A proof of the Hilton–Milner Theorem without computation

by Peter Frankl, Rényi Institute, Budapest, Hungary

Abstract

Let \( n \geq 2k \geq 4 \) be integers and \( \mathcal{F} \) a family of \( k \)-subsets of \( \{1, 2, \ldots, n\} \). It is called intersecting if \( F \cap F' \neq \emptyset \) for all \( F, F' \in \mathcal{F} \). It is called non-trivial if \( \bigcap_{F \in \mathcal{F}} F = \emptyset \). Strengthening the famous Erdős–Ko–Rado Theorem Hilton and Milner proved that \( |\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 \) if \( \mathcal{F} \) is non-trivial and intersecting. We provide a proof by injection of this result.

1 Introduction

Let \( [n] = \{1, \ldots, n\} \) be the standard \( n \)-element set and \( 2^{[n]} \) its power set. Subsets \( \mathcal{F} \subset 2^{[n]} \) are called families. For \( i \in [n] \) we use the standard notations \( \mathcal{F}(i) = \{ F \setminus \{i\} : i \in F \in \mathcal{F}\} \) and \( \mathcal{F}(i) = \{ F : i \notin F \in \mathcal{F}\} \). Note that

\[
|\mathcal{F}| = |\mathcal{F}(i)| + |\mathcal{F}(i)|.
\]

For a positive integer \( t \) the family \( \mathcal{F} \) is said to be \( t \)-intersecting if \( |F \cap F'| \geq t \) for all \( F, F' \in \mathcal{F} \). For \( t = 1 \) we use the term intersecting.

Let us recall the definition of the \( S_{i,j} \) shift, an important operation on families, discovered by Erdős, Ko and Rado [EKR].

Definition 1.1. For \( 1 \leq i < j \leq n \) and a family \( \mathcal{F} \subset 2^{[n]} \) one defines \( S_{i,j}(\mathcal{F}) = \{ S_{i,j}(F) : F \in \mathcal{F}\} \) where

\[
S_{i,j}(F) = \begin{cases}
F' & \text{def} \quad (F \setminus \{j\}) \cup \{i\} \quad \text{if } j \in F, \ i \notin F \text{ and } F' \notin \mathcal{F}, \\
F & \text{otherwise}.
\end{cases}
\]
From the definition $|S_{i,j}(\mathcal{F})| = |\mathcal{F}|$ and $|S_{i,j}(F)| = |F|$ should be obvious. More importantly, if $\mathcal{F}$ is $t$-intersecting then $S_{i,j}(\mathcal{F})$ is $t$-intersecting as well.

If $S_{i,j}(\mathcal{F}) = \mathcal{F}$ for all $1 \leq i < j \leq n$ then $\mathcal{F}$ is called shifted.

Let us use the notation $(a_1, a_2, \ldots, a_r)$ to denote the set $\{a_1, a_2, \ldots, a_r\}$ where $a_1 < a_2 < \cdots < a_r$. For two subsets $F = (a_1, \ldots, a_r)$ and $G = (b_1, \ldots, b_r)$ we say that $F$ is smaller than $G$ if $a_i \leq b_i$ for all $1 \leq i \leq r$. We denote this by $F \prec G$.

It is not hard to see that $F$ is shifted iff for all pairs of $F, G$ with $F \prec G$, $G \in F$ implies $F \in F$. For the proof of this and many other useful properties of shifting cf. [F87].

We shall need the following simple result.

**Proposition 1.2 ([F78]).** $\mathcal{F} \subset 2^{[n]}$ be a shifted $t$-intersecting family. Then (i) and (ii) hold.

(i) For every $F \in \mathcal{F}$ there exists an integer $\ell \geq t$ such that

$$|F \cap [2\ell - t]| \geq \ell.$$

(ii) For all $F, G \in \mathcal{F}$ there exists an integer $h \geq t$ such that

$$|F \cap [h]| + |G \cap [h]| \geq h + t. \tag{1.1}$$

*Note that (1.1) implies $|F \cap G \cap [h]| \geq t$.***

For $F \in \mathcal{F}$ define $\ell(F) = \{\max \ell, t \leq \ell \leq \frac{n}{2} : |F \cap [2\ell]| \geq \ell\}$. Note that if $2|F| \leq n$ then the maximality of $\ell(F)$ implies

$$|F \cap [2\ell(F)]| = \ell(F). \tag{1.2}$$

Let $k \geq s \geq 2$ be integers. Let $\binom{[n]}{k}$ denote the collection of all $k$-subsets of $[n]$.

**Example 1.3.** Define $\mathcal{E}(n, k, s) = \left\{E \in \binom{[n]}{k} : 1 \in E, E \cap [2, s + 1] \neq \emptyset\right\} \cup \left\{F \subset \binom{[2n]}{k} : [2, s + 1] \subset F\right\}$.

Note that $\mathcal{E}(n, k, s)$ is intersecting, $E \cap [2, s + 1] \neq \emptyset$ for all $E \in \mathcal{E}(n, k, s)$ and

$$|\mathcal{E}(n, k, s)| = \binom{n - 1}{k - 1} - \binom{n - s - 1}{k - 1} + \binom{n - s - 1}{k - s}.$$
Theorem 1.4. Let $n \geq 2k \geq 2s \geq 4$. Suppose that $F \subset \binom{[n]}{k}$ is a shifted intersecting family satisfying $F \cap [2, s+1] \neq \emptyset$ for all $F \in \mathcal{F}$. Then

\begin{equation}
|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-s-1}{k-1} + \binom{n-s-1}{k-s}.
\end{equation}

This result is somewhat technical but its proof is rather special. We are going to prove it through an explicit injection from $\mathcal{F}$ into $\mathcal{E}(n, k, s)$.

For sets $A, B$ let $A \triangle B$ denote their symmetric difference. Let us define the map $\alpha : \mathcal{F} \to \mathcal{E}(n, k, s)$ by

$$\alpha(F) = \begin{cases} F & \text{if } 1 \in F \text{ or if } [2, s+1] \subset F, \\ F \triangle [2\ell(F)] & \text{otherwise}. \end{cases}$$

To prove (1.3) it is sufficient to prove the following.

Proposition 1.5. The map $\alpha$ is an injection into $\mathcal{E}(n, k, s)$.

Let us recall two important results concerning intersecting families of $k$-sets.

**Erdős–Ko–Rado Theorem** ([EKR]). Suppose that $n \geq 2k > 0$, $\mathcal{F} \subset \binom{[n]}{k}$ is an intersecting family. Then

\begin{equation}
|\mathcal{F}| \leq \binom{n-1}{k-1}.
\end{equation}

Taking all $k$-sets containing a fixed element shows that (1.4) is best possible.

An intersecting family is called *non-trivial* if there is no element common to all its members. For $k = 1$ there is no non-trivial $k$-intersecting family. For $k = 2$ the only such family is the triangle: $\binom{[3]}{2}$.

**Hilton–Milner Theorem** ([HM]). Suppose that $n \geq 2k \geq 4$ and $\mathcal{F} \subset \binom{[n]}{k}$ is a non-trivial intersecting family. Then

\begin{equation}
|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.
\end{equation}

Recently Hurlbert and Kamat [HK] gave an injective proof for (1.4). We extend their work by providing an injective proof for (1.5). For this we need the following proposition.
Proposition 1.6 ([F87]). Suppose that \( n \geq 2k \geq 4 \), \( \mathcal{F} \subset \binom{[n]}{k} \) is a non-trivial intersecting family of maximal size. Then there exists a non-trivial intersecting family \( \tilde{\mathcal{F}} \subset \binom{[n]}{k} \) such that \( |\tilde{\mathcal{F}}| = |\mathcal{F}| \) and \( \tilde{\mathcal{F}} \) is shifted.

Once one has Proposition 1.6, to establish (1.5) is easy. One only needs to apply the case \( s = k \) of Theorem 1.4 to the family \( \tilde{\mathcal{F}} \). Indeed, since \( \tilde{\mathcal{F}} \) is non-trivial and shifted, \( [2, k + 1] \in \tilde{\mathcal{F}} \) and being intersecting \( F \cap [2, k + 1] \neq \emptyset \) holds for all \( F \in \tilde{\mathcal{F}} \).

Since the proof of Proposition 1.6 is quite short and somewhat hidden in [F87], we reproduce it in Section 2.

Let us mention that there are several other, known proofs of the Hilton–Milner Theorem, cf. [FF], [FT], [M] or [KZ].

2 The proof of Propositions 1.5 and 1.6

We divide the proof into two lemmas. The first shows that for \( F \in \mathcal{F} \setminus \mathcal{E}(n, k, s) \) the image \( \alpha(F) \) is in \( \mathcal{E}(n, k, s) \setminus \mathcal{F} \).

The second shows that \( \alpha \) is an injection.

Lemma 2.1. Suppose that \( F \in \mathcal{F}(\overline{1}) \) and \( [2, s + 1] \not\subset F \). Then (i), (ii) and (iii) hold.

(i) \( 1 \in \alpha(F) \);
(ii) \( \alpha(F) \not\in \mathcal{F} \);
(iii) \( \alpha(F) \cap [2, s + 1] \neq \emptyset \).

Proof. Recall that \( \alpha(F) = F \triangle [2\ell(F)] \). As \( 1 \not\in F \) implies \( 1 \in \alpha(F) \), (i) is true.

(ii) Suppose for contradiction that \( \alpha(F) \in \mathcal{F} \). Apply Proposition 1.2 to \( F \) and \( \alpha(F) \). By (1.2), \( F \cap [2\ell(F)] \) and \( \alpha(F) \cap [2\ell(F)] \) are complementary \( \ell \)-element subsets of \( [2\ell(F)] \). Consequently \( h > 2\ell(F) \).

However, for \( h \geq 2\ell \), \( |F \cap [h]| = |\alpha(F) \cap [h]| \). Thus \( 2|F \cap [h]| \geq h + 1 \) implies

\[
|F \cap [h]| \geq (h + 1)/2.
\]

Thus for \( h + 1 \) as well

\[
|F \cap [h + 1]| \geq (h + 1)/2
\]
and we get a contradiction with the maximality of $\ell(F)$.

(iii) Define $i(F) = \min \{ i : 2 \leq i \leq n, i \notin F \}$. As $\ell(F) \geq 2$, (1.2) implies $i(F) \leq 2\ell(F)$. Also, $[2, s + 1] \not\subset F$ implies $i(F) \leq s + 1$. Consequently $i(F) \in [2\ell(F)]$ and $i(F) \in [2, s + 1]$ hold. Therefore $i(F) \in \alpha(F) \cap [2, s + 1]$. \qed

Lemma 2.2. For distinct $F, F' \in \mathcal{F} \setminus \mathcal{E}(n, k, s)$, $\alpha(F) \neq \alpha(F')$ holds.

Proof. Since $F, F' \notin \mathcal{E}(n, k, s)$, $\alpha(F) = F \triangle [2\ell(F)]$ and $\alpha(F') = F' \triangle [2\ell(F')]$. If $\ell(F) = \ell(F')$ then $\alpha(F) \neq \alpha(F')$ is evident from $F \neq F'$.

By symmetry suppose $\ell(F) < \ell(F')$. The maximality of $\ell(F)$ implies $|F \cap [2\ell(F)]| < \ell(F)$. Using $|F \cap [2\ell(F)]| = \ell(F) = |\alpha(F) \cap [2\ell(F)]|$, $|\alpha(F) \cap [2\ell(F)]| < \ell(F) = |\alpha(F') \cap [2\ell(F')]|$ follows. This proves $\alpha(F) \neq \alpha(F')$. \qed

Since $\alpha(F) = F$ for $F \in \mathcal{F} \cap \mathcal{E}(n, k, s)$, Lemmas 2.1 and 2.2 prove that $\alpha$ is an injection into $\mathcal{E}(n, k, s)$. \qed

The proof of Proposition 1.6. Starting with a non-trivial intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ of maximal size we can keep on applying the $S_{ij}$ shift for various pairs until we run into trouble. The possible trouble is that $S_{ij}(\mathcal{F})$ ceases to be non-trivial, i.e., all its members contain the element $i$. Then $\{i, j\} \cap F \neq \emptyset$ must hold for all $F \in \mathcal{F}$. By symmetry let $i = 1$, $j = 2$.

The maximality of $|\mathcal{F}|$ implies that all $k$-sets $G$ with $\{1, 2\} \subset G \subset [n]$ are in $\mathcal{F}$. Therefore continuing with the $S_{a,b}$ shift for $3 \leq a < b \leq n$ will never produce a trivial intersecting family. Eventually we obtain a non-trivial intersecting family $\mathcal{H}$, $|\mathcal{H}| = |\mathcal{F}|$ such that $S_{a,b}(\mathcal{H}) = \mathcal{H}$ for all $3 \leq a < b \leq n$.

Consequently, both $\{1, 3, 4, \ldots, k + 1\}$ and $\{2, 3, 4, \ldots, k + 1\}$ are in $\mathcal{H}$. Since all $G \in \binom{[n]}{k}$ with $\{1, 2\} \subset G \subset [n]$ are unchanged under the shift $S_{a,b}$ for $3 \leq a < b \leq n$, we infer that $\binom{[k+1]}{k} \subset \mathcal{H}$.

Noting that $\binom{[k+1]}{k}$ is not affected by $S_{i,j}$ for $1 \leq i < j \leq n$, we can continue shifting and eventually obtain a shifted, non-trivial intersecting family of the same size. \qed

3 Concluding remarks

For a family $\mathcal{F} \subset 2^{[n]}$ let $\triangle(\mathcal{F})$ be its maximum degree, that is, $\max_{i} |\mathcal{F}(i)|$. Then $g(\mathcal{F}) = |\mathcal{F}| - \triangle(\mathcal{F})$ is called the diversity of $\mathcal{F}$. With this terminology,
for intersecting families $\mathcal{F}$, $\mathcal{F} \subset \binom{[n]}{k}$, $n \geq 2k$, the Hilton–Milner Theorem shows that $\varrho(\mathcal{F}) \geq 1$ implies $|\mathcal{F}| \leq |\mathcal{E}(n, k, k)| = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.

In [F87a] the author proved that $\varrho(\mathcal{F}) \geq \binom{n-s-1}{k-s-1}$ ($3 \leq s \leq k$) implies $|\mathcal{F}| \leq |\mathcal{E}(n, k, s)|$. Kupavskii and Zakharov [KZ] gave a new proof of this result. It would be desirable to have a proof by injection. Let us note that for $\mathcal{F} \subset \mathcal{G}$ necessarily $\varrho(\mathcal{F}) \leq \varrho(\mathcal{G})$ holds.

In the case of Theorem 1.4, we may replace $\mathcal{F}$ by another family $\mathcal{G}$, $\mathcal{F} \subset \mathcal{G} \subset \binom{[n]}{k}$ where $\mathcal{G}$ is shifted, intersecting and all $G \in \binom{[n]}{k}$ with $[2, s + 1] \subset G$ are members of $\mathcal{G}$. For such a special case Theorem 1.4 provides an injective proof. However the general case seems to be harder.

The proofs in [F87a] and [KZ] rely heavily on the Kruskal–Katona Theorem (cf. [Kr], [Ka]). Therefore we feel that it would be desirable to have a proof by injection for this important result as well.

References


