REGULARITY CONDITIONS AND INTERSECTING HYPERGRAPHS

PETER FRANKL

Abstract. Let \((\mathcal{F}, X)\) be a hypergraph with a transitive group of automorphisms. Suppose further that any four edges of \(\mathcal{F}\) intersect nontrivially. Denoting \(|X|\) by \(n\) we prove \(|\mathcal{F}| = O(2^n)\). We show as well that it is not sufficient to suppose regularity instead of the transitivity of \(\text{Aut}(\mathcal{F})\).

1. Introduction. Let \((\mathcal{F}, X)\) be a hypergraph, i.e. \(\mathcal{F}\) is a family of nonempty subsets of \(X\). Let \(|X| = n\).

We say \(\mathcal{F}\) is \(k\)-intersecting if any \(k\) edges of \(\mathcal{F}\) have a nonempty intersection. Obviously a \(k\)-intersecting hypergraph is \(k'\)-intersecting for every \(k' < k\).

Erdős, Ko and Rado [2] observed that a 2-intersecting hypergraph has at most \(2^{n-1}\) edges, moreover if it is 3-intersecting, then \(|\mathcal{F}| = 2^{n-1}\) is possible only if \(\mathcal{F}\) consists of all the subsets of \(X\) containing a fixed element of \(X\).

Brace and Daykin [1] refined this result by proving that if there is no element of \(X\) which is contained in every edge of the \(k\)-intersecting hypergraph \(\mathcal{F}\), then

\[|\mathcal{F}| < (k + 2)2^{n-k-1}.\] (1)

To obtain equality in (1) we have to take all the subsets of \(X\) which contain at least \(k\) elements of a fixed \((k + 1)\)-subset of \(X\).

For \(k = 2\), (1) gives \(2^{n-1}\). If \(n\) is odd then the family of subsets of \(X\) with cardinality exceeding \(n/2\) gives a 2-intersecting hypergraph with transitive group of automorphisms and cardinality \(2^{n-1}\). For \(n\) even the family still has cardinality \(2^{n-1}(1 + o(1))\).

In §2 we prove the following

Theorem 1. Suppose \(\text{Aut}(\mathcal{F})\) is transitive on \(X\), and \(\mathcal{F}\) is 4-intersecting. Then

\[|\mathcal{F}| = o(2^n).\] (2)

For the proof we need the following result of [3]:

Theorem 2. Suppose any three edges of \((\mathcal{F}, X)\) have at least \(t\) elements in common. Let us set \(b = (\sqrt{5} - 1)/2\). Then we have

\[|\mathcal{F}| < b'2^n.\] (3)

We say \((\mathcal{F}, X)\) is regular if every element of \(X\) is contained in the same number of edges. In §3 we construct \(k\)-intersecting regular hypergraphs with

\[|\mathcal{F}| = 2^n-(2^{k+1}-k-2).\]
The construction uses the $k$-dimensional projective space over $GF(2)$.

2. The proof of Theorem 1. Let us set

$$t = \min \{|F_1 \cap F_2 \cap F_3|: F_1, F_2, F_3 \in \mathcal{F}\}.$$

We consider two cases.

Case (a). $t > \log n - 3 \log \log n$. (log means $\log_2$.) By Theorem 2,

$$|\mathcal{F}| < 2^{(\log n - 3 \log \log n)2^n}. \tag{4}$$

Now (2) follows from (4). For $n > n_0$, (4) implies that $|\mathcal{F}| < 2^n/\sqrt{n}$.

Case (b).

$$t < \log n - 3 \log \log n. \tag{5}$$

Let $F, G, H$ be members of $\mathcal{F}$ satisfying $|F \cap G \cap H| = t$. Let us set

$$\mathcal{D} = \{a(F) \cap a(G) \cap a(H) | a \in \text{Aut}(\mathcal{F})\}.$$  

(By $a(F)$ we denote the elementwise image of $F$ by the automorphism $a$.) By the definition of $\mathcal{D}$ we have $\text{Aut}(\mathcal{F}) \subseteq \text{Aut}(\mathcal{D})$, in particular every element of $X$ is contained in the same number, say $d$, of members of $\mathcal{D}$. By an elementary count

$$d = t|\mathcal{D}|/n. \tag{6}$$

Let us choose pairwise disjoint members $D_1, \ldots, D_m$ of $\mathcal{D}$ such that for every member $D_{m+1}$ of $\mathcal{D}$ at least one of the intersections $D_i \cap D_{m+1}$ ($i = 1, \ldots, m$) is nonempty. As $S = D_1 \cup \cdots \cup D_m$ has cardinality $mt$ and has a nonempty intersection with every member of $\mathcal{D}$, some vertex of $S$ is contained in at least $|\mathcal{D}|/mt$ members of $\mathcal{D}$. Taking (6) into account we obtain

$$m > n/t^2. \tag{7}$$

We assert that for $i = 1, \ldots, m$, and any $F \in \mathcal{F}$ we have $D_i \cap F \neq \emptyset$. Indeed by the definition of $\mathcal{D}$ we can find $F_1, F_2, F_3$ such that $D_i = F_1 \cap F_2 \cap F_3$, and using the 4-intersection property of $\mathcal{F}$ we deduce

$$D_i \cap F = F_1 \cap F_2 \cap F_3 \cap F \neq \emptyset.$$

Now the number of subsets of $X$ which have a nonempty intersection with every $D_i$, $i = 1, \ldots, m$, gives an upper bound for $|\mathcal{F}|$, that is

$$|\mathcal{F}| < 2^n - mt(2^t - 1)^m. \tag{8}$$

From (8) using (5) and (7) we deduce

$$|\mathcal{F}| < 2^n \left(1 - \frac{1}{2^t}\right)^{n/t^2} < 2^n \exp(-\log_2 n) < 2^n/n. \tag{9}$$

Now the statement of the theorem follows from (9). Let us close this paragraph with a conjecture.

Conjecture 1. If in Theorem 1 we replace “4-intersecting” by “3-intersecting”, then (2) remains valid.

3. Construction of $k$-intersecting regular hypergraphs. Let $(\mathcal{F}, Y)$ be the hypergraph consisting of the $(k - 1)$-dimensional subspaces of the $k$-dimensional projective space over $GF(2)$. Then $|Y| = 2^{k+1} - 1$, $|\mathcal{F}| = |Y|$, and every element of $Y$ is contained in exactly $2^k - 1$ of the members of $\mathcal{F}$, which all have cardinality
2^k - 1. Let y be an arbitrary element of Y, and let us define Z = Y - \{y\}, \mathcal{R} = \{P \in \mathcal{P}_Y | y \notin P\}. Then for the hypergraph (\mathcal{R}, Z) we have
\[ |Z| = 2^{k+1} - 2, \quad |\mathcal{R}| = (2^{k+1} - 1) - (2^k - 1) = 2^k, \]
\[ |R| = 2^k - 1 \quad \text{for every } R \in \mathcal{R}. \]
Moreover, as the group of automorphisms of (\mathcal{P}, Y) is doubly transitive on Y, \text{Aut}(\mathcal{R}) is transitive on Z. Hence the hypergraph (\mathcal{R}, Y) is regular. Consequently every point of Z is contained in \[|\mathcal{R}|/|Z| = \frac{1}{2} |\mathcal{R}| = 2^{k-1} \] members of \mathcal{R}. For our purposes the most important property of \mathcal{P}, and so of \mathcal{R}, is that any k members of it have a nonempty intersection.

Let X be an n-element set containing Z, and let us define
\[ \mathcal{S} = \{ F \subset X | (F \cap Z) \in \mathcal{R} \}. \]
As \mathcal{R} is k-intersecting, \mathcal{S} is k-intersecting as well. Using the definition of \mathcal{S} we obtain
\[ |\mathcal{S}| = |\mathcal{R}| 2^{n-|Z|} = 2^k 2^n (2^{k+1} - 2) = 2^n (2^{k+1} - k - 2); \]
\[ |\{ F \in \mathcal{S} | z \in F \}| = \left( \frac{1}{2} |\mathcal{R}| \right) (2^n - |Z|) = \frac{1}{2} |\mathcal{S}| \quad (z \in Z); \]
\[ |\{ F \in \mathcal{S} | x \in F \}| = \left( \frac{1}{2} |\mathcal{R}| \right) (\frac{1}{2} 2^n - |Z|) = \frac{1}{2} |\mathcal{S}| \quad (x \in (X - Z)). \]
So we have constructed a regular, k-intersecting hypergraph on n vertices with \[ 2^n (2^{k+1} - k - 2) \] edges. Could we have done better?

**Conjecture 2.** A regular, k-intersecting hypergraph on n vertices has at most \[ 2^n (2^{k+1} - k - 2) \] edges when k > 3.

The best upper bound we can prove for the moment is \[ 2^n b^2 + 3. \]

**References**


Centre National de la Recherche Scientifique, 15 Quai Anatole France, 75007 Paris, France