On randomly generated non-trivially intersecting hypergraphs

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Abstract

We propose two procedures to choose members of $\binom{[n]}{r}$ sequentially at random to form a non-trivially intersecting hypergraph. In both cases we show what is the limiting probability that if $r = c_n n^{1/3}$ with $c_n \to c$, then the process results a Hilton-Milner-type hypergraph.

1 Introduction

In 1961, Erdős, Ko and Rado [5] proved that if $2r \leq n$, then the edge set E of an intersecting r-uniform hypergraph with vertex set V and $|V| = n$ cannot have larger size than $\binom{n-1}{r-1}$, moreover if $2r < n$, then the only hypergraphs with that many edges are of the form $\{e \in \binom{V}{r} : v \in e\}$ for some fixed $v \in V$. In the past almost five decades, the area of intersection theorems has been widely studied, but randomized versions of the Erdős-Ko-Rado theorem have only attracted the attention of researchers recently. There are mainly two approaches to the randomized problem. Balogh, Bohman and Mubayi [2] considered the problem of finding the largest intersecting hypergraph in the probability space $G^r(n,p)$ of all labeled r-uniform hypergraphs on n vertices where every hyperedge appears randomly and independently with probability $p = p(n)$. In this paper, we follow the approach of Bohman et al. [3], [4]. They considered the following process to generate an intersecting hypergraph by selecting edges sequentially and randomly.

Choose Random Intersecting System

Choose $e_1 \in \binom{[n]}{r}$ uniformly at random. Given $\mathcal{F}_i = \{e_1, ..., e_i\}$ let $\mathcal{A}(\mathcal{F}_i) = \{e \in \binom{[n]}{r} : e \notin \mathcal{F}_i, \forall 1 \leq j \leq i : e \cap e_j \neq \emptyset\}$. Choose e_{i+1} uniformly at random from $\mathcal{A}(\mathcal{F}_i)$. The procedure halts when $\mathcal{A}(\mathcal{F}_i) = \emptyset$ and $\mathcal{F} = \mathcal{F}_i$ is then output by the procedure.

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Theorem 1.1 (Bohman et al. [3]) *Let $\mathcal{E}_{r,n}$ denote the event that $|\mathcal{F}| = \binom{n-1}{r-1}$. Then if $r = c_n n^{1/3}$,*

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{E}_{r,n}) = \begin{cases} 1 & \text{if } c_n \to 0 \\ \frac{1}{1+c^3} & \text{if } c_n \to c \\ 0 & \text{if } c_n \to \infty. \end{cases}$$

Theorem 1.1 states that the probability that the resulting hypergraph will be trivially intersecting (i.e. all of its edges will share a common element) with probability tending to 1 (in other words, *with high probablity*, w.h.p.) provided $r = o(n^{1/3})$. In this paper we will be interested in two processes that generate non-trivially intersecting hypergraphs for this range of r. Before introducing the actual processes, let us state the theorem of Hilton and Milner that determines the size of the largest non-trivially intersecting hypergraph.

Theorem 1.2 (Hilton, Milner [6]) *Let $\mathcal{F} \subset \binom{[n]}{r}$ be a non-trivially intersecting hypergraph with $r \geq 3$, $2r+1 \leq n$. Then $|\mathcal{F}| \leq \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$. The hypergraphs achieving that size are*

(i) *for any r-subset F and $x \in [n] \setminus F$ the hypergraph $\mathcal{F}_{HM} = \{F\} \cup \{G \in \binom{[n]}{r} : x \in G, F \cap G \neq \emptyset\}$,*

(ii) *if $r = 3$, then for any 3-subset S the hypergraph $\mathcal{F}_{\Delta} = \{F \in \binom{[n]}{3} : |F \cap S| \geq 2\}$.*

We will call the hypergraphs described in **(i)** *HM-type* hypergraphs, while hypergraphs $\mathcal{F}$ for which there exists a 3-subset S of $[n]$ such that $\mathcal{F}$ consists of all r-subsets of $[n]$ with $|F \cap S| \geq 2$ will be called 2-3 hypergraphs even if $r > 3$ (the natural generalizations of hypergraphs of the form of **(ii)**).

We now introduce the two processes we will be interested in. In some sense they are the opposite of each other as the first process assures as early as possible (i.e. when picking the third edge e_3) that it produces a non-trivially intersecting hypergraph while the second one is the same as the original process of Bohman et al. as long as it is possible that the process results a non-trivially intersecting hypergraph. The main value of the first model is that the results concerning this model allows us to calculate the probability that the original model of Bohman et al. produces an HM-type hypergraph when $r = \Theta(n^{1/3})$, while the second model seems to be the model that can be obtained with the least modification to the original such that it results a non-trivially intersecting hypergraph for all values of r and n.

Here are the formal definitions.

The Third Round Process

Choose $e_1 \in \binom{[n]}{r}$ uniformly at random. Given $\mathcal{F}_i = \{e_1, ..., e_i\}$ if $i \neq 2$ let $\mathcal{A}(\mathcal{F}_i) = \{e \in \binom{[n]}{r} : e \notin \mathcal{F}_i, \forall 1 \leq j \leq i : e \cap e_j \neq \emptyset\}$ while for $i = 2$ let $\mathcal{A}(\mathcal{F}_2) = \{e \in \binom{[n]}{r} : e \notin \mathcal{F}_2, e \cap e_j \neq \emptyset (j = 1, 2), e \cap e_1 \cap e_2 = \emptyset\}$. Choose e_{i+1} uniformly at random from $\mathcal{A}(\mathcal{F}_i)$. The procedure halts when $\mathcal{A}(\mathcal{F}_i) = \emptyset$ and $\mathcal{F} = \mathcal{F}_i$ is then output by the procedure.

Note that by Lemma 7 in [3] if $O(n^{2/3}) = r = \omega(n^{1/3})$, then w.h.p. $\mathcal{F}_3$ of the original process of Bohman et al. is non-trivially intersecting and thus the two processes are the same w.h.p. The probability of an event $\mathcal{E}$ in the Third Round Process will be denoted by $\mathbb{P}_{3R}(\mathcal{E})$.

The Put-Off Process

Choose $e_1 \in \binom{[n]}{r}$ uniformly at random. Given $\mathcal{F}_i = \{e_1, ..., e_i\}$ let

$$\mathcal{A}(\mathcal{F}_i) = \left\{ e \in \binom{[n]}{r} : e \notin \mathcal{F}_i; \exists \mathcal{G} \text{ non-trivially intersecting with } \{e\} \cup \mathcal{F}_i \subseteq \mathcal{G} \right\},$$

i.e. $\mathcal{A}(\mathcal{F}_i)$ is the set of all edges that can be added to $\mathcal{F}_i$ such that $\{e\} \cup \mathcal{F}_i$ can be extended to a non-trivially intersecting hypergraph. Choose e_{i+1} uniformly at random from $\mathcal{A}(\mathcal{F}_i)$. The procedure halts when $\mathcal{A}(\mathcal{F}_i) = \emptyset$ and $\mathcal{F} = \mathcal{F}_i$ is then output by the procedure.

Note again that by Lemmas 7 and 8 in [3] if $r = \omega(n^{1/3})$, then w.h.p already $\mathcal{F}_{2\log_2 n}$ of the original process of Bohman et al. is non-trivially intersecting and thus the two processes are the same. The probability of an event $\mathcal{E}$ in the Put-Off Process will be denoted by $\mathbb{P}_{PO}(\mathcal{E})$. If the probability of an event $\mathcal{E}$ is the same in the two models or the same bound applies for it in both models, then we will denote this probability by $\mathbb{P}_{3R,PO}(\mathcal{E})$. The probability of an event $\mathcal{E}$ in the original process will be denoted by $\mathbb{P}_{INT}(\mathcal{E})$.

To formulate the main results of the paper we need to introduce the following events:

$\mathcal{E}_{HM}$ stands for the event that the process outputs an HM-type hypergraph while $\mathcal{E}_{\Delta}$ denotes the event that the output is a 2-3 hypergraph.

Theorem 1.3 *If $\omega(1) = r = c_n n^{1/3}$, then*

$$\lim_{n\to\infty} \mathbb{P}_{3R}(\mathcal{E}_{HM}) = \begin{cases} 1 & \text{if } c_n \to 0 \\ \frac{1}{1+c^3/3} & \text{if } c_n \to c \\ 0 & \text{if } c_n \to \infty. \end{cases}$$

Theorem 1.4 *If $3 \leq r$ is a fixed constant, then*

$$\lim_{n\to\infty} \mathbb{P}_{3R}(\mathcal{E}_{HM}) = 1 - \left(\frac{1}{r-1}\right)^3, \qquad \lim_{n\to\infty} \mathbb{P}_{3R}(\mathcal{E}_{\Delta}) = \left(\frac{1}{r-1}\right)^3.$$

Theorem 1.5 *If $r = c_n n^{1/3}$, then*

$$\lim_{n\to\infty} \mathbb{P}_{PO}(\mathcal{E}_{HM}) = \begin{cases} 1 & \text{if } c_n \to 0 \\ \frac{1}{1+c^3} + \frac{c^3}{1+c^3} \cdot \frac{1}{1+c^3/3} & \text{if } c_n \to c \\ 0 & \text{if } c_n \to \infty. \end{cases}$$

Corollary 1.6 *If $r = c_n n^{1/3}$ with $c_n \to c$, then*

$$\mathbb{P}_{INT}(\mathcal{E}_{HM}) = \frac{c^3}{1+c^3} \cdot \frac{1}{1+c^3/3}.$$

The rest of the paper is organized as follows: in the next section we introduce some events that will be useful in the proofs and restate some of the lemmas of [3]. In Section 3, we prove Theorem 1.3 and Corollary 1.6, Section 4 contains the proof of Theorem 1.4 and Section 5 contains the proof of Theorem 1.5

2 Definitions and Lemmas from [3]

We will write $g(n) = o(f(n))$ ($g(n) = \omega(f(n))$) to denote the fact that $\lim_n \frac{g(n)}{f(n)} = 0$ ($\lim_n \frac{g(n)}{f(n)} = \infty$), while $g(n) = O(f(n))$ ($g(n) = \Omega(f(n))$) will mean that there exists a positive number K such that $\frac{g(n)}{f(n)} < K$ ($\frac{g(n)}{f(n)} > K$) for all integers n and $g(n) = \Theta(f(n))$ denotes the fact that both $g(n) = O(f(n))$ and $g(n) = \Omega(f(n))$ hold. Throughout the paper log stands for the logarithm in the natural base e.

We will use the following well-known inequalities: for any x we have $1 + x \leq e^x$ and if x tends to 0, then $1 + x = \exp(x + O(x^2))$. Binomial coefficients will be bounded by $(\frac{a}{b})^b \leq \binom{a}{b} \leq (\frac{ea}{b})^b$. Finally, for binomial random variables we have the following fact (see e.g. [1])

Fact 2.1 If X is a random variable with $X \sim \mathrm{Bi}(n,p)$, then we have $\mathbb{P}(|X - np| > \delta np) \leq 2e^{-\delta^2 np/3}$. In particular, for any constant c with $0 < c < 1$ we have $\mathbb{P}(|X - np| > cnp) = \exp(-\Omega(np))$.

We call a hypergraph with i edges an *i-star* if the pairwise intersections of the edges are the same and have one element which we will call the *kernel* of the i-star.

A hypergraph of 3 edges e_1, e_2, e_3 is a *triangle* if $\cap_{i=1}^3 e_i = \emptyset$ and $|e_i \cap e_j| = 1$ for all $1 \leq i < j \leq 3$. The *base* of a triangle is the 3-set $\{e_i \cap e_j : 1 \leq i < j \leq 3\}$. A hypergraph is a *sunflower* if the intersection of any two of its edges are the same which is the *kernel* of the sunflower. A hypergraph $\mathcal{H}$ of $3r$ edges is an *r-triangle* if $\mathcal{H}$ can be partitioned into 3 sunflowers each of r edges with kernel size 2 such that any 3 edges taken from different sunflowers form a triangle with the same base.

A hypergraph of $2r$ edges $e_1^1, e_1^2, ..., e_r^1, e_r^2$ is an *r-double-broom* if $|\cap_{i=1}^r \cap_{j=1}^2 e_i^j| = 1$, $|e_i^1 \cap e_i^2| = 2$ for all $1 \leq i \leq r$ and $|e_i^j \cap e_{i'}^{j'}| = 1$ for any $i \neq i'$. We call $\cap_{i=1}^r \cap_{j=1}^2 e_i^j$ the *kernel* of the double-broom. The subhypergraph of an r-double-broom consisting of the $d + r$ edges $e_1^1, e_1^2, ..., e_d^1, e_d^2, e_{d+1}^1, ..., e_r^1$ is a *d-partial r-double-broom.* The elements not identical to the kernel that belong to $e_i^1 \cap e_i^2$ are called the *semi-kernels* of the d-partial r-double-broom and the sets e_j^1 without the kernel ($d + 1 \leq j \leq r$) are called the *lonely fingers* of the d-partial r-double-broom.

The following two trivial propositions show what intersecting subhypergraphs of $\mathcal{F}_j$ assure that the output of the process will be 2-3-hypergraph or an HM-type hypergraph.

Proposition 2.2 If an intersecting hypergraph $\mathcal{H}$ contains an r-triangle, then there is only one maximal intersecting hypergraph $\mathcal{H}^*$ containing $\mathcal{H}$ and $\mathcal{H}^*$ is a 2-3-hypergraph. □

Proposition 2.3 If an r-set f does not contain the kernel x of a d-partial r-double-broom B, but meets all sets in B, then f must contain all semi-kernels of B and meet each lonely finger of B in exactly one element. In particular, the only r-set meeting all sets of an r-double-broom not containing the kernel is the set of all semi-kernels and thus

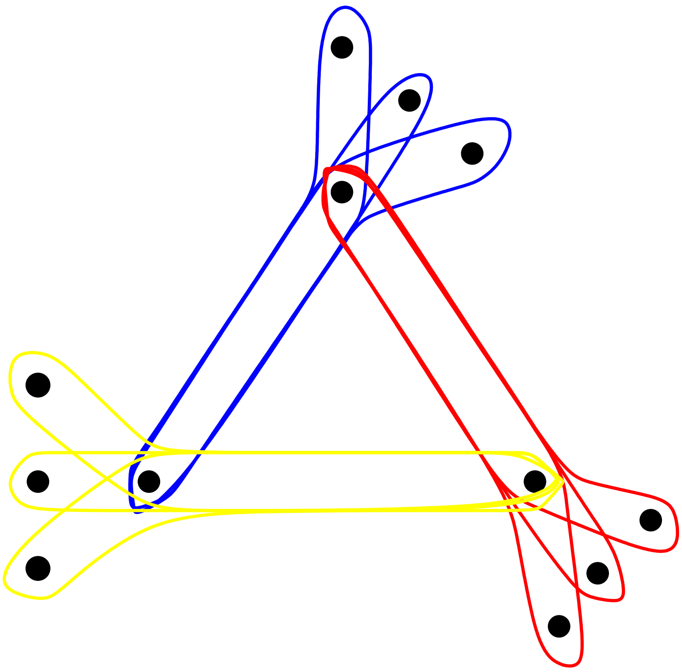

Figure 1: An r-triangle with $r = 3$.

if an intersecting hypergraph $\mathcal{H}$ contains an r-double broom, then there is only one non-trivially intersecting hypergraph $\mathcal{H}^*$ that contains $\mathcal{H}$ and $\mathcal{H}^*$ is an HM-type hypergraph. $\square$

- Let $\mathcal{A}_i$ be the event that $\mathcal{F}_i$ is an i-star.
- Let $\mathcal{A}'_{j,r}$ denote the event that $\mathcal{F}_j$ contains an r-star and there exists at most 1 edge $e \in \mathcal{F}_j$ not containing the kernel of the r-star. In particular, $\mathcal{A}'_{r,r} = \mathcal{A}_r$.
- Let $\mathcal{A}''_{j,r}$ denote the event that $\mathcal{F}_j$ contains an r-double broom and there exists at most 1 edge $e \in \mathcal{F}_j$ not containing the kernel of the r-double broom.
- Let $\mathcal{H}$ denote the event that e_3 contains all of $e_1 \cap e_2$ as well as at least one vertex from $(e_1 \setminus e_2) \cup (e_2 \setminus e_1)$.
- Let Δ denote the event that $\mathcal{F}_3$ is a triangle.
- Let $\Delta_{j,l}$ denote the event that $\mathcal{F}_j$ contains an l-triangle and all edges in $\mathcal{F}_j$ meet the base of this l-triangle in at least 2 elements.
- Let $\mathcal{B}_j$ denote the event that $\bigcap_{e \in \mathcal{F}_j} e \neq \emptyset$.
- Let $\mathcal{C}_{j,1}$ denote the event that $\mathcal{F}_{j+1}$ is a j-star with a *transversal*, a set meeting all sets of the star in 1 element which is different from the kernel of the star.

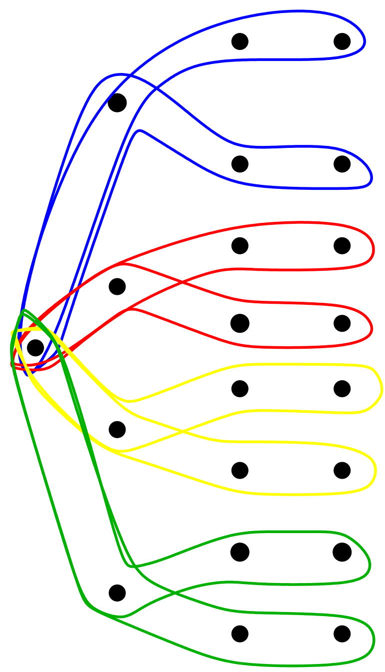

Figure 2: An r-double broom with $r = 4$.

- Let $\mathcal{C}'_{l,j,1}$ denote the event that $\mathcal{F}_l$ contains a j-star T and there is exactly one edge $e \in \mathcal{F}_l$ not containing the kernel of T and e is a transversal of T. In particular, $\mathcal{C}'_{j+1,j,1} = \mathcal{C}_{j,1}$.

- Let $\mathcal{C}''_{l,j,1}$ denote the event that $\mathcal{F}_l$ contains a subhypergraph H of an r-double broom B with $|E(H)| = j$ and there is only one edge $e \in \mathcal{F}_l$ not containing the kernel of H and e is the set of semi-kernels of B.

- Let $\mathcal{D}_j$ denote the event that there exists an $x \in [n]$ such that there is at most one edge $e \in \mathcal{F}_j$ that does not contain x.

- For any event $\mathcal{E}$, the complement of the event is denoted by $\overline{\mathcal{E}}$.

We finish this section by stating some of the lemmas from [3] that we will use in the proofs of Theorem 1.3 and Theorem 1.5.

Lemma 2.4 (Lemma 1 in [3]) *If $r = o(n^{1/2})$, then w.h.p. $\mathcal{A}_2$ holds.*

Lemma 2.5 (Lemma 2 in [3]) *If $r = o(n^{1/2})$, then*

$$\mathbb{P}_{INT}(\mathcal{A}_3) = \frac{1 - o(1)}{1 + \frac{(r-1)^3}{n}(1 + o(1))}.$$

Lemma 2.6 (Lemma 3 in [3]) *If $r = o(n^{2/5})$ and $m = O(n^{1/2}/r)$, then*

$$\mathbb{P}(\mathcal{A}_m|\mathcal{A}_3) = \exp\left(\frac{-m^2r^2}{4n} + o(1)\right).$$

Lemma 2.7 (Lemma 4 in [3]) *If $r = (o^{1/2})$, then*

$$\mathbb{P}(\mathcal{H}|\mathcal{A}_2) = o(1).$$

Lemma 2.8 (Lemma 7 in [3]) *If $\omega(n^{1/3}) = r = o(n^{2/3})$, then*

$$\mathbb{P}(\mathcal{B}_3) = o(1).$$

3 The Third Round Model I. ($r \to \infty$)

In this section we prove Theorem 1.3 and Corollary 1.6. First we give an outline of the proof, then we proceed with lemmas corresponding to the different cases of Theorem 1.3 and at the end of the section we show how to deduce Theorem 1.3 from these lemmas and how Corollary 1.6 follows from Theorem 1.3 and Theorem 1.1.

Outline of the proof: We will use Proposition 2.2 and Proposition 2.3 to calculate the probability of the events $\mathcal{E}_\Delta$ and $\mathcal{E}_{HM}$, while to prove that $\mathcal{E}_{HM}$ does not hold w.h.p. if $r = \omega(n^{1/3})$ we will show that for every vertex x there exist at least 2 edges in $\mathcal{F}_i$ none of them containing x, i.e. $\mathcal{D}_i$ does not hold. The latter will be done by Lemma 3.4 and Lemma 3.5. To show the emergence of an r-double broom we will prove in Lemma 3.3 that it follows from the early appearance of a 3-star of which the probability is calculated in Lemma 3.2.

Our first lemma states that if $r = o(n^{1/2})$, then $\mathcal{F}_3$ is a triangle w.h.p.

Lemma 3.1 *In the Third Round Model, if $r = o(n^{1/2})$, then Δ holds w.h.p.*

Proof.

$$\mathbb{P}_{3R}(\overline{\Delta}|\mathcal{A}_2) \leq \frac{\binom{2r-1}{3}\binom{n-3}{r-3}}{(r-1)^2\binom{n-2r+1}{r-2}} = O\left(\frac{r^2}{n}\prod_{j=0}^{r-4}\frac{n-3-j}{n-2r+1-j}\right) =$$

$$O\left(\frac{r^2}{n}\exp\left(\frac{2r-4}{n-3r+5}(r-3)\right)\right) = o(1).$$

Together with Lemma 2.4, this proves the statement. □

Lemma 3.2, for the Third Round Model, is the equivalent of Lemma 2.5 in [3] for the Intersection Model. It gives the probability that $\mathcal{F}_4$ contains a 3-star.

Lemma 3.2 *In the Third Round Model, if $r = o(n^{1/2})$, then*

$$\mathbb{P}_{3R}(\mathcal{C}_{3,1}|\Delta) = \frac{1-o(1)}{1+\frac{1}{r-2}+\frac{(r-2)^3}{3n}}.$$

Proof. If S is the base of $\mathcal{F}_3$, then the kernel of the 3-star in $\mathcal{F}_4$ can only be an element of S. Thus the number of sets that can extend $\mathcal{F}_3$ to $\mathcal{F}_4$ in such a way that $\mathcal{C}_{3,1}$ should hold is $3(r-2)\binom{n-3r+3}{r-2}$. Let N_i denote the number of sets f in $\mathcal{A}(\mathcal{F}_3)$ with $|f \cap S| = i$ ($i = 0,1,2,3$). Every set f with $|f \cap S| \geq 2$ belongs to $\mathcal{A}(\mathcal{F}_3)$, sets belonging to $\mathcal{A}(\mathcal{F}_3)$ with $|f \cap S| = 1$ must meet one edge of $\mathcal{F}_3$ outside S, while sets disjoint from S that belong to $\mathcal{A}(\mathcal{F}_3)$ must meet all three edges in $\mathcal{F}_3$ outside S. Therefore we have the following bounds on N_i:

$$N_2 = 3\binom{n-3}{r-2} - 3, N_3 = \binom{n-3}{r-3},$$

$$3(r-2)\binom{n-r-1}{r-2} \leq N_1 \leq 3(r-2)\binom{n-4}{r-2},$$

$$(r-2)^3\binom{n-3r+3}{r-3} \leq N_0 \leq (r-2)^3\binom{n-6}{r-3}.$$

By the assumption $r = o(n^{1/2})$ we have $(\frac{n-c_1}{n-c_2 r})^r \leq \exp(O(\frac{r^2}{n})) \to 1$ for any constants c_1, c_2, and thus the lower and upper bounds on N_0 and N_1 are of the same order of magnitude. Hence we obtain

$$\begin{aligned}
\mathbb{P}_{3R}(\mathcal{C}_{3,1}|\Delta) &= \frac{3(r-2)\binom{n-3r+3}{r-2}}{\sum_{i=0}^{3} N_i} \\
&= \frac{3(r-2)\binom{n-3r+3}{r-2}}{3\binom{n-3}{r-2} - 3 + \binom{n-3}{r-3} + \left(3(r-2)\binom{n-4}{r-2} + (r-2)^3\binom{n-6}{r-3}\right)(1+o(1))} \\
&= \frac{1}{\left(\frac{1}{r-2} + o(1) + 1 + \frac{(r-2)^3}{3n}\right)(1+o(1))}.
\end{aligned}$$

□

Lemma 3.3 states that if $\mathcal{F}_j$ contains a 3-star for some small enough j, then $\mathcal{F}_{n^2}$ will contain an r-double broom w.h.p. which by Proposition 2.3 assures that the process outputs an HM-type hypergraph.

Lemma 3.3 *If $r = O(n^{1/3})$ and $j \leq \log n$, then*

$$\mathbb{P}_{3R}((\exists l \leq n^2 : \ \mathcal{A}''_{l,r})|\mathcal{C}'_{j,3,1}) = 1 - o(1).$$

Proof. Suppose $\mathcal{C}'_{j',3,1}$ holds for some j' with $j \leq j' \leq \log n$. Then the number of sets in $\mathcal{A}(\mathcal{F}_{j'})$ containing the kernel of a 3-star S in $\mathcal{F}_{j'}$ is

$$M = \binom{n-1}{r-1} - \binom{n-r-1}{r-1} - j' + 1$$

as they all must meet the transversal t of S already in $\mathcal{F}_{j'}$. Clearly, we have

$$r\binom{n-r-1}{r-2} - j + 1 \leq M \leq r\binom{n-2}{r-2},$$

as for the lower bound we enumerated the r-sets containing the kernel and exactly one element of t, while for he upper bound we counted r times the number of r-sets containing the kernel and one fixed element of t. The number of sets in $\mathcal{A}(\mathcal{F}_{j'})$ not containing the kernel of S is at most

$$(r-2)^3(r-3)\binom{n-5}{r-4}+3(r-1)^2\binom{n-4}{r-3}, \tag{1}$$

where the first term of the sum stands for the sets in $\mathcal{A}(\mathcal{F}_{j'})$ that meet all elements of S outside t (and thus we have to make sure that they meet t as well), while the second term stands for the other sets. Thus the probability that the random process picks an edge not containing the kernel is at most

$$\frac{(r-2)^3(r-3)\binom{n-5}{r-4}+3(r-1)^2\binom{n-4}{r-3}}{r\binom{n-r-1}{r-2}-j+1}\le\frac{4r^5}{n^2}\left(1-\frac{r}{n-2r}\right)^{r-2}+\frac{6r^2}{n}\left(1-\frac{r}{n-2r}\right)^{r-2}=$$

$$O\left(\frac{4r^5}{n^2}+\frac{6r^2}{n}\right).$$

Remember that $\mathcal{D}_k$ denotes the event that there is a vertex x which is contained in all but at most one edge of $\mathcal{F}_k$, thus as $r=O(n^{1/3})$, we obtain that $\mathbb{P}_{PO,3R}(\mathcal{D}_{n^{1/7}}|\mathcal{C}'_{j,3,1})=1-o(1)$.

For $i\ge j$ let α_i denote the maximum number k such that there exist k edges in $\mathcal{F}_i$ that form a subhypergraph of an r-double broom of which the semi-kernels are elements of t, in particular $\alpha_i=2r$ implies the existence of an r-double broom. Let us introduce the following random variables:

$$Z_i=\begin{cases}1 & \text{if } \alpha_i\ne\alpha_{i+1} \text{ or } \alpha_i=2r\\ 0 & \text{otherwise .}\end{cases}$$

The number of edges that would make α_i grow (if $\alpha_i<2r$) is at least $\frac{2r-\alpha_i}{2}\binom{n-\alpha_i(r-2)-r-1}{r-2}$. The total number of edges in $\mathcal{A}(\mathcal{F}_i)$ is at most $r\binom{n-r-1}{r-2}+(r-1)^3\binom{n-4}{r-3}=O(r\binom{n-r-1}{r-2})$ as $r=O(n^{1/3})$. Thus for $j\le i\le n^{1/7}$ we have

$$\begin{aligned}\mathbb{P}_{3R,PO}(Z_i=1|\mathcal{D}_{n^{1/7}},\mathcal{C}'_{j,3,1}) &= \Omega\left(\frac{(2r-\alpha_i)\binom{n-\alpha_i(r-2)-r-1}{r-2}}{r\binom{n-r-1}{r-2}}\right)\\ &= \Omega\left(\frac{2r-\alpha_i}{r}\left(1-\frac{3r^2}{n}\right)^r\right)=\Omega\left(\frac{2r-\alpha_i}{r}\right)\end{aligned}\tag{2}$$

as $r=O(n^{1/3})$. Note that if $\alpha_i=2r$, then by definition $\mathbb{P}(Z_i=1)=1$, thus any lower bound obtained in the $\alpha_i<2r$ case is valid in this case, too.

Let us consider 2 cases:

Case I $r=o(n^{1/15})$

By (2), we have $\mathbb{P}_{3R}(Z_i = 1|\mathcal{D}_{n^{1/7}}, \mathcal{C}'_{j,3}) \geq \Omega(1/r)$, thus

$$\mathbb{P}_{3R}\left(\sum_{i=j}^{n^{1/7}} Z_i < 2r|\mathcal{D}_{n^{1/7}}, \mathcal{C}'_{j,3,1}\right) < \mathbb{P}(\mathrm{Bi}(n^{1/7}, \Omega(1/r)) < 2r) \to 0$$

as $\frac{n^{1/7}}{r} = \omega(r)$ by the assumption $r = o(n^{1/15})$.

Case II $r = \omega(n^{1/16})$
By (2), we obtain

$$\mathbb{P}_{3R}(Z_i = 1|\mathcal{D}_{n^{1/7}}, \mathcal{C}'_{j,3,1}, \alpha_i \leq r/2) = \Theta(1),$$

thus

$$\mathbb{P}_{3R}\left(\sum_{i=j}^{n^{1/7}} Z_i < 2n^{1/20}|\mathcal{D}_{n^{1/7}}, \mathcal{C}'_{j,3,1}\right) < \mathbb{P}(\mathrm{Bi}(n^{1/7}, \Theta(1)) < 2n^{1/20}) \to 0$$

as $2n^{1/20} < r/2$ by the assumption $r = \omega(n^{1/16})$.

For any subhypergraph of an r-double broom there exists a set of at least half the edges that are pairwise disjoint apart from the kernel, thus if $\alpha_i \geq 2n^{1/20}$, then the number of r-sets that do not contain the kernel but meet all edges of $\mathcal{F}_i$ is at most $(r-1)^{n^{1/20}}\binom{n-n^{1/20}}{r-n^{1/20}}$. As before, if there is only one edge in $\mathcal{F}_i$ not containing x, then the number of r-sets in $\mathcal{A}(\mathcal{F}_i)$ containing x is

$$\binom{n-1}{r-1} - \binom{n-r-1}{r-1} - j + 1 \geq r\binom{n-r-1}{r-2}.$$

Hence, we have

$$\mathbb{P}_{3R}(\overline{\mathcal{D}_{n^2}}|\mathcal{C}''_{n^{1/7},2n^{1/20},1}) \leq n^2 \frac{(r-1)^{n^{1/20}}\binom{n-n^{1/20}}{r-n^{1/20}}}{r\binom{n-r-1}{r-2}} \leq n^2\left(\frac{2r^2}{n}\right)^{n^{1/20}} \to 0$$

as $r = o(n^{1/2-\epsilon})$. On the other hand, just as in (2) we have

$$\mathbb{P}_{3R}(Z_i = 1|\mathcal{D}_{n^2}, \mathcal{C}'_{j,3,1}) = \Omega\left(\frac{2r-\alpha_i}{r}\right) = \Omega(1/r)$$

and thus

$$\mathbb{P}_{3R}\left(\sum_{i=j}^{n^2} Z_i < 2r|\mathcal{D}_{n^2}, \mathcal{C}'_{j,3,1}\right) \leq \mathbb{P}(\mathrm{Bi}(n^2, \Omega(1/r)) < 2r) \to 0$$

as $n^2/r = \omega(r)$ since $r = O(n^{1/3})$. □

Lemma 3.4 asserts that if $\omega(n^{1/3}) = r = o(n^{1/2}\log^{1/10} n)$, then all vertices are contained in at most 2 edges of $\mathcal{F}_4$ and therefore the resulting hypergraph of the process cannot be HM-type.

Lemma 3.4 *If $\omega(n^{1/3}) = r = o(n^{1/2}\log^{1/10} n)$, then*

$$\mathbb{P}_{3R,PO}(\mathcal{D}_4) = o(1).$$

Proof. We consider 2 cases:

Case I $\omega(n^{1/3}) = r = o(n^{1/2})$

In this case, Lemma 3.1 states that Δ holds w.h.p., and the computation in Lemma 3.2 shows that e_4 is disjoint from the base of the triangle of $\mathcal{F}_3$ w.h.p.

Case II $\omega(n^{1/2}\log^{-1} n) = r = o(n^{1/2}\log^{1/10} n)$

First note that

$$\frac{\binom{n-r^{1/2}}{r-r^{1/2}}}{\binom{n-3r}{r-3}} \leq \left(\frac{(2r)}{n}\right)^{r^{1/2}} \exp\left(O\left(\frac{r^2}{n}\right)\right). \tag{3}$$

Using (3) and writing $\mathcal{E}_2$ for the event $|e_1 \cap e_2| \geq r^{1/2}$ we have

$$\mathbb{P}_{3R}(\mathcal{E}_2) \leq \frac{\binom{r}{r^{1/2}}\binom{n-r^{1/2}}{r-r^{1/2}}}{\binom{n-r}{r-1}} = (er^{1/2})^{r^{1/2}} \left(\frac{(2r)}{n}\right)^{r^{1/2}} \exp\left(O\left(\frac{r^2}{n}\right)\right),$$

as the denominator bounds from below the number of r-sets meeting e_1 in exactly 1 element, while the enumerator is an upper bound on the number of r-sets meeting e_1 in at least $r^{1/2}$ elements. This bound tends to 0 as $r^{3/2}/n$ tends to 0.

Furthermore, still using (3) and writing $\mathcal{E}_3$ for the event $|(e_1 \cup e_2) \cap e_3| \geq r^{1/2}$, we have

$$\mathbb{P}_{3R}(\mathcal{E}_3|\mathcal{E}_2) \leq \frac{\binom{2r}{r^{1/2}}\binom{n-r^{1/2}}{r-r^{1/2}}}{(r-r^{1/2})^2\binom{n-2r}{r-2}} = (2er^{1/2})^{r^{1/2}} \left(\frac{(2r)}{n}\right)^{r^{1/2}} \exp\left(O\left(\frac{r^2}{n}\right)\right),$$

which tends to 0 for the same reason as the previous bound.

Now note that by the definition of the Third Round process $e_1 \cap e_2 \cap e_3 = \emptyset$ and thus $\mathcal{D}_4$ is equivalent to $e_4 \cap \bigcup_{1 \leq i < j \leq 3} e_i \cap e_j \neq \emptyset$. As $\overline{\mathcal{E}_2}, \overline{\mathcal{E}_3}$ imply $|e_1 \setminus (e_2 \cup e_3)|, |e_2 \setminus (e_1 \cup e_3)|, |e_3 \setminus (e_1 \cup e_2)| \geq r - 2r^{1/2}$, we have

$$\mathbb{P}_{3R}(\mathcal{D}_4|\overline{\mathcal{E}_2}, \overline{\mathcal{E}_3}) \leq \frac{2r^{1/2} r \binom{n-2}{r-2}}{(r-2r^{1/2})^3\binom{n-3r}{r-3}} = \frac{n}{r^{5/2}} \exp\left(O\left(\frac{r^2}{n}\right)\right) = O(n^{6/5-5/4}\log^3 n) \to 0,$$

where for the last equality we used the assumption $\omega(n^{1/2}\log^{-1} n) = r = o(n^{1/2}\log^{1/10} n)$ to obtain $\exp\left(O\left(\frac{r^2}{n}\right)\right) = O(n^{1/5})$ and $r^{5/2} = \Omega(n^{5/4}\log^{-5/2} n)$. □

Lemma 3.5 is the equivalent of Lemma 8 in [4] and the 2 proofs are almost identical.

Lemma 3.5 *If $r = \omega(n^{1/2})$ and $2\log n \leq m \leq \exp(\frac{r^2}{3n})$, then*

$$\mathbb{P}_{3R,PO}(\mathcal{D}_m) = o(1).$$

Proof. Pick the first 3 edges e_1, e_2, e_3 according to any of the 2 processes and then consider m elements of $\binom{[n]}{r} \setminus \{e_1, e_2, e_3\}$ being chosen at random without replacement. The probability that these $m+3$ edges fail to form an intersecting family is at most

$$\left(3m + \binom{m}{2}\right) \frac{\binom{n-r}{r}}{\binom{n}{r}} \leq \frac{m^2}{2} \exp\left(-\frac{r^2}{n}\right) \leq \frac{1}{2} \exp\left(-\frac{r^2}{3n}\right),$$

which tends to 0 as $r = \omega(n^{1/2})$. We know that the Put-Off process and the Random Intersecting Hypergraph process is w.h.p. the same if $r = \omega(n^{1/3})$ and the Third Round process is the same as the Random Intersecting Hypergraph process from the fourth round by definition. Thus conditioning on e_1, e_2, e_3, the distribution of $\mathcal{F}_{m+3}$ will be the same as picking m distinct r-sets uniformly at random if we further condition on the event of probability $1 - o(1)$ that the randomly picked sets together with the first 3 edges form an intersecting hypergraph. Thus we obtain that

$$\mathbb{P}_{3R,PO}(\mathcal{D}_m) \leq \frac{1}{2} \exp\left(-\frac{r^2}{3n}\right) + o(1) + nm \left(\frac{\binom{n-1}{r-1}}{\binom{n}{r}}\right)^{m-1} =$$

$$O\left(\exp\left(-\frac{r^2}{3n}\right)\right) + o(1) + mr^{m-1}n^{-m} \leq O\left(\exp\left(-\frac{r^2}{3n}\right)\right) + o(1) + mn2^{-m}.$$

□

Proof of Theorem 1.3. We consider several cases.

Let $\omega(1) = r = c_n n^{1/3}$. If c_n tends to 0, then from Lemma 3.1 and Lemma 3.2 it follows that $\mathcal{C}_{3,1}$ holds w.h.p and then Lemma 3.3 together with Proposition 2.3 finishes the proof of this case.

If $c_n \to c$, then Lemma 3.1 states that Δ holds almost surely. According to Lemma 3.2 the probability that $\mathcal{C}_{3,1}$ holds is $\frac{1}{1+c^3}(1 + o(1))$ and the proof of Lemma 3.2 shows that if $\mathcal{C}_{3,1}$ does not hold, then nor does $\mathcal{D}_4$ thus $\mathcal{E}_{HM}$ cannot happen. Again, Lemma 3.3 together with Proposition 2.3 finishes the proof of this case.

Finally, if c_n tends to infinity then for $\omega(n^{1/3}) = r = o(n^{1/2} \log^{1/10} n)$ Lemma 3.4 while for $\omega(n^{1/2}) = r \leq n/2$ Lemma 3.5 proves that the probability of $\mathcal{E}_{HM}$ is $o(1)$. □

Proof of Corollary 1.6: Bohman et al. in [3] prove that conditioned on the event $\mathcal{A}_3$, a trivially intersecting family is the output of the original process w.h.p. Lemma 2.4 and Lemma 2.7 give that conditioned on the event that A3 does not hold, we have $\bigcap_{e \in \mathcal{F}_3} e = \emptyset$ with probability tending to 1, that is, e_3 is chosen according to the rule of the Third Round Process. Thus the following equality holds:

$$\lim_{n\to\infty} \mathbb{P}_{INT}(\mathcal{E}_{HM}) = \left(1 - \lim_{n\to\infty} \mathbb{P}_{INT}(\mathcal{A}_3)\right) \cdot \lim_{n\to\infty} \mathbb{P}_{3R}(\mathcal{E}_{HM}).$$

Lemma 2.5 and Theorem 1.3 complete the proof. □

4 The Third Round Model II. (r constant)

In this section we consider the Third Round Model when r is a fixed constant and we prove Theorem 1.4. We will use one of the lemmas proved in the previous section and we will also need 2 new ones.

Lemma 4.1 states that $\mathcal{F}_{\log n}$ contains either a 3-star or a 2-triangle w.h.p. but not both.

Lemma 4.1 *If $3 \leq r$ is a constant, then*

$$\mathbb{P}_{3R}(\Delta_{\log n,2}|\Delta) = \left(\frac{1}{r-1}\right)^3 (1+o(1)),$$
$$\mathbb{P}_{3R}(\mathcal{C}'_{\log n,3,1}|\Delta) = 1 - \left(\frac{1}{r-1}\right)^3 (1+o(1)),$$
$$\mathbb{P}_{3R}(\Delta_{\log n,2} \wedge \mathcal{C}'_{\log n,3,1}|\Delta) = o(1).$$

Proof. Let S be the base of $\mathcal{F}_3$ if Δ holds and for any $3 \leq j \leq \log n$ and $i = 0,1,2,3$ let $\mathcal{A}(\mathcal{F}_{j,i})$ denote the r-sets in $\mathcal{A}(\mathcal{F}_j)$ that meet S in i elements and let $M_{j,i} = |\mathcal{A}(\mathcal{F}_{j,i})|$. We will prove that for any $3 \leq j \leq \log n$ the process picks an edge either from $\mathcal{A}(\mathcal{F}_{j,1})$ or from $\mathcal{A}(\mathcal{F}_{j,2})$ w.h.p. Furthermore, if for some $3 \leq j \leq \log n$ the process picks an edge from $\mathcal{A}(\mathcal{F}_{j,1})$, then $\mathcal{C}'_{\log n,3,1}$ holds w.h.p., while if this is not the case (i.e. the process picks an edge from $\mathcal{A}(\mathcal{F}_{j,2})$ for all $3 \leq j \leq \log n$), then $\Delta_{\log n,2}$ holds w.h.p. We will need some bounds on $M_{j,i}$. Clearly, we have

$$M_{j,3} \leq \binom{n-3}{r-3}, \qquad M_{j,0} \leq (r-2)^3 \binom{n-6}{r-3}.$$

For the first inequality we counted all r-sets containing S, while for the second inequality we used that if an r-set $f \in \mathcal{A}(\mathcal{F}_j)$ is disjoint from S, then f must meet e_1, e_2, e_3 outside S. Thus as r is constant we have $M_{j,0}, M_{j,3} = O(n^{r-3})$.

Let Δ'_j denote the event that for all edges e of $\mathcal{F}_j$ we have $|e \cap S| \geq 2$. Clearly Δ'_3 holds. If Δ'_j holds, then every edge f with $|f \cap S| = 2$ belongs to $\mathcal{A}(\mathcal{F}_j)$, therefore we have $M_{j,2} \geq 3\binom{n-3}{r-2} - j = \Theta(n^{r-2})$. By comparing this to the bounds on $M_{j,0}$ and $M_{j,3}$ we obtain that if there is a $j \leq \log n$ such that $e_{j+1} \notin \mathcal{A}(\mathcal{F}_{j,2})$, then $e_{j+1} \in \mathcal{A}(\mathcal{F}_{j,1})$ w.h.p.

We claim that if Δ'_j holds, then all edges of $\mathcal{F}_j$ are pairwise disjoint outside S w.h.p. Indeed, for any $3 \leq j' \leq j$ the number of r-sets $f \in \mathcal{A}(\mathcal{F}_{j'})$ that meet $\cup_{e \in \mathcal{F}_{j'}} e \setminus S$ and $|f \cap S| = 2$ is at most $3(r-2)\binom{n-4}{r-3}\log n = O(n^{r-3}\log n)$ as we can choose in 3 ways which 2 elements of S belong to f, which other element of $\cup_{e \in \mathcal{F}_{j'}} e \setminus S$ belongs to f and $|\cup_{e \in \mathcal{F}_{j'}} e \setminus S| \leq (r-2)\log n$ as $j' \leq \log n$. Thus the probability that $e_{j'+1}$ meets $\cup_{e \in \mathcal{F}_{j'}} e \setminus S$ for some $j' \leq \log n$ is $O(\log^2 n/n)$. Thus if Δ''_j denotes the event that Δ'_j holds and the edges in $\mathcal{F}_j$ are pairwise disjoint outside $\cup_{e \in \mathcal{F}_j} e \setminus S$, then we have just seen that Δ'_j implies Δ''_j w.h.p.

To calculate the probability that there is a $j \leq \log n$ for which $e_{j+1} \in \mathcal{A}(\mathcal{F}_{j,1})$ and to have more insight on the process we introduce some more notations. Let $S_j = \{s \in$

$S : d_j(s) = j-1\}$ and $h_j = |S_j|$, i.e. S_j is the set of elements which are contained in all but at most one edge of $\mathcal{F}_j$ and therefore they still might become the kernel of a possible HM-type extension of $\mathcal{F}_j$. As Δ holds, we have $S_3 = S$ and $h_3 = 3$. Note that if for some $j \le \log n$ the event Δ''_j holds and we have $h_j = 0$, then $\Delta_{j,2}$ holds.

Let us suppose that Δ''_j holds and let us consider e_{j+1}. We distinguish several possibilities (as we already ruled out the occurrence of an edge from $\mathcal{A}(\mathcal{F}_{j,0}) \cup \mathcal{A}(\mathcal{F}_{j,3})$ w.h.p., we omit these possibilities):

1. $e_{j+1} \in \mathcal{A}(\mathcal{F}_{j,1}), e_{j+1} \cap S \notin S_j$, that is, the only common element x of e_{j+1} and S is not contained in at least 2 edges of $\mathcal{F}_j$. There are at most $(3-h_j)(r-2)^2\binom{n-5}{r-3} = \Theta(n^{r-3})$ possibilities for such an e_{j+1}, as we can pick $e_{j+1} \cap S$ in $3-h_j$ ways and e_{j+1} must meet all the edges not containing x (there are at least 2 of them) outside S where those edges are pairwise disjoint. Thus the probability of picking such an edge is $O(1/n)$.

2. $e_{j+1} \in \mathcal{A}(\mathcal{F}_{j,1}), e_{j+1} \cap S \in S_j$ and e_{j+1} does not create a 3-star. We can pick the only element x of $e_{j+1} \cap S_j$ in h_j ways. We know that e_{j+1} must meet the edge of $\mathcal{F}_3$ that does not contain x outside S and to not create a 3-star e_{j+1} must meet at least one of the edges of $\mathcal{F}_3$ containing x outside S as well. These sets are pairwise disjoint outside S, thus there are at most $h_j(r-2)(2r-4)\binom{n-5}{r-3} = \Theta(n^{r-3})$ possibilities again, thus the probability of this to happen is $O(1/n)$.

3. $e_{j+1} \in \mathcal{A}(\mathcal{F}_{j,1}), e_{j+1} \cap S \in S_j$ and e_{j+1} creates a 3-star, i.e. $\mathcal{C}'_{j+1,3,1}$ holds. Number of possibilities: again, we can pick the only element x of $e_{j+1} \cap S_j$ in h_j ways and e_{j+1} must meet the edge of $\mathcal{F}_3$ that does not contain x outside S. Thus there are at most $h_j(r-2)\binom{n-4}{r-2}$ possibilities and if e_{j+1} does not meet any of the edges of $\mathcal{F}_3$ containing x outside S, then a 3-star is created, thus there are at least $h_j(r-2)\binom{n-3r+3}{r-2}$ to choose e_{j+1}. Therefore the number of possibilities is $h_j(r-2)\binom{n}{r}(1+\Theta(1/n))$.

4. $e_{j+1} \in \mathcal{A}(\mathcal{F}_{j,2}), h_{j+1} = h_j$, that is, the only element x of S which is not in e_{j+1} does not belong to S_j. To pick x we have $3-h_j$ possibilities and the fact that the other 2 elements of S do belong to e_{j+1} assures that e_{j+1} intersects all edges in $\mathcal{F}_j$, thus the number of possible e_{j+1}'s is $(3-h_j)\binom{n}{r-2}(1+\Theta(1/n))$.

5. $e_{j+1} \in \mathcal{A}(\mathcal{F}_{j,2}), h_{j+1} = h_j - 1$. The only difference to the previous case is that now we have to pick x from S_j and thus the number of possible e_{j+1}'s is $h_j\binom{n-3}{r-2}(1+\Theta(1/n))$.

The probability that the first two possibilities happen at least once for some $j \le \log n$ is $O(\frac{\log n}{n})$ thus either (3) happens or the process always picks an edge according to (4) or (5). The probability that possibility (4) happens with $h_j < 3$ at least $\frac{\log n}{2}$ times while (3) does not occur at all is $O(3^{-\log n})$. Thus we obtain that for some $j \le \log n$ either possibility (3) happens or we will have $h_j = 0$ which is equivalent to $\Delta_{j,2}$. The probability that if $h_j > 0$, then possibility (5) happens before possibility (3) is

$$\frac{h_j\binom{n-3}{r-2}}{h_j\binom{n-3}{r-2} + h_j(r-2)\binom{n}{r-2}}(1+O(1/n))^{\log n} = \frac{1}{r-1}(1+o(1)).$$

Thus the probability that $\Delta_{j,2}$ holds for some $j \le \log n$ is

$$\left(\frac{1}{r-1}\right)^3 (1+o(1)),$$

while the probability that $\mathcal{C}'_{j,3,1}$ holds for some $j \le \log n$ is

$$1-\left(\frac{1}{r-1}\right)^3 (1+o(1)).$$

Note that Lemma 3.3 implies $\mathbb{P}(\mathcal{C}'_{\log n,3,1}|\mathcal{C}'_{j,3,1}) = 1-o(1)$ (in fact, it states that $\mathcal{C}'_{j,3,1}$ implies the appearance of an r-double broom, which is much more than $\mathcal{C}'_{\log n,3,1}$). Also, $\mathbb{P}_{3R}(\overline{\Delta_{j'+1,2}}|\Delta_{j',2}) = O(1/n)$ for any $j \le j' \le \log n$ as the number of r-sets f meeting all edges of $\mathcal{F}_{j'}$ and intersecting S in one element is at most $3(r-2)^2\binom{n-5}{r-3} = \Theta(n^3)$ (pick $\{x\} = S \cap f$ in 3 ways and f must meet the 2 edges of the 2-triangle, assured by $\Delta_{j',2}$, that do not contain x) thus $\mathbb{P}_{3R}(\Delta_{\log n,2}|\Delta_{j,2}) = O(\frac{\log n}{n})$. □

Lemma 4.1 asserts that, in the case of constant r, $\mathcal{F}_{\log n}$ contains either a 3-star or a 2-triangle w.h.p. By Lemma 3.3, in the former case $\mathcal{F}_j$ contains an r-double broom w.h.p. for sufficiently large j and thus by Proposition 2.3 assures that the process results an HM-type hypergraph. The next lemma states that in the latter case $\mathcal{F}_{2\log n}$ contains an r-triangle w.h.p. and thus by Proposition 2.2 assures that the process outputs a 2-3 hypergraph.

Lemma 4.2 *If $3 \le r$ is a constant, then*

$$\mathbb{P}_{3R}(\Delta_{2\log n,r}|\Delta_{\log n,2}) = 1-o(1).$$

Proof. With the notation of Lemma 4.1, the event $\Delta_{\log n,2}$ implies that for any $j \ge \log n$ we have

$$M_{j,0} \le (r-2)^6\binom{n-9}{r-6} \qquad M_{j,1} \le \binom{n-5}{r-3} \qquad M_{j,3} \le \binom{n-3}{r-3}.$$

Furthermore, if all edges in $\mathcal{F}_j$ intersect the base S of the 2-triangle contained in $\mathcal{F}_{\log n}$ in at least 2 elements, then

$$3\binom{n-3}{r-2} - j \le M_{j,2} \le 3\binom{n-3}{r-2}.$$

Thus the probability, that $\mathcal{F}_{2\log n}$ will contain an edge e with $|e \cap S| \neq 2$ is $O(\frac{\log n}{n})$.

Let β_j denote the largest integer k such that $\mathcal{F}_j$ contains a subhypergraph of an r-triangle with k edges, in particular $\beta_j = 3r$ if and only if $\mathcal{F}_j$ contains an r-triangle. Let us introduce the following random variable

$$W_j = \begin{cases} 1 & \text{if } \beta_j \neq \alpha_{j+1} \text{ or } \beta_j = 3r \\ 0 & \text{otherwise .} \end{cases}$$

If $\beta_j < 3r$ and if all edges in $\mathcal{F}_j$ intersect the base S of the 2-triangle contained in $\mathcal{F}_{\log n}$ in at least 2 elements, then

$$\mathbb{P}_{3R}(W_j = 1) \geq \frac{\binom{n-\beta_j(r-2)-3}{r-2}}{\sum_{i=0}^{3} M_{j,i}} \geq 1/3 - o(1).$$

Therefore

$$\mathbb{P}_{3R}(\beta_{2\log n} < 3n|\Delta_{\log n,2}) = \mathbb{P}_{3R}\left(\sum_{j=\log n}^{2\log n} W_j < 3r|\Delta_{\log n,2}\right) \leq$$

$$\mathbb{P}(\mathrm{Bi}(\log n, 1/3 - o(1)) < 3r) + O\left(\frac{\log n}{n}\right).$$

□

Proof of Theorem 1.4. Lemma 3.1 assures that Δ holds w.h.p. Lemma 4.1 and Lemma 4.2 together with Proposition 2.2 and Proposition 2.3 proves Theorem 1.4. □

5 The Put-Off Model

In this section, we prove Theorem 1.5. The proof is similar to that of Theorem 1.3 as again for a large enough j we will prove the existence of an r-double broom in $\mathcal{F}_j$ to assure that $\mathcal{E}_{HM}$ happens. The main difference between the 2 models is that while in the Third Round Model the set of semi-kernels of the r-double broom is already determined after e_4 is picked in the Put-Off Model there might be lots of possibilities for this set even later.

Let us define $j_1 = \min\{\binom{n-r-1}{r-2}^{1/4}, n^2\}$ and $j_2 = \min\{\binom{n-r-1}{r-2}^{1/2}, n^3\}$. Our first lemma states that if $\mathcal{F}_3$ is a 3-star, then w.h.p. the kernel of this 3-star is contained in all edges of $\mathcal{F}_{j_2}$.

Lemma 5.1 *If $r = O(n^{1/3})$, then*

$$\mathbb{P}_{PO}(\mathcal{B}_{j_2}|\mathcal{A}_3) = 1 - o(1).$$

Proof. Observe that if $\mathcal{B}_j$ holds, then the number of sets in $\mathcal{A}(\mathcal{F}_j)$ containing an element of $\bigcap_{e\in\mathcal{F}_j} e$ is at least $r\binom{n-r-1}{r-2} - j$. The number of r-sets containing a fixed element x and meeting a fixed r-set $f \in \mathcal{A}(\mathcal{F}_j)$ in exactly one element with $x \notin f$ is $r\binom{n-r-1}{r-2}$ and even if all of them belong to $\mathcal{F}_j$ then the others are in $\mathcal{A}(\mathcal{F}_j)$.

Suppose first that $r = o(n^{1/4})$. Then by Lemma 2.6 we may assume that $\mathcal{A}_r$ holds and thus for any $j \geq r$ the number of sets in $\mathcal{A}(\mathcal{F}_j)$ not containing the kernel of $\mathcal{F}_j$ is at most $(r-1)^r$. Thus we have

$$\mathbb{P}_{PO}(\overline{\mathcal{B}_{j+1}}|\mathcal{B}_j, \mathcal{A}_3) \leq \frac{(r-1)^r}{r\binom{n-r-1}{r-2} - j} \leq \frac{(r-1)^r}{(r-1)\binom{n-r-1}{r-2}}.$$

If r is constant, then multiplying the last ratio by $j_2 \le \binom{n-r-1}{r-2}^{1/2}$ still gives a bound that tends to 0. If r tends to infinity, then $n^3 \frac{(r-1)^r}{(r-1)\binom{n-r-1}{r-2}} \to 0$ holds as $r = o(n^{1/2-\epsilon})$.

Suppose now that $r = \omega(n^{1/5})$ and thus $j_2 = n^3$. Then by Lemma 2.6 we know that $\mathcal{A}_{n^{1/5}}$ holds w.h.p. In this case for any j with $n^{1/5} \le j \le n^3 = j_2$ the number of sets in $\mathcal{A}(\mathcal{F}_j)$ not containing the kernel of $\mathcal{F}_{n^{1/5}}$ is at most $(r-1)^{n^{1/5}}\binom{n-n^{1/5}-1}{r-n^{1/5}}$ and thus we have

$$\mathbb{P}_{PO}(\overline{\mathcal{B}_{j+1}}|\mathcal{B}_j, \mathcal{A}_3) \le \frac{(r-1)^{n^{1/5}}\binom{n-n^{1/5}-1}{r-n^{1/5}}}{r\binom{n-r-1}{r-2} - j} \le \frac{(r-1)^{n^{1/5}}\binom{n-n^{1/5}-1}{r-n^{1/5}}}{(r-1)\binom{n-r-1}{r-2}} \le$$

$$\left(\frac{2r^2}{n}\right)^{n^{1/5}} \cdot \prod_{j=0}^{r-n^{1/5}-1} \frac{n-n^{1/5}-1-j}{n-r-1-j} = \left(\frac{2r^2}{n}\right)^{n^{1/5}} \exp\left(O\left(\frac{r^2}{n}\right)\right).$$

Therefore

$$\mathbb{P}_{PO}(\overline{\mathcal{B}_{n^3}}|\mathcal{A}_3) \le n^3 \left(\frac{2r^2}{n}\right)^{n^{1/5}} \exp\left(O\left(\frac{r^2}{n}\right)\right) + \mathbb{P}_{PO}(\overline{\mathcal{A}_{n^{1/5}}}|\mathcal{A}_3) \to 0.$$

□

Lemma 5.2 asserts that if the edges of $\mathcal{F}_3$ form a 3-star, then $\mathcal{F}_{j_1}$ contains an r-star and its kernel belongs to all but at most one edge of $\mathcal{F}_{j_1}$.

Lemma 5.2 *If $r = O(n^{1/3})$, then*

$$\mathbb{P}_{PO}(\mathcal{A}'_{j_1,r}|\mathcal{A}_3) = 1 - o(1).$$

Proof. We consider two cases.

Case I $r = o(n^{1/4})$

In this case Lemma 2.6 shows that $\mathbb{P}_{PO}(\mathcal{A}_r|\mathcal{A}_3) = 1-o(1)$ and we are done as $\mathcal{A}_r = \mathcal{A}'_{r,r}$ and $\mathcal{A}'_{r,r} \wedge \mathcal{B}_{n^3}$ implies $\mathcal{A}'_{j,r}$ for any $n^3 \ge j \ge r$.

Case II $\omega(n^{1/5}) = r = O(n^{1/3})$

In this case $j_1 = n^2$. Using Lemma 2.6 and Lemma 5.1 we can assume that $\mathcal{A}_{n^{1/5}}$ and $\mathcal{B}_{n^3}$ hold. Let x denote the kernel of $\mathcal{F}_{n^{1/5}}$. Let us define $\mathcal{T} = \{t \in \binom{[n]}{r} : x \notin t, t \cap e_i \neq \emptyset \forall i, 1 \le i \le n^{1/5}\}$. Clearly, we have

$$|\mathcal{T}| \le (r-1)^{n^{1/5}}\binom{n-n^{1/5}-1}{r-n^{1/5}}.$$

For any $t \in \mathcal{T}$ and $j \ge n^{1/5}$ let $\nu_{t,j}$ denote the maximum number k such that there exists a k-star in $\mathcal{F}_j$ such that t is a transversal of the sets of the k-star. Clearly, the event $\mathcal{A}'_{j,r}$ holds if and only if there is a $t \in \mathcal{T}$ with $\nu_{j,t} = r$.

Let us introduce the following random variables:

$$X_{j,t} = \begin{cases} 1 & \text{if } \nu_{j,t} \neq \nu_{j+1,t} \text{ or } \nu_{j,t} = r \text{ or } t \notin \mathcal{A}(\mathcal{F}_j) \cup \mathcal{F}_j \\ 0 & \text{otherwise .} \end{cases}$$

Let us write furthermore $X_t = \sum_{j=n^{1/5}}^{n^2} X_{j,t}$. Observe that if $X_t \geq r$ for all $t \in \mathcal{T}$, then $\mathcal{A}'_{n^2,r}$ holds. Indeed, by the definition of the Put-Off Process there must always be a $t^* \in \mathcal{T}$ which belongs $\mathcal{A}(\mathcal{F}_j) \cup \mathcal{F}_j$ and for this t^* the fact $X_{t^*} \geq r$ shows that there exists an r-star in $\mathcal{F}_{n^{1/3}}$ of which t^* is a transversal.

Let us bound from below the probability $\mathbb{P}_{PO}(X_{j,t} = 1)$ for any t and $j \geq n^{1/5}$. If $t \notin \mathcal{A}(\mathcal{F}_j) \cup \mathcal{F}_j$ and $\nu_{j,t} < r$ (as otherwise $X_{j,t} = 1$ for sure), then let us fix a $\nu_{j,t}$-star B showing this. The kernel of B must be x. The number of r-sets containing x but otherwise disjoint from $\bigcup_{b \in B} b$ that meet t in exactly one element is

$$(r - \nu_{j,t}) \binom{n - \nu_{j,t}(r-1) - 2}{r-2} \geq \binom{n - (r-1)^2 - 2}{r-2}.$$

On the other hand, for any $j \geq n^{1/5}$ we have $\mathcal{A}(\mathcal{F}_j) \subseteq \{e \in \binom{[n]}{r} : x \in e\} \cup \mathcal{T}$. Since $|\mathcal{T}| \leq (r-1)^{n^{1/5}} \binom{n-n^{1/5}-1}{r-n^{1/5}} \leq \binom{n-1}{r-1}$, we have $|\mathcal{A}(\mathcal{F}_j)| \leq 2\binom{n-1}{r-1}$ and thus

$$\mathbb{P}_{PO}(X_{j,t} = 1) \geq \frac{\binom{n-(r-1)^2-2}{r-2}}{2\binom{n-1}{r-1}} \geq \frac{r-1}{2(n-r-1)} \frac{\binom{n-r^2}{r-2}}{\binom{n-1}{r-2}} \geq \frac{r}{3n} \left(\frac{n - r^2 - r + 3}{n - r^2} \right)^{r-2} =$$

$$\Omega\left(\frac{r}{n}\left(1 - \frac{r^2}{n}\right)^r\right) = \Omega\left(\frac{r}{n}\left(1 - \frac{1}{r}\right)^r\right) = \Omega\left(\frac{r}{n}\right).$$

Therefore we have

$$\mathbb{P}_{PO}(\exists t \in \mathcal{T} : X_t \leq r) \leq |\mathcal{T}| \mathbb{P}\left(\mathrm{Bi}\left(n^2, \Omega\left(\frac{r}{n}\right)\right) \leq r\right) \leq \exp(O(r \log n)) \exp(-\Omega(rn)) \to 0.$$

□

Lemma 5.3 states that if $\mathcal{F}_{j_1}$ contains an r-star, then $\mathcal{F}_{j_2}$ contains an r-double broom w.h.p. and thus with Proposition 2.3 assures that the process outputs an HM-type hypergraph.

Lemma 5.3 *If $r = O(n^{1/3})$, then we have*

$$\mathbb{P}_{PO}(\mathcal{A}''_{j_2,r} | \mathcal{B}_{n^3}, \mathcal{A}'_{n^2,r}) = 1 - o(1).$$

Proof. The proof is similar to that of Case II in the proof of Lemma 5.2. For any $j \geq j_1$ let μ_j denote the largest integer d such that $\mathcal{F}_j$ contains a d-partial r-double broom. Since an r-star is a 0-partial r-double broom, μ_j is well-defined. Let us introduce the following random variables:

$$Y_j = \begin{cases} 1 & \text{if } \mu_j \neq \mu_{j+1} \text{ or } \mu_j = r \\ 0 & \text{otherwise .} \end{cases}$$

Let us write furthermore $Y = \sum_{j=j_1}^{j_2} Y_j$. Let us sharpen the upper bound on $|\mathcal{A}(\mathcal{F}_j)|$ from Lemma 5.2 by considering a μ_j-partial r-double broom B with kernel x. Every set f in $\mathcal{A}(\mathcal{F}_j)$ must meet $\bigcup_{b\in B} b\setminus\{x\}$ as otherwise $\{f\}\cup\mathcal{F}_j$ would contain an $(r+1)$-star which is impossible to extend to a non-trivially intersecting hypergraph. Thus the number of sets in $\mathcal{A}(\mathcal{F}_j)$ containing x is at most $2r^2\binom{n-2}{r-2}$. The number of sets in $\mathcal{A}(\mathcal{F}_j)$ not containing x is at most $(r-1)^r$ as $\mathcal{A}'_{j,r}$ holds. Thus

$$|\mathcal{A}(\mathcal{F}_j)| \leq 2r^2\binom{n-2}{r-2} + (r-1)^r \leq 3r^2\binom{n-2}{r-2}.$$

On the other hand, by Proposition 2.3 every $t \in \mathcal{A}(\mathcal{F}_j)$ with $x \notin t$ contains all the semi-kernels of B and meets all the lonely fingers of B. Note that there is at least one such set by the definition of the Put-Off Process, say t_j. Observe that any r-set e containing x with $|t_j \cap e| = 1, |e \cap \bigcup_{b\in B} b| = 2$ and $t_j \cap e$ belonging to a lonely finger of B is in $\mathcal{A}(\mathcal{F}_j)$, furthermore if such a set is chosen to be e_{j+1} by the Put-Off process, then $\mu_{j+1} = \mu_j + 1$.

Thus we obtain

$$\mathbb{P}_{PO}(Y_j = 1) \geq \frac{(r-\mu_j)\binom{n-r(r-1)-1}{r-2}}{3r^2\binom{n-2}{r-2}} \geq \frac{1}{3r^2}\frac{\binom{n-r^2}{r-2}}{\binom{n-1}{r-2}} \geq \Omega\left(\frac{1}{r^2}\right)\left(1-\frac{r^2}{n-2r}\right)^r = \Omega\left(\frac{1}{r^2}\right),$$

as $r = O(n^{1/3})$.

Now we have

$$\mathbb{P}_{PO}(\overline{\mathcal{A}''_{j_2,r}}|\mathcal{B}_{j_2}, \mathcal{A}'_{j_1,r}) \leq \mathbb{P}_{PO}(Y < r|\mathcal{B}_{j_2}, \mathcal{A}'_{j_1,r}) \leq \mathbb{P}\left(\mathrm{Bi}\left(j_2 - j_1, \Omega\left(\frac{1}{r^2}\right)\right) < r\right) \to 0,$$

as if r is constant, then $1/r^2$ is constant and $j_2 - j_1 \to \infty$, while if $r = \omega(1)$, then $j_2 - j_1 = n^3 - n^2$. $\square$

Proof of Theorem 1.5. Let $c_n n = r < n/2$. If c_n tends to 0, then Lemma 2.5, Lemma 5.1, Lemma 5.2 and Lemma 5.3 assures that for a suitably chosen j $\mathcal{F}_j$ contains an r-double-broom w.h.p. and thus, by Proposition 2.3, $\mathcal{E}_{HM}$ holds w.h.p.

If c_n tends to c, then Lemma 2.5 states that the probability that $\mathcal{A}_3$ holds tends to $\frac{1}{1+c^3}$ in which case, just as for $r = \omega(n^{1/3})$, $\mathcal{E}_{HM}$ holds. Lemma 2.7 assures that $\overline{\mathcal{B}_3}$ holds with probability tending to $1 - \frac{1}{1+c^3}$. As if $\overline{\mathcal{B}_3}$ holds, then the Third Round Model and the Put-Off Model coincide and thus by Theorem 1.3 we obtain that $\mathbb{P}_{PO}(\mathcal{E}_{HM}|\mathcal{B}_3) = \mathbb{P}_{3R}(\mathcal{E}_{HM})(1+o(1))$ tends to $\frac{1}{1+c^3/3}$. Hence $\mathbb{P}_{PO}(\mathcal{E}_{HM})$ tends to $\frac{1}{1+c^3} + (1 - \frac{1}{1+c^3})\frac{1}{1+c^3/3}$ as claimed in the theorem.

If c_n tends to infinity, then note that for $\omega(n^{1/3}) = r = o(n^{2/3})$ a consequence of Lemma 2.8 is that w.h.p the Third Round Model and the Put-Off Model are the same, while Lemma 3.5 proves Theorem 1.5 for $\omega(n^{1/2}) = r < n/2$. $\square$

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