A structural result for 3-graphs

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Abstract. Suppose that $\mathcal{F} \subset \binom{[n]}{3}$ contains no three sets whose intersection is empty and their union has size at most 6. We prove a structure theorem for such families which easily implies the best possible bound, $|\mathcal{F}| \leq \binom{n-1}{2}$. 
Let \([n] = \{1, 2, \ldots, n\}\). For an integer \(k, 0 \leq k \leq n\) let \(\binom{n}{k}\) denote the collection of all \(k\)-element subsets of \([n]\). A family \(\mathcal{F} \subseteq \binom{n}{k}\) is called a \(k\)-graph.

**Definition 1.** The \(k\)-element sets \(A, B, C\) are called a **Katona-triple** if both \(A \cap B \cap C = \emptyset\) and \(|A \cup B \cup C| \leq 2k\) hold.

Let \(m(n, k)\) denote the maximum of \(|\mathcal{F}|\) for \(\mathcal{F} \subseteq \binom{n}{k}\) over all \(\mathcal{F}\) without a Katona-triple.

For \(n < \frac{3}{2}k\), \(A \cap B \cap C \neq \emptyset\) for all choices of \(A, B, C \in \binom{n}{k}\). Consequently, \(m(n, k) = \binom{n}{k}\) holds.

For \(\frac{3}{2}k \leq n \leq 2k\) the second condition, \(|A \cup B \cup C| \leq 2k\) is satisfied automatically. Thus \(m(n, k) = \binom{n-1}{k-1}\) follows from the following.

**Theorem 1** (Frankl [Fra76], 1976). Let \(r \geq 3\) be an integer, \(n \geq \frac{r}{r-1}k\) and suppose that \(\mathcal{F} \subseteq \binom{n}{k}\) is \(r\)-wise intersecting. That is, \(F_1 \cap \ldots \cap F_r \neq \emptyset\) for all \(F_1, \ldots, F_r \in \mathcal{F}\). Then
\[
|\mathcal{F}| \leq \binom{n-1}{k-1}.
\]

Moreover, in case of equality, for some fixed element \(x \in [n]\) one has \(\mathcal{F} = \{F \in \binom{n}{k} : x \in F\}\).

**Remark 1.** The statement about uniqueness was not stated in [Fra76] but it is proved, e. g., in [Fra87].

What happens for \(n > 2k\)? This was a problem asked by Katona and answered by Frankl and Füredi [FF83] who proved the following result.

**Theorem 2** ([FF83], 1983). Suppose that \(\mathcal{F} \subseteq \binom{n}{k}\), \(\mathcal{F}\) contains no Katona-triple and \(n > k^2 + 3k\). Then (1) holds and the only optimal family is the star.

Frankl and Füredi conjectured that the same holds true for all \(n > 2k\) as well. They proved it in [FF83] for \(k = 3\) and claim it for \(k = 4, 5\) (without proof). Mubayi [Mub06] proved this conjecture for all \(k\) and \(n > 2k\) via an entirely different proof. For four and more sets cf. [Mub07] and [MR09].
Our aim is to use a different approach and derive (1) in the first non-trivial case, \( k = 3 \) from a structure theorem.

**Definition 2.** Let \( \mathcal{F} \subset 2^{[n]} \) be a family, \( F \in \mathcal{F} \). The subset \( G \subset F \) is called **unique** if there is no different \( F' \in \mathcal{F} \) with \( G \subset F' \).

**Theorem 3** (Bollobás [Bol65], 1963). Suppose that for every member \( H \) of the family \( \mathcal{H} \subset 2^{[n]} \), \( G(H) \subset H \) is a unique subset. Then

\[
\sum_{H \in \mathcal{H}} \frac{1}{\binom{n-|H-G(H)|}{|G(H)|}} \leq 1
\]

holds.

Since for an antichain \( \mathcal{H} \) and for all \( H \in \mathcal{H} \) the choice \( G(H) = H \) provides a unique subset, (2) generalizes the famous LYM-inequality.

**Corollary 1.** Suppose that \( \mathcal{H} \subset \binom{[n]}{k} \) and every \( H \in \mathcal{H} \) has a unique \((k-1)\)-subset \( G(H) \subset H \). Then

\[
|\mathcal{H}| \leq \binom{n-1}{k-1}
\]

holds.

Indeed, for such \( H, G(H) \) each term in (2) is exactly \( \frac{1}{k-1} \).

One can show that equality is possible only for the full star of a fixed vertex. Let us mention that (3) can be proved without using (2). One proof is using linear independence. Another one is using a weight function \( w(H, G) \) for all pairs \( G \subset H \in \mathcal{H} \), \( |G| = k - 1 \). Assigning weights 1 to \((H, G)\) for \( G = G(H) \) and \( \frac{1}{n-k+1} \) for \( G \neq G(H) \) assures that for \( G \in \binom{[n]}{k-1} \)

\[
\sum_{H \in \mathcal{H}} w(H, G) \leq 1
\]

and (3) follows by simple calculation.

Let us state our main result.
Theorem 4. Suppose that \( F \subseteq \binom{[n]}{3} \) contains no Katona-triple. Then \( F \) can be partitioned into two families \( \mathcal{H} \) and \( \mathcal{B} \) and the ground set \([n]\) into two disjoint subsets \( Y \) and \( Z \) such that

- \( \mathcal{H} \subseteq \binom{Y}{3} \) and every \( H \in \mathcal{H} \) contains a unique 2-element set,
- \( \mathcal{B} \subseteq \binom{Z}{3} \) and \( \mathcal{B} \) is the vertex-disjoint union of \( \frac{|Z|}{4} \) complete 3-graphs on 4 vertices.

Proof of the theorem:

Let us define
\[
\mathcal{H} = \left\{ H \in F : \exists G = G(H) \in \binom{[n]}{2} \text{ such that } G \text{ is unique} \right\}.
\]

Set \( \mathcal{B} = F - \mathcal{H} \). Note that for all \( B \in \mathcal{B} \) and every \( b \in B \), there exists \( F = F(B, b) \), \( F \neq B \) such that \( (B - \{ b \}) \) is contained in \( F \). A priori there might be several choices for such an \( F \). However we prove the following.

Lemma 1. For every \( B \in \mathcal{B} \) there exists an element \( c \in ([n] - B) \) such that
\[
F(B, b) = (B - \{ b \}) \cup \{ c \}
\]
holds for each \( b \in B \).

Proof: Let \( B = \{ b_1, b_2, b_3 \} \) and let \( c_1, c_2, c_3 \) be such that \( F(B, b_i) = (B - b_i) \cup \{ c_i \} \) holds. Note that
\[
F(B, b_1) \cup F(B, b_2) \cup F(B, b_3) = \{ b_1, b_2, b_3 \} \cup \{ c_1, c_2, c_3 \},
\]
i. e., it consist of at most six elements. Since it is not a Katona-triple, the intersection of these three sets is non-empty. Let \( c \) denote the common element. Then \( c = c_i, i = 1, 2, 3 \) and the uniqueness of the choice of \( c_i \) follows. \( \Box \)
By the lemma every $B \in \mathcal{B}$ gives rise to a complete 3-graph on four vertices (with vertex set $B \cup \{c\}$). Since every 2-subset of a complete 3-graph on four vertices is contained in two edges, $(B - \{b_i\}) \cup \{c\}$ is in $\mathcal{B}$ for all $i = 1, 2, 3$.

The following lemma was essentially proved already in [FF83].

**Lemma 2.** If $F, F' \in \mathcal{F}$ satisfy $F \cap F' = \{y\}$ for some $y \in [n]$ then $(F' - \{y\})$ is a unique subset.

**Proof:** Suppose the contrary and let $F'' \in \mathcal{F}$ satisfy $F' \neq F''$ and $(F' - \{y\}) \subset F''$. Then $y \notin F''$ implies that $F \cap F' \cap F'' = \emptyset$. Since $F \cup F' \cup F'' = F \cup F' \cup (F'' - F')$, the size of the union is at most six. That is $F, F', F''$ form a Katona-triple, a contradiction. 

Let $D \in \binom{[n]}{3}$ satisfy that $(D \setminus 3) \subset F$. Let us prove the following.

**Proposition 1.** For every $F \in \mathcal{F}$ either $F \subset D$ or $F \cap D = \emptyset$ holds.

**Proof:** Suppose $F \not\subset D$. If $|F \cap D| = 1$ or 2 then we have exactly two choices for $B \in \binom{D}{3}$ satisfying $|F \cap B| = 1$. Setting $F' = B$ and using $B \in \mathcal{B}$ this contradicts Lemma 2. The only remaining possibility is $F \cap D = \emptyset$. 

To conclude the proof of the theorem just let $Z$ be the union of all $D \in \binom{[n]}{3}$ with $(D \setminus 3) \subset \mathcal{F}$, and set $Y = ([n] - Z)$.

In view of the Bollobás Theorem,

$$|\mathcal{F}| \leq \left(\frac{|Y| - 1}{2}\right) + 4 \cdot \left\lfloor \frac{|Z|}{4} \right\rfloor \leq \left(\frac{|Y| - 1}{2}\right) + n - |Y|.$$ 

For $n \geq 5$ the right hand side is maximized for $|Y| = n$ and its maximal value is $\binom{n-1}{2}$.

It would be interesting to find a structure theorem for $k$-graphs with $k \geq 4$ without Katona-triples that implies the $\binom{n-1}{k-1}$ upper bound, i. e., $m(n, k) = \binom{n-1}{k-1}$. 

References


