds for each $r = 1, \ldots, k$. We prove that case (ii) holds again. Let $T = S_1 \cup \cdots \cup S_k$. We have

$$x([k]) + x(T) \leq 0 \quad \text{with } T \subset [n]-[k], \ |T| = k^2-k. \quad (2)$$

Define $t := \lfloor (n - |T|)/k \rfloor$. Let $Y$ consist of the first $kt$ elements of $X$ and note that $Y$ is disjoint from $\bar{T} = \{x_i: i \in T\}$. We have $|Y| = kt \geq n - |T| - k + 1 = n - k^2 + 1$. If all the elements of $Y$ are nonnegative, then using $n \geq (3/2)n^2$, we obtain

$$|\mathcal{P}(X, k)| \geq |\mathcal{P}(Y, k)| \geq \binom{n-k^2+1}{k} \geq 2 \binom{n-k^2}{k-1}.$$

If there are negative elements in $Y$ then the monotonicity of the $x_i$’s implies that every $x_j \in X - Y - \bar{T}$ is negative. Also, (2) gives that $x(T) \leq 0$, so $x(X) = 0$ implies $x(Y) \geq 0$.

Now we are ready to apply a simple but very useful averaging argument due to Katona [3]. Let $Y' = [kt]$, so that $Y = \{x_i: i \in Y'\}$. Let $Y' = P_1 \cup \cdots \cup P_t$ be an arbitrary partition of $Y'$ into $k$-element sets. We claim that at least two of the $P_j$’s are in $\mathcal{P}(Y, k)$. Indeed, $x(Y) \geq 0$ gives that there exists a $P_j$ with $x(P_j) \geq 0$. Using (2) and $x(P_j) \leq x([k])$ we obtain

$$x([n]) = 0 \geq x([k]) + x(T) \geq x(P_j) + x(T) \geq x(P_j) + x([n] - Y').$$

This gives $x(Y' - P_j) \geq 0$, so there must be another $P_j \in \mathcal{P}(Y, k)$.

Thus we have shown that, in an arbitrary partition of $Y$ into $t$ $k$-sets, at least 2 members of the partition have nonnegative sum. By Katona’s argument this implies

$$|\mathcal{P}(Y, k)| \geq \frac{2}{t} \binom{|Y'|}{k} = \frac{1}{2} \binom{|Y| - 1}{k-1},$$

leading to
\[ |P(X, k)| \geq |P(Y, k)| \geq 2\binom{|Y| - 1}{k - 1} = 2\binom{tk - 1}{k - 1} \geq 2\binom{n - k^2}{k - 1}. \]

An easy calculation shows that
\[ 2\binom{n - k^2}{k - 1} \geq \binom{n - 1}{k - 1} \]
holds for \( n \geq (3/2)k^3 \), completing the proof of the theorem.

3. Some remarks

Although our results are somewhat weaker than those of Alon, Huang and Sudakov [1], the proof is considerably simpler. In [1] instead of conclusion (ii) an exact result is proven. Let us mention that their proof is basically the same as the new proof given for the Hilton–Milner Theorem in [2]. To keep the paper short, we contented ourselves with the slightly weaker assertion (ii). Note that the core of our proof is the following fact.

**Fact 3.1.** Suppose that \( T \) is a subset of \([n] - [k] \), \( |T| < n - 3k \), satisfying
\[ x([k]) + x(T) \leq 0. \]
Then \( |P(X - T)| \geq 2\binom{n - |T| - k}{k - 1} \) holds.

To obtain a quadratic bound—matching that of [1], one would need the size of \( T \) from (2) to be linear in \( k \), which does not seem to be easy to obtain. However, we hope to return to this problem with some new bounds characterizing sequences with \( |P(X, k)| = O(n^{k-1}) \).

**References**