

Extremal Combinatorics

The Erdős-Ko-Rado theorem - polynomial method proof

Balázs Patkós

We present a third proof of the Erdős-Ko-Rado theorem that uses the polynomial method. The main idea of this method is to assign polynomials $p_F(x)$ to every set $F \in \mathcal{F}$ and show that these polynomials are linearly independent in the appropriate vector space V . If this is so, then $|\mathcal{F}| \leq \dim(V)$ follows. Let us start with a general lemma giving necessary conditions for polynomials to be independent.

Lemma 1. *Let $p_1(x), p_2(x), \dots, p_m(x) \in \mathbb{F}^n[x]$ be polynomials and $v_1, v_2, \dots, v_m \in \mathbb{F}^n$ be vectors such that $p_i(v_i) \neq 0$ and $p_i(v_j) = 0$ holds for all $1 \leq j < i \leq m$. Then the polynomials are linearly independent.*

Proof. Suppose that $\sum_{i=1}^m c_i p_i(x) = 0$. As $p_i(v_1) = 0$ for all $1 < i$ we obtain $c_1 p_1(v_1) = 0$ and therefore $c_1 = 0$ holds. We proceed by induction on j . If $c_h = 0$ holds for all $h < j$, then using this and $p_i(v_j) = 0$ for all $i > j$, we obtain $c_j p_j(v_j) = 0$ and therefore $c_j = 0$. ■

Recall that the characteristic vector v_F of a set $F \subseteq [n]$ is the binary vector of length n with i th entry 1 if and only if $i \in F$. By definition, we have $v_F \cdot v_G = |F \cap G|$ for any pair of sets $F, G \subseteq [n]$. Therefore if $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting family, then the polynomials $p_F(x) = \prod_{i=1}^{k-1} (x \cdot v_F - i)$ and the characteristic vectors v_F satisfy the conditions of Lemma 1 (independently of the order of sets of \mathcal{F}). What is the dimension of the smallest vector space that contains these polynomials? As characteristic vectors have only 0 and 1 entries, we can replace all powers x_j^h by x_j in the p_F 's and the polynomials p'_F obtained will still satisfy the conditions of Lemma 1. The vector space generated by these polynomials is a subspace of $V = \langle \{x_{i_1} \cdot x_{i_2} \cdot \dots \cdot x_{i_h}, 1 \leq i_1 < i_2 < \dots < i_h \leq n, 0 \leq h \leq k-1\} \rangle$ and clearly, we have $\dim(V) = \sum_{i=0}^{k-1} \binom{n}{i}$ holds. This yields the weak bound $|\mathcal{F}| \leq \sum_{i=0}^{k-1} \binom{n}{i}$. The next proof that improves this and attains the bound of Theorem ?? is due to Füredi, Hwang, and Weichsel [?] and uses an idea that first appeared in a paper of Blokhuis [?].

Third proof of Theorem ??. Our plan is to add $\sum_{i=0}^{k-1} \binom{n}{i} - \binom{n-1}{k-1} = 2 \sum_{j=0}^{k-2} \binom{n-1}{j}$ many sets to \mathcal{F} and define corresponding polynomials and vectors such that the conditions of Lemma 1 are still satisfied. More precisely, we will also change some of the polynomials

corresponding to sets in \mathcal{F} , and replace the characteristic vectors by other vectors. Let $a \in [n]$ be arbitrary and let us define $\mathcal{A} = \mathcal{F}_0 \cup \mathcal{H} \cup \mathcal{F}_1 \cup \mathcal{G}$ where $\mathcal{F}_0 = \{F \in \mathcal{F} : a \notin F\}$, $\mathcal{H} = \{H \subset [n] : a \notin H, 0 \leq |H| \leq k-2\}$, $\mathcal{F}_1 = \{F \in \mathcal{F} : a \in F\}$, $\mathcal{G} = \{G \subset [n] : a \in G : 1 \leq |G| \leq k-1\}$. Clearly, $|\mathcal{H}| = |\mathcal{G}| = \sum_{j=0}^{k-2} \binom{n-1}{j}$ holds. We define the following polynomials:

- For $F \in \mathcal{F}_0$ let $p_F(x) = \prod_{i=1}^{k-1} (v_F \cdot x - i)$, therefore for a set $S \subseteq [n]$ we have $p_F(v_S) = 0$ if and only if $1 \leq |(F \cap S)| \leq k-1$. As opposed to what we had before, let $u_F = v_{\overline{F} \setminus \{a\}}$.
- For $H \in \mathcal{H}$ we define $p_H(x) = (\mathbf{1} \cdot x - (n-k-1)) \prod_{h \in H} x_h$, where $\mathbf{1}$ denotes the vector of length n with all 1 entries. For a set $S \subseteq [n]$ we have $p_H(v_S) = 0$ if and only if $|S| = n-k-1$ or $H \not\subseteq S$. We let $u_H = v_H$.
- For $F \in \mathcal{F}_1$ we consider $p_F(x) = \prod_{i=0}^{k-2} (v_{F \setminus \{a\}} \cdot x - i)$; therefore for a set $S \subseteq [n]$ we have $p_F(v_S) = 0$ if and only if $0 \leq |(F \setminus \{a\}) \cap S| \leq k-2$. Let $u_F = v_{F \setminus \{a\}}$.
- For $G \in \mathcal{G}$ we define $p_G(x) = \prod_{g \in G} x_g$. For a set $S \subseteq [n]$ we have $p_G(v_S) = 0$ if and only if $G \not\subseteq S$. We let $u_G = v_G$.

Again, as all vectors are binary, we can change every power x_i^s to x_i in these polynomials, so they are contained in the vector space of polynomials with n real variables that have degree at most 1 in each variable. Therefore $|\mathcal{F}| + |\mathcal{H}| + |\mathcal{G}| = |\mathcal{A}| \leq \sum_{i=0}^{k-1} \binom{n}{i}$ and thus $|\mathcal{F}| \leq \binom{n-1}{k-1}$ provided these polynomials are independent. To check that the conditions of Lemma 1 hold, we still need to define an order of these sets and polynomials. First we enumerate the polynomials p_F belonging to sets in \mathcal{F}_0 in arbitrary order, then the polynomials p_H belonging to sets in \mathcal{H} such that if $|H| < |H'|$ then H comes before H' . Then come polynomials p_F belonging to sets in \mathcal{F}_1 in arbitrary order, finally polynomials p_G belonging to sets in \mathcal{G} such that if $|G| < |G'|$ then G comes before G' . We verify that the conditions of Lemma 1 hold by a simple case analysis.

- For $F, F' \in \mathcal{F}_0$ we have $p_F(u_F) \neq 0$ as $F \cap \overline{F} = \emptyset$ and $p_F(u'_F) = 0$ as $1 \leq |F' \cap (\overline{F} \setminus \{a\})| \leq k-1$ if $F \neq F'$. Indeed, $1 \leq |F' \cap \overline{F}| \leq k-1$ as \mathcal{F} is intersecting, and also we have $a \notin F'$. For $H \in \mathcal{H}$ we have $p_H(u_F) = 0$ as $|\overline{F} \setminus \{a\}| = n-k-1$. For $F_1 \in \mathcal{F}_1$ we have $p_{F_1}(u_F) = 0$ as $|(F_1 \setminus \{a\}) \cap (\overline{F} \setminus \{a\})| \leq k-2$. Indeed, $|F_1 \cap \overline{F}| \leq k-1$ as \mathcal{F} is intersecting, and a is contained in both. For $G \in \mathcal{G}$ we have $p_G(u_F) = 0$ as $G \not\subseteq \overline{F} \setminus \{a\}$ since $a \in G$ for all $G \in \mathcal{G}$.
- For $H, H' \in \mathcal{H}$ with $|H| \leq |H'|$ we have $p_H(u_H) \neq 0$ as $H \subseteq H'$ and $|H| \leq k-2 < n-k-1$ (this is the only time we use the assumption $2k \leq n$). We also have $p_{H'}(u_H) = 0$ as $H' \not\subseteq H$. For $F \in \mathcal{F}_1$ we have $p_F(u_H) = 0$ as $|H| \leq k-2$ and for $G \in \mathcal{G}$ we have $p_G(u_H) = 0$ as $G \not\subseteq H$ since $a \in G, a \notin H$.
- Trivially, for $F, F' \in \mathcal{F}_1$ we have $p_F(u_F) \neq 0$ and $p_{F'}(u_F) = 0$. For $G \in \mathcal{G}$ we have $p_G(u_F) = 0$ as $G \not\subseteq F \setminus \{a\}$ since $a \in G$ for all $G \in \mathcal{G}$.

- For $G, G' \in \mathcal{H}$ with $|G| \leq |G'|$ we have $p_G(u_G) \neq 0$ as $G \subseteq G'$, and $p_{G'}(u_G) = 0$ as $G' \not\subseteq G$.

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