Quasi-orthogonal subalgebras of $4 \times 4$ matrices

Hiromichi Ohno$^{1,4}$, Dénes Petz$^{2,5,6}$ and András Szántó$^{3,6}$

$^4$ Graduate School of Information Sciences, Tohoku University
Aobaku, Sendai 980-8579, Japan

$^5$ Alfréd Rényi Institute of Mathematics,
H-1364 Budapest, POB 127, Hungary

$^6$ Department for Mathematical Analysis, BUTE,
H-1521 Budapest, POB 91, Hungary

Abstract: Maximal Abelian quasi-orthogonal subalgebras form a popular research problem. In this paper quasi-orthogonal subalgebras of $M_4(\mathbb{C})$ isomorphic to $M_2(\mathbb{C})$ are studied. It is proved that if 4 such subalgebras are given, then their orthogonal complement is always a commutative subalgebra. In particular, 5 such subalgebras do not exist. A conjecture is made about the maximal number of pairwise quasi-orthogonal subalgebras of $M_{2n}(\mathbb{C})$.

Key words: Quasi-orthogonality, Pauli matrices, Cartan decomposition.
MSC: 15A30, 81R05.

1 Introduction

The motivation of this paper comes from the algebraic or matrix formalism of finite quantum systems. An $n$-level quantum system is described by the algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices. The matrix algebra of a composite system consisting of an $n$ level and an $m$-level system is $M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) \simeq M_{nm}(\mathbb{C})$. A subalgebra of $M_k(\mathbb{C})$ corresponds to a subsystem of a $k$-level quantum system.

$^1$E-mail: ono@ims.is.tohoku.ac.jp.
$^2$E-mail: petz@math.bme.hu. Partially supported by the Hungarian Research Grant OTKA T068258.
$^3$E-mail: szbandi@math.bme.hu. Partially supported by the Hungarian Research Grant OTKA TS-49835.
In this paper subalgebras always contain the identity and closed under the adjoint operation of matrices, that is, they are unital *-subalgebras. The algebra $M_k(\mathbb{C})$ can be endowed by the inner product $\langle A, B \rangle = \text{Tr}(A^*B)$ and it becomes a Hilbert space.

A kind of quantum mechanical background gives motivation for the following definition [7]. Two subalgebras $\mathcal{A}(1)$ and $\mathcal{A}(2)$ of $M_k(\mathbb{C})$ are called quasi-orthogonal if $\text{Tr}A_1A_2 = 0$, whenever $A_i \in \mathcal{A}(i)$ and $\text{Tr}A_1 = \text{Tr}A_2 = 0$. Since the intersection of $\mathcal{A}(1)$ and $\mathcal{A}(2)$ contain the identity, they cannot be orthogonal, but $\mathcal{A}_1 \ominus \mathbb{C}I \perp \mathcal{A}_1 \ominus \mathbb{C}I$ can happen, and this is exactly the quasi-orthogonality. In the literature of quantum mechanics the terminology complementarity is used instead of quasi-orthogonality, see [1, 5].

The analysis of pairwise quasi-orthogonal maximal Abelian subalgebras is a popular subject [2, 3]. If $\mathcal{A} \subset M_k(\mathbb{C})$ is a maximal Abelian subalgebra and $W$ is a unitary, then $\mathcal{A}$ and $W\mathcal{A}W^*$ are quasi-orthogonal if and only if $W$ is a Hadamard matrix. The maximal number of pairwise quasi-orthogonal maximal Abelian subalgebras is an open problem [9].

A different problem is the analysis of pairwise quasi-orthogonal non-commutative subalgebras [6]. If $\mathcal{A} \subset M_{k^2}(\mathbb{C})$ is isomorphic to $M_k(\mathbb{C})$, then the commutant $\mathcal{A}'$ of $\mathcal{A}$ is quasi-orthogonal to $\mathcal{A}$. Another example of two quasi-orthogonal subalgebras isomorphic to $M_k(\mathbb{C})$ was shown in [6]. The maximal number of such (pairwise) quasi-orthogonal subalgebras is not known except for the case $k = 2$. Then the maximum is 4 as this was proved in [8]. The aim of the present paper is to study the structure of the 4 pairwise quasi-orthogonal subalgebras. The analysis of the structure gives that the quasi-orthogonal complement of 4 (pairwise) quasi-orthogonal subalgebras is always a maximal Abelian subalgebra.

Although we are mostly concentrate on subalgebras of $M_4(\mathbb{C})$, we try to extend the results to subalgebras of $M_{2^n}(\mathbb{C})$. Let $m(n)$ be the maximal number of pairwise quasi-orthogonal subalgebras of $M_{2^n}(\mathbb{C})$ which are isomorphic to $M_2(\mathbb{C})$. We show that

$$m(n) \geq \frac{4^n - 1}{3} - 1$$

and we conjecture that the inequality is actually an equality.

2 Preliminaries

A natural orthogonal basis of $M_2(\mathbb{C})$ consists of the Pauli matrices:

$$\sigma_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Computation in the Pauli basis is convenient.
For $x, y \in \mathbb{R}^3$ and
\[ x \cdot \sigma := \sum_{i=1}^{3} x_i \sigma_i, \quad y \cdot \sigma := \sum_{i=1}^{3} y_i \sigma_i \]
we have
\[ (x \cdot \sigma)(y \cdot \sigma) = \langle x, y \rangle \sigma_0 + i(x \times y) \cdot \sigma, \quad (1) \]
where $x \times y$ is the vectorial product in $\mathbb{R}^3$.

If we want to construct a subalgebra of $M_3(\mathbb{C})$ which is isomorphic to $M_2(\mathbb{C})$, then it is enough to find $S_1, S_2, S_3 \in M_3(\mathbb{C})$ such that $S_j$ is a self-adjoint unitary ($1 \leq j \leq 3$) and $S_3 = -iS_1S_2$. When a triplet $(S_1, S_2, S_3)$ satisfies these conditions, it will be called a Pauli triplet. For such a triplet $\text{Tr } S_i = 0$ and $\text{Tr } S_iS_j = 0$ for $i \neq j$. The latter relation is interpreted as the orthogonality of $S_i$ and $S_j$. Given a Pauli triplet $(S_1, S_2, S_3)$, the linear mapping defined as
\[ \sigma_0 \mapsto I, \quad \sigma_1 \mapsto S_1, \quad \sigma_2 \mapsto S_2, \quad \sigma_3 \mapsto -iS_1S_2 \]
is an algebraic isomorphism between $M_2(\mathbb{C})$ and the linear span of the operators $I, S_1, S_2$ and $S_3$.

Although our aim is to study subalgebras of $M_3(\mathbb{C})$, the next result is in a more general setting. If $e, f, g$ are vectors of a Hilbert space, then the linear operator $|e\rangle\langle f|$ acts as $|e\rangle\langle f|g := \langle f, g\rangle e$.

**Theorem 1** Let $E_i$ be an orthonormal basis in $M_n(\mathbb{C})$ and let $W = \sum_i E_i \otimes W_i \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ be a unitary. The subalgebra $W(\mathbb{C}I \otimes M_m(\mathbb{C}))W^*$ is quasi-orthogonal to $\mathbb{C}I \otimes M_m(\mathbb{C})$ if and only if
\[ \frac{m}{n} \sum_k |W_k\rangle\langle W_k| \]
is the identity mapping on $M_m(\mathbb{C})$. This condition cannot hold if $m < n$ and in the case $n = m$ the condition means that $\{W_k : 1 \leq k \leq n^2\}$ is an orthonormal basis in $M_m(\mathbb{C})$.

**Proof:** Assume that $A, B \in M_m(\mathbb{C})$ and $\text{Tr } B = 0$. Then the condition
\[ W(I \otimes A^*)W^* \perp (I \otimes B) \]
is equivalently written as
\[ \text{Tr } (W(I \otimes A)W^*(I \otimes B)) = \sum_{k,l} \text{Tr } (E_kE_l^\ast)\text{Tr } (W_kAW_l^*B) = \sum_k \text{Tr } (W_kAW_k^*B) = 0. \]

Putting $B = \text{Tr } (B)I_m/m$ in place of $B$, we get
\[ \sum_k \text{Tr } (W_kAW_k^*B) = \frac{\text{Tr } B}{m} \sum_k \text{Tr } (W_kAW_k^*) \]
for every $B \in M_m(\mathbb{C})$. Let $\mathcal{E}_2 : M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) \rightarrow M_m(\mathbb{C})$ be the linear mapping defined as

$$\mathcal{E}_2(K \otimes L) = \frac{\text{Tr} K}{n} L.$$ 

Since $\mathcal{E}_2$ is unit-preserving and $W$ is a unitary,

$$I_m = \mathcal{E}_2(W^*W) = \mathcal{E}_2\left( \sum_{k,l} E_k^* E_l \otimes W_k^* W_l \right) = \frac{1}{n} \sum_{k,l} \text{Tr} (E_k^* E_l) W_k^* W_l = \frac{1}{n} \sum_k W_k^* W_k,$$

and we arrive at the relation

$$\sum_k \text{Tr} W_k A W_k^* B = \frac{n}{m} \text{Tr} A \text{Tr} B. \quad (2)$$

We can transform this into another equivalent condition in terms of the left multiplication, right multiplication and $|W_k\\rangle \langle W_k|$ operators.

For $A, B \in M_m(\mathbb{C})$, the operator $R_A$ is the right multiplication by $A$ and the operator $L_B$ is the left multiplication by $B$: $R_A, L_B : M_m(\mathbb{C}) \rightarrow M_m(\mathbb{C})$, $R_A X = X A$, $L_B X = B X$. If $\lambda_i$’s are the eigenvalues of $A$ and $\mu_j$’s are the eigenvalues of $B$, then $\lambda_i \mu_j$’s are the eigenvalues of $R_A L_B$. Therefore

$$\text{Tr} R_A L_B = \left( \sum_i \lambda_i \right) \left( \sum_j \mu_j \right) = \text{Tr} A \text{Tr} B.$$ 

We have

$$\sum_k \text{Tr} |W_k\\rangle \langle W_k| R_A L_B = \sum_k \langle W_k, R_A L_B W_k \rangle = \sum_k \text{Tr} W_k A W_k^* B$$

$$= \frac{n}{m} \text{Tr} A \text{Tr} B = \frac{n}{m} \text{Tr} R_A L_B$$

for every $A, B \in M_m(\mathbb{C})$. Since the operators $R_A L_B$ linearly span the space of all linear operators on $M_m(\mathbb{C})$, we have

$$\frac{m}{n} \sum_k |W_k\\rangle \langle W_k| = I_m^2.$$ 

This is our statement. $\square$

3 Main results

Assume that $\{A(i)\}_{i=0}^3$ is a family of pairwise quasi-orthogonal subalgebras of $M_4(\mathbb{C})$ which are isomorphic to $M_2(\mathbb{C})$. The commutant of $A(i)$ will be denoted by $A(i)'$. Our aim is to describe the relation of the subalgebras $\{A(i)\}_{i=0}^3$ and $\{A(i)’\}_{i=0}^3$. 4
Without restricting the generality, we may assume that \( \mathcal{A}(0) = CI \otimes M_2(\mathbb{C}) \). Then the commutant of \( \mathcal{A}(0) \) is \( \mathcal{A}(0)' = M_2(\mathbb{C}) \otimes CI \), moreover there are unitaries \( W_i \) such that
\[
W_i \mathcal{A}(0) W_i^* = \mathcal{A}(i) \quad \text{and} \quad \mathcal{A}(j)' = W_j \mathcal{A}(0)' W_j^* \quad (1 \leq j \leq 3).
\]

**Theorem 2** Let \( \mathcal{A} \) and \( \mathcal{B} \) be quasi-orthogonal subalgebras of \( M_4(\mathbb{C}) \) which are isomorphic to \( M_2(\mathbb{C}) \). Then the intersection \( \mathcal{A}' \cap \mathcal{B} \) is an at least two dimensional subspace of \( M_4(\mathbb{C}) \).

**Proof:** We may assume that \( \mathcal{A} = \mathcal{A}(0) = CI \otimes M_2 \).

The \( 4 \times 4 \) matrices
\[
C = \begin{bmatrix}
a & 0 & 0 & b \\
0 & c & d & 0 \\
0 & d & c & 0 \\
b & 0 & 0 & a
\end{bmatrix}
\]
form a commutative algebra \( \mathcal{C} \). Since
\[
\sum_{i=0}^{3} c_i \sigma_i \otimes \sigma_i = \begin{bmatrix}
c_0 + c_3 & 0 & 0 & c_1 - c_2 \\
0 & c_0 - c_3 & c_1 + c_2 & 0 \\
0 & c_1 + c_2 & c_0 - c_3 & 0 \\
c_1 - c_2 & 0 & 0 & c_0 + c_3
\end{bmatrix},
\]
\( \mathcal{C} \) is the linear span of the matrices \( \sigma_i \otimes \sigma_i, 0 \leq i \leq 3 \). (These are the matrices which are diagonal in the so-called Bell basis.)

The algebra \( \mathcal{C} \) plays a special role. Any unitary in \( M_4(\mathbb{C}) \) can be written in the form
\[
(L_1 \otimes L_2) N (L_3 \otimes L_4),
\]
where \( L_1, L_2, L_3, L_4 \) are \( 2 \times 2 \) unitaries and the unitary \( N \) is in \( \mathcal{C} \). This is called Cartan decomposition, see equation (11) in [11] or [4].

There is a unitary \( W \in M_4(\mathbb{C}) \) such that
\[
W (CI \otimes M_2(\mathbb{C})) W^* = \mathcal{B}.
\]
\( W \) has a Cartan decomposition (4). Since the subalgebra \( W (CI \otimes M_2(\mathbb{C})) W^* \) does not depend on \( L_3 \) and \( L_4 \), we may assume that \( L_3 = L_4 = I \). Moreover, the quasi-orthogonality of \( W (CI \otimes M_2(\mathbb{C})) W^* \) and \( CI \otimes M_2(\mathbb{C}) \) does not depend on \( L_1 \) and \( L_2 \). The quasi-orthogonality is determined by the factor \( N \in \mathcal{C} \). Since the matrices \( E_i = \sigma_i / \sqrt{2} \) form a basis in \( M_2(\mathbb{C}) \), Theorem 1 is conveniently applied for the unitary \( N = \sum_{i=0}^{3} c_i \sigma_i \otimes \sigma_i \), choose \( W_i \) as \( c_i \sqrt{2} \sigma_i \). The theorem gives that
\[
2 \sum_{i=0}^{3} |c_i|^2 |\sigma_i \rangle \langle \sigma_i|
\]
is the identity mapping on $M_2(\mathbb{C})$ which implies $|c_i|^2 = 1/4$ ($0 \leq i \leq 3$). In a trigonometric approach, let

$$
\begin{align*}
  c_0 &= \cos \alpha \cos \beta \cos \gamma + i \sin \alpha \sin \beta \sin \gamma, \\
  c_1 &= \cos \alpha \sin \beta \sin \gamma + i \sin \alpha \cos \beta \cos \gamma, \\
  c_2 &= \sin \alpha \cos \beta \sin \gamma + i \cos \alpha \sin \beta \cos \gamma, \\
  c_3 &= \sin \alpha \sin \beta \cos \gamma + i \cos \alpha \cos \beta \sin \gamma.
\end{align*}
$$

In order to get a proper unitary, two of the values of $\cos^2 \alpha, \cos^2 \beta$ and $\cos^2 \gamma$ equal 1/2 and the third one may be arbitrary. Let $\mathcal{N}$ be the set of all matrices such that the parameters $\alpha, \beta$ and $\gamma$ satisfy the above condition, in other words two of the three values are of the form $\pi/4 + k\pi/2$. ($k$ is an integer.) Let

$$
\mathcal{N}_i := \{ N \in \mathcal{N} : \alpha \text{ is arbitrary, } \beta = \pi/4 + k_1\pi/2, \text{ and } \gamma = \pi/4 + k_2\pi/2 \} \quad (5)
$$

and define $\mathcal{N}_2$ and $\mathcal{N}_3$ similarly. ($\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$). Since the subalgebra $\mathcal{N}(\mathbb{C}I \otimes M_2(\mathbb{C}))N^*$ does not depend on the integers $k_1$ and $k_2$, we simply take $k_1 = k_2 = 0$. This makes computations a bit more convenient. One computes that

$$
N_i(I \otimes \sigma_i)N_i^* = \pm \sigma_i \otimes I
$$

for $N_i \in \mathcal{N}_i$ [10]. It follows that

$$
(L_1 \otimes L_2)N_i(I \otimes \sigma_i)N_i^*(L_1^* \otimes L_2^*) = \pm L_1\sigma_iL_1^* \otimes I
$$

for every unitary $N_i \in \mathcal{N}_i$. Therefore $L_1\sigma_iL_1^* \otimes I \in \mathcal{A}(0)^{\prime} \cap \mathcal{B}$. \hfill \nobreak \Box

The theorem immediately gives that the maximal number of pairwise quasi-orthogonal subalgebras isomorphic to $M_2(\mathbb{C})$ is at most 4. Moreover, if $\{ \mathcal{A}(j) \}_{j=0}^3$ are such subalgebras, then each subalgebra $\mathcal{A}(i)^{\prime} \cap \mathcal{A}(j)$ is two-dimensional for $i \neq j$. Here is an example of 4 such subalgebras together with the commutants, each of them is determined by Pauli triplets:

$$
\begin{array}{cccccccc}
\sigma_0 \otimes \sigma_1 & \sigma_0 \otimes \sigma_2 & \sigma_0 \otimes \sigma_3 & \sigma_1 \otimes \sigma_0 & \sigma_2 \otimes \sigma_0 & \sigma_3 \otimes \sigma_0 \\
\sigma_1 \otimes \sigma_0 & \sigma_2 \otimes \sigma_1 & \sigma_3 \otimes \sigma_1 & \sigma_0 \otimes \sigma_1 & \sigma_1 \otimes \sigma_2 & \sigma_1 \otimes \sigma_3 \\
\sigma_2 \otimes \sigma_0 & \sigma_1 \otimes \sigma_2 & -\sigma_3 \otimes \sigma_2 & \sigma_2 \otimes \sigma_1 & \sigma_2 \otimes \sigma_3 & -\sigma_0 \otimes \sigma_2 \\
\sigma_3 \otimes \sigma_0 & \sigma_1 \otimes \sigma_3 & \sigma_2 \otimes \sigma_3 & \sigma_0 \otimes \sigma_3 & \sigma_3 \otimes \sigma_1 & \sigma_3 \otimes \sigma_2
\end{array}
$$

(6)

Our next aim is to describe the structure of 4 such algebras in general.

**Theorem 3** Assume that $\{ \mathcal{A}(i) \}_{i=0}^3$ is a family of pairwise quasi-orthogonal subalgebras of $M_4(\mathbb{C})$ which are isomorphic to $M_2(\mathbb{C})$. For every $0 \leq i \leq 3$, there exists a Pauli triplet $\mathcal{A}(i, j)$ ($j \neq i$) such that $\mathcal{A}(i)^{\prime} \cap \mathcal{A}(j)$ is the linear span of $I$ and $\mathcal{A}(i, j)$. Moreover, the subspace linearly spanned by

$$
I \quad \text{and} \quad \left( \bigcup_{i=0}^3 \mathcal{A}(i) \right)^{\perp}
$$

is a maximal Abelian subalgebra.
Proof: Since the intersection $\mathcal{A}(0)^t \cap \mathcal{A}(j)$ is a 2-dimensional commutative subalgebra, we can find a self-adjoint unitary $A(0, j)$ such that $\mathcal{A}(0)^t \cap \mathcal{A}(j)$ is spanned by $I$ and $A(0, j) = x(0, j) \cdot \sigma \otimes I$, where $x(0, j) \in \mathbb{R}^3$. Due to the quasi-orthogonality of $\mathcal{A}(1), \mathcal{A}(2)$ and $\mathcal{A}(3)$, the unit vectors $x(0, j)$ are pairwise orthogonal (see (1)). The matrices $A(0, j)$ anti-commute:

\[
A(0, i)A(0, j) = i(x(0, i) \times x(0, j)) \cdot \sigma \otimes I = -i(x(0, j) \times x(0, i)) \cdot \sigma \otimes I = -A(0, j)A(0, i)
\]

for $i \neq j$. Moreover,

\[
A(0, 1)A(0, 2) = i(x(0, 1) \times x(0, 2)) \cdot \sigma
\]

$x(0, 1) \times x(0, 2) = \pm x(0, 3)$ because $x(0, 1) \times x(0, 2)$ is orthogonal to both $x(0, 1)$ and $x(0, 2)$. If necessary, we can change the sign of $x(0, 3)$ such that $A(0, 1)A(0, 2) = iA(0, 3)$ holds.

Starting with the subalgebras $\mathcal{A}(1)^t$, $\mathcal{A}(2)^t$, $\mathcal{A}(3)^t$ we can construct similarly the other Pauli triplets. In this way, we arrive at the 4 Pauli triplets, the rows of the following table:

<table>
<thead>
<tr>
<th></th>
<th>$A(0, 1)$</th>
<th>$A(0, 2)$</th>
<th>$A(0, 3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A(1, 0)$</td>
<td>$*$</td>
<td>$A(1, 2)$</td>
<td>$A(1, 3)$</td>
</tr>
<tr>
<td>$A(2, 0)$</td>
<td>$A(2, 1)$</td>
<td>$*$</td>
<td>$A(2, 3)$</td>
</tr>
<tr>
<td>$A(3, 0)$</td>
<td>$A(3, 1)$</td>
<td>$A(3, 2)$</td>
<td>$*$</td>
</tr>
</tbody>
</table>

(7)

When $\{A(i)\}_{i=0}^4$ is a family of pairwise quasi-orthogonal subalgebras, then the commutants $\{A(i)^t\}_{i=0}^4$ are pairwise quasi-orthogonal as well. $A(j)^{''} = A(j)$ and $A(i)^t$ have nontrivial intersection for $i \neq j$, actually the previously defined $A(i, j)$ is in the intersection. For a fixed $j$ the three unitaries $A(i, j)$ ($i \neq j$) form a Pauli triplet up to a sign. (It follows that changing sign we can always reach the situation where the first three columns of table (7) form Pauli triplets. $A(0, 3)$ and $A(1, 3)$ anti-commute, but it may happen that $A(0, 3)A(1, 3) = -A(2, 3)$.)

Let $C_0 := \{ \pm A(i, j)A(j, i) : i \neq j \} \cup \{ \pm I \}$ and $C := C_0 \cup iC_0$. We want to show that $C$ is a commutative group (with respect to the multiplication of unitaries).

Note that the products in $C_0$ have factors in symmetric position in (7) with respect to the main diagonal indicated by stars. Moreover, $A(i, j) \in \mathcal{A}(j)$ and $A(j, k) \in \mathcal{A}(j)^t$, and these operators commute.

We have two cases for a product from $C$. Taking the product of $A(i, j)A(j, i)$ and $A(u, v)A(v, u)$, we have

\[
(A(i, j)A(j, i))(A(i, j)A(j, i)) = I
\]

in the simplest case, since $A(i, j)$ and $A(j, i)$ are commuting self-adjoint unitaries. It is slightly more complicated if the cardinality of the set $\{i, j, u, v\}$ is 3 or 4. First,

\[
(A(1, 0)A(0, 1))(A(3, 0)A(0, 3)) = A(0, 1)(A(1, 0)A(3, 0))A(0, 3) = \pm i(A(0, 1)A(2, 0))A(0, 3)
\]
\[
= \pm i A(2,0)(A(0,1)A(0,3)) = \pm A(2,0)A(0,2),
\]
and secondly,
\[
(A(1,0)A(0,1))(A(3,2)A(2,3)) = \pm i A(1,0)A(0,2)(A(0,3)A(3,2))A(2,3) = \pm i A(1,0)A(0,2)A(3,2)A(0,3)A(2,3) = \pm A(1,0)A(0,2)A(3,2)A(1,3) = \pm i A(1,0)(A(1,2)A(1,3)) = \pm A(1,0)A(1,0) = \pm I \tag{8}
\]
So the product of any two operators from \( C \) is in \( C \).

Now we show that the subalgebra \( C \) linearly spanned by the unitaries \( \{ A(i,j)A(j,i) : i \neq j \} \cup \{ I \} \) is a maximal Abelian subalgebra.

Since we know the commutativity of this algebra, we estimate the dimension. It follows from (8) and the self-adjointness of \( A(i,j)A(j,i) \) that
\[
A(i,j)A(j,i) = \pm A(k,\ell)A(\ell,k)
\]
when \( i, j, k \) and \( \ell \) are different. Therefore \( C \) is linearly spanned by \( A(0,1)A(1,0), \ A(0,2)A(2,0), \ A(0,3)A(3,0) \) and \( I \). These are 4 different self-adjoint unitaries.

Finally, we check that the subalgebra \( C \) is quasi-orthogonal to \( \mathcal{A}(i) \).

If the cardinality of the set \( \{i,j,k,\ell\} \) is 4, then we have
\[
\text{Tr} \ A(i,j)(A(i,j)A(j,i)) = \text{Tr} \ A(j,i) = 0
\]
and
\[
\text{Tr} \ A(k,\ell)A(i,j)A(j,i) = \mp \text{Tr} \ A(k,\ell)A(k,\ell)A(\ell,k) = \pm \text{Tr} \ A(\ell,k) = 0.
\]
Moreover, because \( \mathcal{A}(k) \) is quasi-orthogonal to \( \mathcal{A}(i) \), we also have \( A(i,k) \perp A(j,i) \), so
\[
\text{Tr} \ A(i,\ell)(A(i,j)A(j,i)) = \pm i \text{Tr} \ A(i,k)A(j,i) = 0.
\]

From this we can conclude, that
\[
A(k,\ell) \perp A(i,j)A(j,i)
\]
for all \( k \neq \ell \) and \( i \neq j \). \( \square \)

Finally, we note that there is only one subalgebra of \( M_4(\mathbb{C}) \) isomorphic to \( M_2(\mathbb{C}) \) that is quasi-orthogonal to all \( \mathcal{A}(i), (i = 0, 1, 2) \).

The subalgebras \( \{\mathcal{A}(i)\}_{i=0}^2 \) determine the matrices \( \{A(i,j) : i \neq j, i,j \in \{0,1,2\}\} \). Since two anti-commuting matrices define the third element of a Pauli triplet, the matrices \( \{A(j,3)\}_{j \neq 3} \) are also determined. The matrices \( \{A(j,3)\}_{j \neq 3} \) must form a Pauli triplet and fix the fourth subalgebra.
The edges between two vertices represent the one-dimensional traceless intersection of the two subalgebras corresponding two vertices. The three edges starting from a vertex represent a Pauli triplet.

4 Possible extension

Next we consider the pairwise quasi-orthogonal subalgebras $\mathcal{A}_i \simeq M_2(\mathbb{C})$ in $M_{2^n}(\mathbb{C})$. The question is their maximal number $m(n)$.

The traceless subspaces of $M_2(\mathbb{C})$ and $M_{2^n}(\mathbb{C})$ are 3-dimensional and $(4^n - 1)$-dimensional, respectively. Therefore,

$$m(n) \leq \frac{4^n - 1}{3} =: N_n.$$ 

Below, we construct $N_n - 1$ pairwise quasi-orthogonal subalgebras. We conjecture that this is the true value of $m(n)$. Theorem 3 contains the case $n = 2$.

The Hilbert space $M_{2^n}(\mathbb{C})$ has a natural orthogonal basis

$$\sigma_{i_1} \otimes \sigma_{i_2} \otimes \ldots \otimes \sigma_{i_n} =: (i_1, i_2, \ldots, i_n),$$

where $i_j = 0, 1, 2, 3$ and $1 \leq j \leq n$. We put

$$P_n = \{(i_1, i_2, \ldots, i_n) : 0 \leq i_j \leq 3, 1 \leq j \leq n\} \setminus \{I\}.$$ 

A triplet $(A_1, A_2, A_3) \in P_n^3$ is called a weak Pauli triplet if $A_1 A_2 = \pm i A_3$. and $(A_1, A_2, A_3) \in P_n^3$ is a commuting triplet if $A_1 A_2 = \pm A_3$. The linear span of elements of a weak Pauli triplet and $I$ is a subalgebra isomorphic to $M_2(\mathbb{C})$. 

9
Assume that $A = (A_1, A_2, A_3) \in P_3^n$ is a commuting triplet. Then we can construct three pairwise disjoint weak Pauli triplets: $A^{(1)} := (\sigma_1 \otimes A_1, \sigma_2 \otimes A_2, \sigma_3 \otimes A_3)$ and $A^{(2)} := (\sigma_2 \otimes A_1, \sigma_3 \otimes A_2, \sigma_1 \otimes A_3)$ and $A^{(3)} := (\sigma_3 \otimes A_1, \sigma_1 \otimes A_2, \sigma_2 \otimes A_3)$ in $P_{n+1}^3$. Therefore, to construct pairwise quasi-orthogonal subalgebras isomorphic to $M_2(\mathbb{C})$, it is useful to consider weak Pauli triplets and commuting triplets.

**Example 1** There are 5 pairwise disjoint commuting triplets in $P_2^3$. Indeed,

$$((0, 1), (1, 0), (1, 1)), \quad ((0, 2), (2, 0), (2, 2)), \quad ((0, 3), (3, 0), (3, 3)), \quad ((1, 2), (2, 3), (3, 1)), \quad ((1, 3), (2, 1), (3, 2)).$$

There are 21 pairwise disjoint commuting triplets in $P_3^3$. Indeed,

$$((1, 0, 1), (2, 0, 3), (3, 0, 2)), \quad ((1, 0, 2), (2, 0, 1), (3, 0, 3)), \quad ((0, 1, 1), (0, 2, 3), (0, 3, 2)), \quad ((0, 1, 3), (0, 1, 0), (0, 3, 3)), \quad ((0, 2, 2), (0, 2, 0), (0, 0, 2)), \quad ((0, 3, 1), (0, 3, 0), (0, 0, 1)), \quad ((3, 2, 1), (3, 0, 0), (2, 0, 1)), \quad ((1, 1, 2), (1, 2, 1), (1, 3, 0)), \quad ((1, 2, 2), (2, 3, 0), (3, 1, 2)), \quad ((1, 2, 3), (2, 1, 0), (3, 2, 2)).$$

We show that $P_n$ can be decomposed into commuting triplets.

**Theorem 4** For each $n \geq 2$, there is a family of commuting triplets

$$\{A^{(i)} = (A^{(i)}_1, A^{(i)}_2, A^{(i)}_3)\}_{i=1}^{N_n} \subset P_n^3$$

such that

$$\bigcup_{i=1}^{N_n} A^{(i)} = P_n.$$

**Proof:** In the case $n = 2$ and $n = 3$, it is already proven above. Assume it is proven in the case $n = k$, and we consider the case $n = k + 2$. Let $\{A^{(i)}\}_{i=1}^{N_k}$ and $\{B^{(j)}\}_{j=1}^{N_k}$ be the family of commuting triplets satisfying the theorem in the case of $n = 2$ and $n = k$, respectively. Then, for each $A^{(i)} = (A^{(i)}_1, A^{(i)}_2, A^{(i)}_3)$ and $B^{(j)} = (B^{(j)}_1, B^{(j)}_2, B^{(j)}_3)$, we can construct three commuting triplets in $P_{k+2}^3$, that is, $(A^{(i)}_1 \otimes B^{(j)}_1, A^{(i)}_2 \otimes B^{(j)}_2, A^{(i)}_3 \otimes B^{(j)}_3)$ and $(A^{(i)}_1 \otimes B^{(j)}_2, A^{(i)}_2 \otimes B^{(j)}_3, A^{(i)}_3 \otimes B^{(j)}_1)$ and $(A^{(i)}_1 \otimes B^{(j)}_3, A^{(i)}_2 \otimes B^{(j)}_1, A^{(i)}_3 \otimes B^{(j)}_2)$. Moreover, we have other commuting triplets, i.e., $(A^{(i)}_1 \otimes I_k, A^{(i)}_2 \otimes I_k, A^{(i)}_3 \otimes I_k)$ and $(I_2 \otimes B^{(j)}_1, I_2 \otimes B^{(j)}_2, I_2 \otimes B^{(j)}_3)$. Consequently, we have $5 + N_k + 3 \cdot 5 \cdot N_k = N_{k+2}$ commuting triplets. Since $\bigcup_{i=1}^{N_k} A^{(i)} = P_k$ and $\bigcup_{j=1}^{N_k} B^{(j)} = P_k$, $\{A^{(i)}_1, A^{(i)}_2, A^{(i)}_3\}_{i=1}^{5}$ and $\{B^{(j)}_1, B^{(j)}_2, B^{(j)}_3\}_{j=1}^{N_k}$ are distinct. Hence, we obtain the union of the above $N_{k+2}$ commuting triplets is $P_{k+2}$. □

The good point of this construction is that it is easy to use the induction.

**Theorem 5** There exist $N_n - 1$ quasi-orthogonal subalgebras in $M_{2^n}(\mathbb{C})$. 10
Proof: The case \( n = 2 \) is already proven in Theorem 3. Assume it is proven for \( n = k \), and we consider the case \( n = k + 1 \).

From Theorem 4, let \( \{ A^{(i)} = (A_{1}^{(i)}, A_{2}^{(i)}, A_{3}^{(i)}) \}_{i=1}^{N_k} \) be commuting triplets in \( P_{k}^{3} \) such that \( \bigcup_{i=1}^{N_k} A^{(i)} = P_{k} \). Then we have \( 3N_k \) pairwise disjoint weak Pauli triplets, that is, 
\[
(\sigma_1 \otimes A_1^{(i)}, \sigma_2 \otimes A_2^{(i)}, \sigma_3 \otimes A_3^{(i)}), (\sigma_2 \otimes A_1^{(i)}, \sigma_3 \otimes A_2^{(i)}, \sigma_1 \otimes A_3^{(i)}), (\sigma_3 \otimes A_1^{(i)}, \sigma_1 \otimes A_2^{(i)}, \sigma_2 \otimes A_3^{(i)})
\]
Furthermore, we obtain another weak Pauli triplet \((\sigma_1 \otimes I_k, \sigma_2 \otimes I_k, \sigma_3 \otimes I_k)\). These \( 3N_k + 1 \) weak Pauli triplets are pairwise disjoint. Moreover, the complement space of above \( 3N_k + 1 \) Pauli triplets is \( CI \otimes M_{2^n}(\mathbb{C}) \). Indeed, since \( \bigcup_{i=1}^{N_k} A^{(i)} = P_{k} \), we have
\[
\{ (\sigma_1 \otimes A_1^{(i)}, \sigma_2 \otimes A_2^{(i)}, \sigma_3 \otimes A_3^{(i)}), (\sigma_2 \otimes A_1^{(i)}, \sigma_3 \otimes A_2^{(i)}, \sigma_1 \otimes A_3^{(i)}), \\
(\sigma_3 \otimes A_1^{(i)}, \sigma_1 \otimes A_2^{(i)}, \sigma_2 \otimes A_3^{(i)}),(\sigma_1 \otimes I_k, \sigma_2 \otimes I_k, \sigma_3 \otimes I_k) : 1 \leq i \leq N_k \}
\]
Therefore, the complement space is \( CI \otimes M_{2^n}(\mathbb{C}) \) spanned by
\[
\{ \sigma_0 \otimes \sigma_{j_1} \otimes \ldots \otimes \sigma_{j_k} : j_i = 0, 1, 2, 3, \ 1 \leq l \leq k \}.
\]

Now we use the assumption that there are \( N_k - 1 \) pairwise disjoint weak Pauli triplets \( B^{(i)} = (B_1^{(i)}, B_2^{(i)}, B_3^{(i)}) \) in \( M_{2^n}(\mathbb{C}) \) (\( 1 \leq i \leq N_k - 1 \)). Then 
\[
(\sigma_0 \otimes B_1^{(i)}, \sigma_0 \otimes B_2^{(i)}, \sigma_0 \otimes B_3^{(i)})
\]
give pairwise disjoint weak Pauli triplets in \( P_{k+1}^{3} \). Summing up, we have \( 3N_{k+1} + 1 = 4N_k = N_{k+1} - 1 \) pairwise disjoint weak Pauli triplets. \( \square \)

Similarly, we can prove the following. If there exist \( N_n \) pairwise quasi-orthogonal subalgebras in \( M_{2^n}(\mathbb{C}) \) for some \( n \), then there exist \( N_k \) pairwise quasi-orthogonal subalgebras in \( M_{2^k}(\mathbb{C}) \) for all \( k \geq n \).

References


