The Katona theorem for vector spaces

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A R T I C L E   I N F O

Article history:
Received 18 October 2011
Available online 7 June 2013

Dedicated to Gyula O.H. Katona on his 70th

birthday

Keywords:
Katona’s t-intersection theorem
Avoiding just one intersection
Inclusion matrix
Linear algebra methods

A B S T R A C T

We present a vector space version of Katona’s t-intersection theorem [Katona, 1964 [12]]. Let V be the n-dimensional vector space over a finite field, and let F be a family of subspaces of V. Suppose that \( \dim(F \cap F') \geq t \) holds for all \( F, F' \in \mathcal{F} \). Then we show that \( |\mathcal{F}| \leq \sum_{k=0}^{n} \binom{n}{k} \) for \( n + t = 2d \), and \( |\mathcal{F}| \leq \sum_{k=0}^{n} \binom{n}{k} + \binom{n-1}{d} \) for \( n + t = 2d + 1 \). We also consider the case when the condition \( \dim(F \cap F') \geq t \) is replaced with \( \dim(F \cap F') \neq t - 1 \).

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1. Introduction

In 1964, Katona published his t-intersection theorem [12], which is one of the most basic results in extremal set theory. It has been extended in many ways, one of them being a result concerning a set-system avoiding just one intersection due to Frankl and Füredi [6]. In this article, we show vector space versions of these results using the linear algebra method.

We begin by recalling Katona’s original theorem. Let \( X = \{1, 2, \ldots, n\} \) and let \( \binom{X}{k} \) denote the set of all \( k \)-element subsets of \( X \). Let

\[
\mathcal{P}(X) = \bigcup_{k=0}^{n} \binom{X}{k}
\]

be the power set of \( X \). We say that a family of subsets \( \mathcal{F} \subset \mathcal{P}(X) \) is t-intersecting if \( |F \cap F'| \geq t \) holds for all \( F, F' \in \mathcal{F} \). Let us define a t-intersecting family \( \mathcal{K}(n, t) \) of subsets as follows. For \( n + t = 2d \), let \( \mathcal{K}(n, t) = \bigcup_{k=d}^{n} \binom{X}{k} \). For \( n + t = 2d + 1 \), choose an \( (n - 1) \)-element subset \( Y \subset X \), and set \( \mathcal{K}(n, t) = \bigcup_{k=d+1}^{n} \binom{X}{k} \cup \binom{Y}{d} \). Then Katona’s t-intersection theorem states the following.

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1 The second author was supported by JSPS KAKENHI 20340022 and 25287031.

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http://dx.doi.org/10.1016/j.jcta.2013.05.007
Theorem 1. (See [12].) Let $1 \leq t \leq n$ and let $\mathcal{F} \subset \mathcal{P}(X)$ be $t$-intersecting. Then $|\mathcal{F}| \leq |\mathcal{K}(n, t)|$. Moreover if $t > 1$ then equality holds iff $\mathcal{F}$ is isomorphic to $\mathcal{K}(n, t)$.

For a family of subsets $\mathcal{F}$ of $X$ and $0 \leq u \leq n$ we define the $u$-th shadow $\Delta_u(\mathcal{F})$ of $\mathcal{F}$ by

$$\Delta_u(\mathcal{F}) = \left\{ G \in \binom{X}{u} : G \subset F \text{ for some } F \in \mathcal{F} \right\}.$$  

The following result is a key tool for the original proof of Theorem 1.

Theorem 2. (See [12].) Let $1 \leq t \leq k \leq n$ and let $\mathcal{F} \subset \binom{X}{k}$ be $t$-intersecting. Then, for $k - t \leq u \leq k$, we have

$$|\Delta_u(\mathcal{F})|/|\mathcal{F}| \geq \binom{2k-t}{u}/\binom{2k-t}{k}.$$  

Now we present vector space versions of the above theorems. Fix the $q$-element field $\mathbb{F}_q$ and let $V$ be the $n$-dimensional vector space over this field. Let $[\binom{V}{k}]$ denote the set of all $k$-dimensional subspaces of $V$, let $[\binom{V}{k}^{1}] = \{ F \in \binom{V}{k} : |F| = q^k \}$, and let

$$\mathcal{L}(V) = \bigcup_{k=0}^{n} [\binom{V}{k}]$$

be the lattice of subspaces of $V$ with respect to inclusion. We say that a family of subspaces $\mathcal{F} \subset \mathcal{L}(V)$ is $t$-intersecting if $\dim(F \cap F') \geq t$ holds for all $F, F' \in \mathcal{F}$. For $0 \leq u \leq n$ we define the $u$-th shadow $\Delta_u[\mathcal{F}]$ of $\mathcal{F}$ by

$$\Delta_u[\mathcal{F}] = \left\{ G \in [\binom{V}{u}] : G \subset F \text{ for some } F \in \mathcal{F} \right\}.$$  

Then the corresponding result to Theorem 2 is as follows.

Theorem 3. Let $1 \leq t \leq k \leq n$ and let $\mathcal{F} \subset \binom{V}{k}$ be $t$-intersecting. Then, for $k - t \leq u \leq k$, we have

$$|\Delta_u[\mathcal{F}]|/|\mathcal{F}| \geq \left\lceil \frac{2k-t}{u} \right\rceil/\left\lceil \frac{2k-t}{k} \right\rceil.$$  

Let us define a $t$-intersecting family $\mathcal{K}[n, t]$ of subspaces as follows. For $n + t = 2d$, let $\mathcal{K}[n, t] = \bigcup_{k=d}^{d} \binom{V}{k}$. For $n + t = 2d + 1$, choose an $(n-1)$-dimensional subspace $W \subset V$, and set $\mathcal{K}[n, t] = \left( \bigcup_{k=d+1}^{d} \binom{W}{k} \right) \cup \binom{W}{d}$. Using Theorem 3 we will obtain the following vector space version of Katona’s theorem.

Theorem 4. Let $1 \leq t \leq n$ and let $\mathcal{F} \subset \mathcal{L}(V)$ be $t$-intersecting. Then $|\mathcal{F}| \leq |\mathcal{K}[n, t]|$. Moreover if $t > 1$ then equality holds iff $\mathcal{F}$ is isomorphic to $\mathcal{K}[n, t]$.

We say that a family of subsets $\mathcal{F} \subset \mathcal{P}(X)$ is $(t-1)$-avoiding if $|F \cap F'| \neq t-1$ holds for all distinct $F, F' \in \mathcal{F}$. Notice that if $\mathcal{F}$ is $t$-intersecting then it is $(t-1)$-avoiding. In 1975, Erdős [4] asked what happens if in Theorem 1 we weaken the condition “$t$-intersecting” to “$(t-1)$-avoiding.” Define a $(t-1)$-avoiding family $\mathcal{K}^*(n, t-1)$ of subsets of $X$ by $\mathcal{K}^*(n, t-1) = \mathcal{K}(n, t) \cup \bigcup_{k=t-1}^{n} \binom{X}{k}$. In [5], Frankl conjectured that this construction gives the maximum possible size for $n > n_0(t)$, and he proved this for the case $t = 2$ (1-avoiding families) for all $n$. This conjecture was solved by Frankl and Füredi in 1984 [6] using the so-called “linear algebra method.” We present the corresponding vector space version. To state our result, we need some definitions. We say that a family of subspaces $\mathcal{F} \subset \mathcal{L}(V)$ is $(t-1)$-avoiding if $\dim(F \cap F') \neq t-1$ holds for all distinct $F, F' \in \mathcal{F}$. Define a $(t-1)$-avoiding family $\mathcal{K}^*[n, t-1]$ of subspaces of $V$ by $\mathcal{K}^*[n, t-1] = \mathcal{K}[n, t] \cup \bigcup_{k=t-1}^{n} \binom{V}{k}$.
Theorem 5. Let \( t \geq 1 \), \( n > n_0(t) \), and let \( \mathcal{F} \subset \mathcal{L}(V) \) be \((t - 1)\)-avoiding. Then \(|\mathcal{F}| \leq |K^{n \setminus t - 1}| \). Moreover if \( t > 1 \) then equality holds iff \( \mathcal{F} \) is isomorphic to \( K^{n \setminus t - 1} \).

Since a \( t \)-intersecting family is always a \((t - 1)\)-avoiding family, the following result is an obvious extension of Theorem 3 (for the case \( k \geq 2t - 1 \)), which will be used to prove Theorem 5.

Theorem 6. Let \( t \geq 1 \), \( n \geq k \geq 2t - 1 \), and let \( \mathcal{F} \subset \binom{V}{k} \) be \((t - 1)\)-avoiding. Then, for \( k - t \leq u \leq k \), we have

\[
|\Delta_u[\mathcal{F}]|/|\mathcal{F}| \geq \left\lfloor \frac{2k - t}{u} \right\rfloor / \left\lfloor \frac{2k - t}{k} \right\rfloor.
\]

In [6] the corresponding set-system version of Theorem 6 is conjectured to be true but it is proved only under the assumption of \( k > k_0(t) \). This is because the proof relies on a result of Frankl and Singhi [10] stating that every \( k \)-uniform, \((t - 1)\)-avoiding family of subsets is \((k - t)\)-independent, provided \( k > k_0(t) \). (We will define “\((k - t)\)-independence” in Section 2.) This proof, in turn, uses a divisibility property of integers which requires \( k > k_0(t) \). On the other hand, we will use some basic properties of the cyclotomic polynomials to show that every \( k \)-uniform, \((t - 1)\)-avoiding family of subspaces is \((k - t)\)-independent provided \( k \geq 2t - 1 \) (Lemma 5). In this sense, Theorem 6 is an example where a vector space version of a theorem has a stronger result than a set-system version, with a simpler proof.

Finally we mention the maximum size of \( k \)-uniform, \((t - 1)\)-avoiding families. As for the case \( k \geq 2t - 1 \), we only have the following weaker bound, which is stated in [8] without a proof. (In [8] they claimed that Theorem 7 follows from their Theorem 1.1, but this is true only for \( t \)-intersecting families.)

Theorem 7. Let \( t \geq 1 \), \( n \geq k \geq 2t - 1 \), and let \( \mathcal{F} \subset \binom{V}{k} \) be \((t - 1)\)-avoiding. Then \(|\mathcal{F}| \leq \binom{n}{k-t} \).

Frankl and Graham [8] conjecture that if \( k \geq 2t \) then the upper bound can be improved to \( \binom{n-t}{k-t} \). (Theorem 7 for the case \( k = 2t - 1 \) is almost sharp as described below.) On the other hand, Frankl and Füredi [7] obtained the sharp upper bound \( \binom{n-t}{k-t} \) for the corresponding set-system version, provided \( k \geq 2t \) and \( n > n_0(k) \). The proof technique used in [7] is more combinatorial, and different from that in [6].

For the case \( k \leq 2t - 1 \) we will derive the following result from Theorem 7.

Theorem 8. Let \( t \geq 1 \), \( 2t - 1 \geq k > t - 1 \), \( n \geq k \), and let \( \mathcal{F} \subset \binom{V}{k} \) be \((t - 1)\)-avoiding. Then \(|\mathcal{F}| \leq \binom{n}{k-t} / \binom{2k-t}{k-t} \).

Theorem 8 is asymptotically tight as \( n \to \infty \) for fixed \( t,k \). We show the tightness (Theorem 9 in Section 4) using a packing result of Rödl [14].

We will use the linear algebra method to prove our results. The proofs are similar to those in [6], but we will follow the formulation in the Babai–Frankl book [2]. The key idea is an independence of row vectors of the inclusion matrix. This idea was already used by Frankl and Graham in [8], and we could use their results but we choose to give direct and elementary proofs for self-completeness.

This paper is organized as follows. In Section 2 we prepare some basic tools for the linear algebra method, and prove Theorem 3 and Theorem 4 (the Katona theorem for vector spaces). Then in Section 3 we consider families avoiding just one intersection, and prove Theorem 5 and Theorem 6. In Section 4 we focus on uniform families and prove Theorem 7 and Theorem 8.

2. The Katona theorem for vector spaces

In this section, we prepare some basic tools for the linear algebra method, and prove Theorem 3 and Theorem 4.
Let $V$ be the $n$-dimensional vector space over $\mathbb{F}_q$. For $0 \leq i \leq k \leq n$, $\mathcal{F} \subset [V]^i$, and $\mathcal{G} \subset [V]^j$, define the inclusion matrix $M(\mathcal{F}, \mathcal{G})$ as follows. This is an $|\mathcal{F}| \times |\mathcal{G}|$ matrix whose $(F, G)$-entry $m(F, G)$, where $F \in \mathcal{F}$ and $G \in \mathcal{G}$, is defined by

$$m(F, G) = \begin{cases} 1 & \text{if } F \supseteq G, \\ 0 & \text{if } F \nsubseteq G. \end{cases}$$

For $\mathcal{F} \subset [V]^k$ and $0 \leq j \leq i \leq k$, simple counting yields


In fact, the $(F, J)$-entry of (1), where $F \in \mathcal{F}$ and $J \in [V]_j$, counts

$$\# \{ I \in [V]: J \subset I \subset F \}.$$ 

In particular, (1) shows the following.

**Lemma 1.** Let $0 \leq j \leq i \leq k$ and $\mathcal{F} \subset [V]^k$. Then colsp $M(\mathcal{F}, [i]^j)$ is contained in colsp $M(\mathcal{F}, [V]^j)$, where colsp $M$ denotes the column space of $M$ over $\mathbb{Q}$.

We say that $\mathcal{F} \subset [V]^k$ is $s$-independent if the rows of $M(\mathcal{F}, [V]^j)$ are linearly independent over $\mathbb{Q}$, that is, the inclusion matrix has full row-rank. In this case, $|\mathcal{F}| \leq \binom{n}{s}$ immediately follows.

**Lemma 2.** (See [8].) Let $0 \leq s \leq u \leq k$ and let $\mathcal{F} \subset [V]^k$ be $s$-independent. Then

$$|\Delta_u[\mathcal{F}]|/|\mathcal{F}| \geq \frac{k+s}{u} \binom{k+s}{k}.$$

**Proof.** Let $A \oplus B = V$ denote the direct sum, that is, $A \cap B = \{0\}$ and $\text{span}(A, B) = V$. For each line $x \in [V]^1$, choose $W = W_x \in [W]_{k-1}$ so that $x \oplus W = V$. Let

$$\mathcal{F}_x = \{ G \in [W]_{k-1}: x \oplus G \in \mathcal{F} \} \subset [W]_{k-1}.$$

**Claim 1.** $\mathcal{F}_x \subset [W]_{k-1}$ is $s$-independent, that is, rank $M(\mathcal{F}_x, [W]_s) = |\mathcal{F}_x|$.

We postpone the proof of Claim 1, and we first prove the lemma by induction on $k$ assuming Claim 1. Inequality (2) trivially holds for the following three cases: $s = 0$, $u = s$, and $u = k$. So let $1 \leq s < u < k$ and assume that (2) is true for $k - 1$. By Claim 1 we can apply the induction hypothesis to $\mathcal{F}_x \subset [W]_{k-1}$, and we get

$$|\Delta_{u-1}[\mathcal{F}_x]| \geq |\mathcal{F}_x| \binom{k-1+s}{u-1} \binom{k-1+s}{k-1}.$$ 

By counting $\# \{ (x, F) \in [V]^1 \times \mathcal{F}: x \subset F \}$ in two ways, namely, by counting the number of edges in the corresponding bipartite graph from each side, we have

$$\sum_{x \in [V]^1} |\mathcal{F}_x| = \binom{k}{1}|\mathcal{F}|.$$
Similarly by counting $\#\{(x, G) \in [V]^s \times \Delta_u[\mathcal{F}] : x \subset G\}$, we have

$$\sum_{x \in [V]^s} |\Delta_{u-1}[\mathcal{F}_x]| = \begin{bmatrix} 1 \\ u \end{bmatrix} |\Delta_u[\mathcal{F}]|.$$  \hfill (5)

Using (5), (3), and (4), we get

$$|\Delta_u[\mathcal{F}]| = \begin{bmatrix} 1 \\ u \end{bmatrix} \sum_x |\Delta_{u-1}[\mathcal{F}_x]| \geq \begin{bmatrix} 1 \\ u \end{bmatrix} \sum_x |\mathcal{F}_x| \begin{bmatrix} k-1+s \\ k-1 \\ u-1 \end{bmatrix} \begin{bmatrix} k+s \\ u \\ k \end{bmatrix}.$$  \hfill (4)

This shows that (2) is true for $k$ as well, and completes the induction.

So all we need is to prove Claim 1. Fix $x \in [V]^s$ and let $W \in [n-1]$ be such that $x \oplus W = V$. Divide $[V]^s$ into two parts $[V]^s = C \cup D$, where $C$ is the set of $s$-dimensional subspaces of $V$ not containing $x$, and the remaining part is $D = \{x \oplus T : T \in [W]_{s-1}\}$. (Then $|C| = q^s[n-1]$ and $|D| = [n-1]$.) Let

$$\mathcal{F}^x = \{F \in \mathcal{F}: x \subset F\} \subset [V]^s.$$  \hfill (7)

We divide the columns of $M(\mathcal{F}^x, [V]^s)$ into two blocks:

$$M\left(\mathcal{F}^x, \begin{bmatrix} V \\ s \end{bmatrix}\right) = (M(\mathcal{F}^x, C) | M(\mathcal{F}^x, D)).$$  \hfill (6)

A subspace $S \in [V]^s$ can be represented by an $s \times n$ matrix in reduced echelon form with no zero rows (see, e.g., [3]), and let $\text{ref}(S)$ denote the matrix. We can associate $D$ with matrices for which leading 1 in the last row occurs in the last column. Then there is a natural bijection from $D$ to $[W]_{s-1}$ by taking the $(s-1) \times (n-1)$ principal minor of $\text{ref}(S)$. Thus we may assume that

$$M(\mathcal{F}^x, D) = M\left(\mathcal{F}^x, \begin{bmatrix} W \\ s-1 \end{bmatrix}\right).$$

This together with Lemma 1 gives

$$\text{colsp} M(\mathcal{F}^x, D) = \text{colsp} M\left(\mathcal{F}^x, \begin{bmatrix} W \\ s-1 \end{bmatrix}\right) \subset \text{colsp} M\left(\mathcal{F}^x, \begin{bmatrix} W \\ s \end{bmatrix}\right).$$  \hfill (7)

If $S \in C$ then the $s \times (n-1)$ principal minor of $\text{ref}(S)$ determines a subspace in $[W]_{s}$, which gives a map $\varphi: C \to [W]^s$. And for each $S \in [W]^s$, we have $|\varphi^{-1}(S)| = q^s$ because $\varphi(S) = \varphi(S')$ iff $\text{ref}(S)$ and $\text{ref}(S')$ differ only in the last column. Thus columns corresponding to $S$ and $S'$ in $M(\mathcal{F}^x, C)$ are the same iff $\varphi(S) = \varphi(S')$, and $M(\mathcal{F}^x, C)$ can be viewed as $q^s$ copies of $M(\mathcal{F}^x, [V]^s)$. Hence we have

$$\text{colsp} M(\mathcal{F}^x, C) = \text{colsp} M\left(\mathcal{F}^x, \begin{bmatrix} W \\ s \end{bmatrix}\right).$$  \hfill (8)

By (7) and (8) with (6), it follows $\text{colsp} M(\mathcal{F}^x, [V]^s) \subset \text{colsp} M(\mathcal{F}^x, [W]^s)$, and

$$\text{rank} M\left(\mathcal{F}^x, \begin{bmatrix} V \\ s \end{bmatrix}\right) = \text{rank} M\left(\mathcal{F}^x, \begin{bmatrix} W \\ s \end{bmatrix}\right).$$  \hfill (9)

The opposite inequality is trivial, thus we have equality in (9). On the other hand, since $\mathcal{F}$ is $s$-independent, $\mathcal{F}^x$ is also $s$-independent and $\text{rank} M(\mathcal{F}^x, [V]^s) = |\mathcal{F}^x|$. Thus equality in (9) yields that $\text{rank} M(\mathcal{F}^x, [W]^s) = |\mathcal{F}^x|$. Finally, noting that $|\mathcal{F}^x| = |\mathcal{F}_x|$ and $M(\mathcal{F}^x, [V]^s) = M(\mathcal{F}_x, [V]^s)$, we have $\text{rank} M(\mathcal{F}_x, [W]^s) = |\mathcal{F}_x|$ as needed. This completes the proof of Claim 1 and Lemma 2. \hfill \square
Lemma 3. (See [8].) Let $1 \leq t \leq k$ and let $\mathcal{F} \subset \left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right]$ be $t$-intersecting. Then $\mathcal{F}$ is $(k-t)$-independent.

Proof. Let

$$f(x) = \prod_{t \leq i < k} \left(\begin{array}{c} x - i \\ 1 \end{array}\right) = \prod_{t \leq i < k} \frac{q^{x-i} - 1}{q-1}.$$ \hfill (10)

By setting $y = q^x$, we can rewrite $f(x)$ as a polynomial $g(y)$ of degree $k-t$ in $\mathbb{Q}[y]$, that is,

$$f(x) = g(y) = \prod_{t \leq i < k} \frac{y^{t-i} - 1}{q-1}.$$ 

Let $\phi_s(y) = \prod_{i=0}^{s-1} \frac{y^{q^i} - 1}{y^{q^i} - 1} = \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right]$. Then $\phi_0(y), \ldots, \phi_{k-t}(y)$ form a basis of the vector space (over $\mathbb{Q}$) of polynomials of degree $k-t$ with variable $y$. Thus we can determine $a_0, a_1, \ldots, a_{k-t} \in \mathbb{Q}$ uniquely so that

$$g(y) = \sum_{s=0}^{k-t} a_s \phi_s(y).$$

In other words, noting that $\phi_s(y) = \prod_{i=0}^{s-1} \frac{q^{x-t} - 1}{q^{x-t} - 1} = \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right]$, we can determine $a_0, \ldots, a_{k-t}$ so that

$$f(x) = \sum_{s=0}^{k-t} a_s \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right].$$ \hfill (11)

Now define an $|\mathcal{F}| \times |\mathcal{F}|$ matrix $A$ by

$$A = \sum_{s=0}^{k-t} a_s M(\mathcal{F}, \left[\begin{smallmatrix} V \\ s \end{smallmatrix}\right])M(\mathcal{F}, \left[\begin{smallmatrix} V \\ V \setminus s \end{smallmatrix}\right])^T.$$

For $F, F' \in \mathcal{F}$, the $(F, F')$-entry of $A$ is

$$\sum_{s=0}^{k-t} a_s \# \left\{ W \in \left[\begin{smallmatrix} V \\ s \end{smallmatrix}\right] : W \subset F \cap F' \right\} = \sum_{s=0}^{k-t} a_s \left\{ \dim(F \cap F') \right\}.$$ 

This equals $f(\dim(F \cap F'))$ by (11). Moreover, using the $t$-intersecting property with (10), we have

$$f(\dim(F \cap F')) = \begin{cases} 0 & \text{if } F \neq F', \\ f(k) & \text{if } F = F'. \end{cases}$$

Thus $A$ is a diagonal matrix with no zero diagonal entries, and $\text{rank} A = |\mathcal{F}|$.

On the other hand, it follows from Lemma 1 that the colsp $M(\mathcal{F}, \left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right])$ is contained in colsp $M(\mathcal{F}, \left[\begin{smallmatrix} V \\ k-s \end{smallmatrix}\right])$ for $0 \leq s < k-t$, and so colsp $A$ is contained in colsp $M(\mathcal{F}, \left[\begin{smallmatrix} V \\ k-s \end{smallmatrix}\right])$. This gives $\text{rank} M(\mathcal{F}, \left[\begin{smallmatrix} V \\ k-s \end{smallmatrix}\right]) \geq \text{rank} A = |\mathcal{F}|$. Thus $M(\mathcal{F}, \left[\begin{smallmatrix} V \\ k-s \end{smallmatrix}\right])$ has full row-rank, namely, $\mathcal{F}$ is $(k-t)$-independent. \hfill \Box

Proof of Theorem 3. By Lemma 3, $\mathcal{F}$ is $(k-t)$-independent. So letting $s = k - t$ in Lemma 2, we get the desired inequality. \hfill \Box

Proof of Theorem 4. We start with the following simple counting fact.

Claim 2. Let $A \in \left[\begin{smallmatrix} V \\ a \end{smallmatrix}\right]$. Then $\# \left\{ B \in \left[\begin{smallmatrix} V \\ n-a \end{smallmatrix}\right] : A \oplus B = V \right\} = q^{a(a-1)}$. 

Proof. We may assume that \( \text{ref}(A) = (O \quad I_a) \). Then \( A \oplus B = V \) gives \( \text{ref}(B) = (I_{n-a} \quad *) \), and there are \( q^{a(n-a)} \) ways for choosing the * part. \( \square \)

Let \( G = G(a, n-a) \) be a bipartite graph with the vertex partition \( V(G) = [V_1] \cup [V_{n-a}] \) and the edge set \( E(G) = \{ (A, B) : A \oplus B = V \} \). Then, by Claim 2, this is a \( q^{a(n-a)} \)-regular graph. For a vertex subset \( A \subset [V_1] \), let \( N_G(A) = \{ B \in [V_{n-a}] : (A, B) \in E(G) \text{ for some } A \in A \} \) denote the neighborhood of \( A \). We count the number of edges between \( A \) and \( N_G(A) \) in two ways. Then this number is exactly \( q^{a(n-a)} |A| \) on one hand, and at most \( q^{a(n-a)} |N_G(A)| \) on the other hand. Namely, we have the Hall condition:

\[
|A| \leq |N_G(A)|.
\]

Thus the bipartite graph has a perfect matching, which can be stated as follows.

**Lemma 4.** There is a bijection \( \psi : [V_1] \rightarrow [V_{n-a}] \) such that \( A \oplus \psi(A) = V \) holds for all \( A \in [V_1] \).

We will use \( \psi(A) \) as a “complement” of \( A \) here, and also in the proof of Theorem 8 later. (Notice that the orthogonal space \( A^\perp \) does not necessarily satisfy \( A \oplus A^\perp = V \). The authors thank one of the referees for notifying this fact.)

Let \( F_k = F \cap [V_1] \), \( f_k = |F_k|, d = \lfloor (n + t)/2 \rfloor \), and \( a = k - t + 1 \).

**Claim 3.** \( |\Delta_a[F_k]| + |F_{n-a}| \leq \left\lfloor \frac{n}{n-a} \right\rfloor \) for \( t \leq k \leq d \).

**Proof.** Let \( G \subset \Delta_a[F_k] \) and \( G \subset F \subset F_k \). Using Lemma 4 let \( H = \psi(G) \subset [V_{n-a}] \). Since \( V = G \oplus H \subset F + H \subset V \) we have \( n = \dim(F + H) \) and

\[
\dim(F \cap H) = \dim F + \dim H - \dim(F + H) = k + (n-a) - n = t - 1.
\]

Then it follows from the \( t \)-intersecting property of \( F \) that \( H = \psi(G) \not\subset F_{n-a} \). This gives \( \psi(\Delta_a[F_k]) \cap F_{n-a} = \emptyset \), and

\[
|\Delta_a[F_k]| + |F_{n-a}| = |\psi(\Delta_a[F_k])| + |F_{n-a}| \leq \left\lfloor \frac{n}{n-a} \right\rfloor
\]

as desired. \( \square \)

We notice for later use in the proof of Theorem 5 in Section 3 that the proof above did not use the full \( t \)-intersecting property, but only the \( (t-1) \)-avoiding property.

Let \( t \leq k < d \). Applying Theorem 3 with \( u = a = k - t + 1 \), we have

\[
|\Delta_a[F_k]| \geq |F_k| \begin{bmatrix} 2k - t \\ a \end{bmatrix}/\begin{bmatrix} 2k - t \\ k \end{bmatrix} = \frac{q^k - 1}{q^a - 1} f_k.
\]

Since \( \frac{q^k - 1}{q^a - 1} \geq 1 \) iff \( a \leq k \), that is, \( t \geq 1 \), it follows that \( |\Delta_a[F_k]| \geq f_k \) with equality holding iff \( F_k = \emptyset \) or \( t = 1 \). Then we can infer from Claim 3 that

\[
f_k + f_{n-a} \leq |\Delta_a[F_k]| + |F_{n-a}| \leq \left\lfloor \frac{n}{n-a} \right\rfloor.
\]  \( (12) \)

(This is true for \( k = d \) as well but we will not use this case.) Moreover if \( t > 1 \) then \( f_k + f_{n-a} = \left\lfloor \frac{n}{n-a} \right\rfloor \) iff \( f_k = 0 \) and \( f_{n-a} = \left\lfloor \frac{n}{n-a} \right\rfloor \) where \( t \leq k < d \).

First consider the case \( n + t = 2d \). Applying \( (12) \) for \( k = t, t+1, \ldots, d-1 \) we have
By (14), we have
\[ f(x) = \frac{(q^{k-1} - q^x)(q^{k-2} - q^x) \cdots (q^t - q^x)}{(q^{k-t} - 1)(q^{k-t-1} - 1) \cdots (q - 1)}. \] (14)

As we did in the proof of Lemma 3 we can write \( f(x) = \sum_{s=0}^{k-t} a_s [x] \) for some \( a_0, \ldots, a_{k-t} \in \mathbb{Q} \). Define an \( |\mathcal{F}| \times |\mathcal{F}| \) matrix \( A \) by
\[
A = \sum_{s=0}^{k-t} a_s M \begin{pmatrix} V \end{pmatrix} M \begin{pmatrix} V \end{pmatrix}^T.
\]

Then, for \( F, F' \in \mathcal{F} \), the \( (F, F') \)-entry of \( A \) is \( \sum_{s=0}^{k-t} a_s [\dim(F \cap F')] = f(\dim(F \cap F')) \).

By (14), we have \( f(x) = 0 \) for \( x = t, t + 1, \ldots, k - 1 \), and
\[
f(k) = \prod_{j=1}^{k-t} \frac{q^{k-j} - q^k}{q^j - 1} = \prod_{j=1}^{k-t} \frac{q^{k-j}(1 - q^j)}{q^j - 1}
= (-1)^{k-t} q^{k(k-1)/2} (k-2)+(k-1)/2.
\] (15)

3. Avoiding just one intersection

In this section we prove Theorem 5 and Theorem 6.

Lemma 5. Let \( t \geq 1, k \geq 2t - 1 \), and let \( \mathcal{F} \subset \binom{V}{k} \) be \( (t - 1)\)-avoiding. Then \( \mathcal{F} \) is \( (k - t)\)-independent.

Proof. We proceed as in Lemma 3 but using a different \( f(x) \), that is,
\[
f(x) = \frac{(q^{k-1} - q^x)(q^{k-2} - q^x) \cdots (q^t - q^x)}{(q^{k-t} - 1)(q^{k-t-1} - 1) \cdots (q - 1)}. \] (14)
For the remaining values except \( t \) \(-\) 1, namely, for \( x = 0, 1, \ldots, t - 2 \), we have \( f(x) = q^{v(k-t)}[k-x-1]_{k-t} \) and

\[
\begin{bmatrix} k-x-1 \\ k-t \end{bmatrix} = \begin{bmatrix} k-x-1 \\ t-x-1 \end{bmatrix} = \frac{(q^{k-x-1} - 1) \cdots (q^{k-t+1} - 1)}{(q^{t-x-1} - 1) \cdots (q - 1)} \in \mathbb{Z}[q]. \tag{16}
\]

Recall that \( q^a - 1 = \prod_{j \in \mathbb{N}} \Phi_j(q) \), where \( \Phi_j(q) \in \mathbb{Z}[q] \) is the \( j \)-th cyclotomic polynomial. Let us look at the RHS of (16). The numerator contains \( \Phi_{k-t+1}(q) \) as a factor coming from \( q^{k-t+1} - 1 \). On the other hand, \( j = t-x-1 \) is the maximum \( j \) such that \( \Phi_j(q) \) appears in the denominator as a factor. Using \( x \geq 0 \) and \( k \geq 2t - 1 \) we have \( t-x-1 \leq t-1 < k-t+1 \). So \( \Phi_{k-t+1}(q) \) does not appear in the denominator. Since cyclotomic polynomials are pairwise relatively prime, it follows from (16) that \( \Phi_{k-t+1}(q) \) divides \( [k-x-1]_{k-t} \), namely,

\[ \Phi_{k-t+1}(q) \mid f(x) \text{ for } x = 0, 1, \ldots, t - 2. \]

But \( \Phi_{k-t+1}(q) \) does not divide \( f(k) \in \mathbb{Z}[q] \) by (15). Note also that \( f(t-1) \) never appears in \( A \) because of the \((t-1)\) -avoiding property. Consequently it follows that \( f(\dim(F \cap F')) \in \mathbb{Z}[q] \) and

\[
\mathbf{\Phi}_{k-t+1}(q) | f(x) \quad \text{if } F \neq F',
\]

\[
\mathbf{\Phi}_{k-t+1}(q) | f(x) \quad \text{if } F = F'.
\]

This means that \( A \) is a diagonal matrix with no zero diagonal entries in the residue ring \( \mathbb{Z}[q]/(\Phi_{k-t+1}(q)) \), and thus \( \text{rank } A = |F| \). On the other hand, it follows from (1) and the definition of \( A \) that \( \text{colsp } A \subset \text{colsp } M(F, [V]_{k-t}) \). Therefore we have

\[
|F| = \text{rank } A \leq \text{rank } M \left( F, \left[ \begin{array}{c} V \\ k-t \end{array} \right] \right) \leq |F|.
\]

Thus \( \text{rank } M(F, [V]_{k-t}) = |F| \), namely, \( F \) is \((k-t)\) -independent. \( \square \)

**Proof of Theorem 6.** This follows from **Lemma 2** and **Lemma 5**. \( \square \)

**Proof of Theorem 5.** Let \( F_k = F \cap [V]_k \), \( f_k = |F_k|, d = [(n+t)/2] \), and \( a = k - t + 1 \). Let \( t \leq k < d \). By **Theorem 6**, we have

\[
|A_t |F_k| | \geq |F_k| | \begin{bmatrix} 2k-t \\ a \end{bmatrix} \left/ \begin{bmatrix} 2k-t \\ k \end{bmatrix} \right| = \frac{q^k - 1}{q^a - 1} f_k,
\]

and so \( |A_t |F_k| \geq f_k \) with equality holding iff \( F_k = \emptyset \) or \( t = 1 \). Then we can infer from **Claim 3** (see the notice right after the proof of **Claim 3**) that

\[
f_k + f_{n-a} \leq \frac{q^k - 1}{q^a - 1} f_k + f_{n-a} \leq \left\lfloor \frac{n}{n-a} \right\rfloor. \tag{17}
\]

Moreover if \( t > 1 \) then \( f_k + f_{n-a} = \left\lfloor \frac{n}{n-a} \right\rfloor \) iff \( f_k = 0 \) and \( f_{n-a} = \left\lfloor \frac{n}{n-a} \right\rfloor \) where \( t \leq k < d \). For \( k < t \) we will use a trivial upper bound \( f_k \leq \left\lfloor \frac{n}{k} \right\rfloor \).

Case 1. \( n + t = 2d \).

We have

\[
|F| = \sum_{k=0}^{n} f_k = \sum_{k=t}^{n-1} f_k + \sum_{k=0}^{t-1} f_k = f_n + (f_t + f_{n-1}) + (f_{t+1} + f_{n-2}) + \cdots + (f_{d-1} + f_d) + \sum_{k=0}^{t-1} f_k. \tag{18}
\]
First suppose that \( f_{t-1} = 0 \). Then applying (17) for \( k = t, t+1, \ldots, d-1 \) we have
\[
|F| \leq \left[ \frac{n}{t} \right] + \left[ \frac{n}{t-1} \right] + \left[ \frac{n}{t} \right] + \cdots + \left[ \frac{n}{1} \right] + \sum_{k=0}^{t-2} \frac{n}{k} = |K^d[n, t-1]|. 
\]

If \( t > 1 \) then equality holds iff \( f_k = \left[ \frac{n}{k} \right] \) for \( 0 \leq k < t-1 \), \( f_k = 0 \) for \( t-1 \leq k < d \) and \( f_k = \left[ \frac{n}{k} \right] \) for \( d \leq k \leq n \), namely, \( F \) is isomorphic to \( K^d[n, t-1] \).

Next suppose that \( f_{t-1} \neq 0 \), that is, there is an \( F_0 \in \mathcal{F}_{t-1} \). Since \( \mathcal{F} \) is \( (t-1) \)-avoiding, no subspace containing \( F_0 \) can be a member of \( \mathcal{F} \), which implies that
\[
f_k \leq \left[ \frac{n}{k} \right] - \left[ \frac{n}{k} - (t-1) \right] 
\]
for \( k \geq t \). In particular we have
\[
f_d \leq N - M,
\]
where \( N = \left[ \frac{n}{d} \right] \), \( M = \left[ \frac{n-(t-1)}{d-t} \right] \). Setting \( k = d-1 \) in (17) we have
\[
\alpha f_{d-1} + f_d \leq N,
\]
where \( \alpha = \frac{d^{-1} - 1}{d-1} \geq 1 \). So \( f_{d-1} \leq \frac{1}{\alpha} (N - f_d) \). Thus we have
\[
f_{d-1} + f_d \leq \frac{1}{\alpha} (N - f_d) + f_d = \frac{1}{\alpha} \left( N + \left( \frac{1}{\alpha} - 1 \right) \right) f_d 
\]
\[
\leq \frac{1}{\alpha} N + \left( \frac{1}{\alpha} - 1 \right) (N - M) = N - \left( \frac{1}{\alpha} \right) M.
\]
Hence we have
\[
f_{t-1} + f_{d-1} + f_d \leq \left[ \frac{n}{t-1} \right] + \left[ \frac{n}{d} \right] - \frac{q^{d-1} - q^{d-t}}{q^{d-1} - 1} t \left[ \frac{n}{d} - (t-1) \right].
\]
The RHS is less than \( \left[ \frac{n}{d} \right] \) for \( n > n_0(t) \). Using this with (17) for \( k = t, t+1, \ldots, d-2 \) we can infer from (18) that \( |F| < |K^d[n, t-1]| \).

Case 2. \( n + t - 1 = 2d \).

If \( F, F' \in \left[ \frac{n}{d} \right] \) then \( \dim(F \cap F') \geq t-1 \). Since \( \mathcal{F} \) is \( (t-1) \)-avoiding, \( \mathcal{F}_d \subset \left[ \frac{n}{d} \right] \) is actually \( t \)-intersecting. So we can use a result in [11] to get
\[
f_d = |\mathcal{F}_d| \leq \left[ \frac{n-1}{d} \right]. \tag{19}
\]
Moreover if \( t > 1 \) then equality holds iff \( \mathcal{F}_d = \left[ \frac{n}{d} \right] \) for some \((n-1)\)-dimensional subspace \( W \subset V \). Write \( |\mathcal{F}| \) as follows:
\[
|\mathcal{F}| = \sum_{k=0}^{n} f_k = f_n + \sum_{k=t}^{n-1} f_k + \sum_{k=0}^{t-1} f_k 
\]
\[
= f_n + \left( (f_t + f_{t-1}) + (f_{t+1} + f_{t-2}) + \cdots + (f_{d-1} + f_d) \right) + f_d + f_{t-1} + \sum_{k=0}^{t-2} f_k.
\]

First suppose that \( f_{t-1} = 0 \). We use (17) for \( k = t, t+1, \ldots, d-1 \), (19) for \( k = d \), and \( f_k \leq \left[ \frac{n}{k} \right] \) for the remaining \( k \). In this way we get \( |\mathcal{F}| \leq |K^d[n, t-1]| \). Moreover if \( t > 1 \) then equality holds iff \( f_k = \left[ \frac{n}{k} \right] \) for \( 0 \leq k < t-1 \), \( f_k = 0 \) for \( t-1 \leq k < d \), \( f_k = \left[ \frac{n}{k} \right] \) for \( d + 1 \leq k \leq n \), and \( \mathcal{F}_d = \left[ \frac{n}{d} \right] \) for some \((n-1)\)-dimensional subspace \( W \), namely, \( \mathcal{F} \) is isomorphic to \( K^d[n, t-1] \).
Next suppose that \( f_{t-1} \neq 0 \). Then we can argue as in Case 1 to conclude that \( f_{t-1} + f_{d-1} + f_{d+1} < [\frac{n}{d+1}] \) for \( n > n_0(t) \), which gives \( |\mathcal{F}| < |\mathcal{K}^n[n, t-1]| \). This completes the proof of Theorem 5. \( \square \)

If we change the definition of a \((t-1)\)-avoiding family \( \mathcal{F} \) so that \( \dim(F \cap F') \neq t-1 \) is required for all \( F, F' \in \mathcal{F} \), then \( f_{t-1} = 0 \) follows from this new definition. In this case the above proof shows that Theorem 5 holds without assuming \( n > n_0(t) \).

4. Uniform families

In this section we prove Theorem 7 and Theorem 8. Then we will show that Theorem 8 is asymptotically sharp using a packing result of Rödl.

Proof of Theorem 7. This is a direct consequence of Lemma 5. \( \square \)

In the rest of this section we follow the proof in [7]. Recall the bijection \( \psi \) from Lemma 4.

Proof of Theorem 8. Let \( b = 2t - k - 1 \). For \( B \in \binom{V}{b} \) let

\[
\mathcal{F}(B) := \left\{ C \in \binom{\psi(B)}{k-b} : B \oplus C \in \mathcal{F} \right\}.
\]

Then we have

\[
\sum_{B \in \binom{V}{b}} |\mathcal{F}(B)| = \frac{k^b}{b!} |\mathcal{F}|. \tag{20}
\]

Let \( \tilde{k} = k - b = 2k - 2t + 1 \) and \( \tilde{t} - 1 = (t - 1) - b = k - t \). Then \( \mathcal{F}(B) \) is a \( \tilde{k} \)-uniform, \((\tilde{t} - 1)\)-avoiding family with \( \tilde{k} = 2\tilde{t} - 1 \). (In fact if there are \( C_1, C_2 \in \mathcal{F}(B) \) such that \( \dim(C_1 \cap C_2) = \tilde{t} - 1 \), then \( F_i = B \oplus C_i \in \mathcal{F} \) \( (i = 1, 2) \) but \( \dim(F_1 \cap F_2) = b + (\tilde{t} - 1) = t - 1 \), a contradiction.) Thus, by Theorem 7, we have

\[
|\mathcal{F}(B)| \leq \binom{n-b}{k-t} = \frac{n-b}{k-t} \binom{k}{t-1} \binom{2k-t}{t-1} \tag{21}
\]

Now it follows from (20) and (21) that

\[
|\mathcal{F}| \leq \frac{n}{b} \frac{n-b}{k-t} / \frac{k}{t-1} = \frac{n}{t-1} \frac{2k-t}{k} / \frac{2k-t}{t-1}
\]

as needed. \( \square \)

The bound in Theorem 8 is asymptotically sharp. Namely, we have the following.

Theorem 9. Let \( t \geq 1 \) and \( k > t - 1 \) be fixed. Then for every \( \epsilon > 0 \) there is an \( n_0 \) such that for all \( n > n_0 \) and \( V = \mathbb{F}_q^n \) there is a \((t-1)\)-avoiding family \( \mathcal{F} \subset \binom{V}{k} \) with \( |\mathcal{F}| > (1 - \epsilon) \frac{n}{t-1} \frac{2k-t}{k} / \frac{2k-t}{t-1} \).

To prove Theorem 9 we need the following variant of the packing theorem of Rödl [14].

Theorem 10. Let \( r \) and \( s \) be fixed. Then for every \( \epsilon > 0 \) there is an \( n_0 \) such that for all \( n > n_0 \) and \( V = \mathbb{F}_q^n \) there is a family \( \mathcal{H} \subset \binom{V}{r} \) which satisfies \( \dim(H \cap H') < s \) for all \( H, H' \in \mathcal{H} \) and \( |\mathcal{H}| > (1 - \epsilon) \frac{n}{s} / \frac{r}{t} \).

Proof of Theorem 9. By Theorem 10 we can take a family \( SS \subset \binom{V}{2k-t} \) with \( \dim(S \cap S') < t - 1 \) for all \( S, S' \in SS \) and \( |SS| \sim \frac{n}{t-1} / \frac{2k-t}{t-1} \) as \( n \to \infty \). Let \( \Delta_k(SS) \). Then \( \mathcal{F} \) is \((t-1)\)-avoiding and \( |\mathcal{F}| \sim \frac{2k-t}{k} |SS| \) because \( k > t - 1 \). Thus \( \mathcal{F} \) satisfies the desired properties. \( \square \)
Finally we remark that Theorem 10 is derived from the following result stating that almost regular hypergraphs have almost perfect matchings. This result was originally obtained by Frankl and Rödl [9] and we use a stronger version given by Pippenger (see [1] or [13]).

**Theorem 11.** (See [9,1,13].) Let $F \subset \binom{X}{k}$ satisfy the following.

1. There is $D$ such that $\#\{F \in F : x \in F\} = D$ for all $x \in X$.
2. For all $\{x, y\} \in \binom{X}{2}$, $\#\{F \in F : \{x, y\} \subset F\} = o(D)$ as $D \to \infty$.

Then there exist pairwise disjoint $F_1, \ldots, F_m \in F$ with $m \sim \frac{|X|}{k}$ (as $D \to \infty$ and hence $|X| \to \infty$).

**Proof of Theorem 10.** Let $X = \binom{V}{r}$ and $k = \binom{s}{r}$. Define $\mathcal{F} := \{E \in \binom{X}{k} : R \in E \subset \binom{V}{r}\}$. Then $\mathcal{F}$ is $D$-regular, where $D = \binom{n-s-1}{r-s-1}$. Moreover, for a pair $\{x, y\} \subset X$, we have

$$\#\{F \in \mathcal{F} : \{x, y\} \subset F\} \leq \frac{n-s-1}{r-s-1} = o(D).$$

In fact if $n \to \infty$ for fixed $r$ and $s$, then $D \to \infty$ and $\frac{n-s-1}{r-s-1} \to 0$, namely, $\frac{n-s-1}{r-s-1} = o(D)$. Thus, by Theorem 11, we have a matching $F_1, \ldots, F_m \in F$ with $m \sim \binom{n}{s}/\binom{s}{r}$. For $1 \leq i \leq m$ we can write $F_i = \binom{R_i}{s}$. Then $\mathcal{H} := \{R_1, \ldots, R_m\} \subset \binom{V}{r}$ satisfies the desired properties of Theorem 10.

**Acknowledgments**

The authors thank the referees for their careful reading and many helpful comments.

**References**