Extremal Combinatorics
t-intersecting, k-Sperner families

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First we prove the result of Milner that determines the maximum possible size of an intersecting Sperner family $\mathcal{F} \subseteq 2^{[n]}$. Obviously, if n is odd, then $\binom{[n]}{\lceil n/2 \rceil}$ is intersecting Sperner and by Sperner's theorem, it has maximum size even among all antichains. Therefore the important part of the following theorem is when n is even. The proof we present here is due to Katona [7].

Theorem 1 (Milner, [8])**.** *If $\mathcal{F} \subseteq 2^{[n]}$ is an intersecting Sperner family, then $|\mathcal{F}| \le \binom{n}{\lceil \frac{n+1}{2} \rceil}$.*

Proof. First we prove the following lemma.

Lemma 2. *Let σ be a cyclic permutation of $[n]$ and let $G_1, G_2, \dots, G_r$ be intervals of σ that form an intersecting Sperner family. Then the following inequality holds:*

$$\sum_{i=1}^{r} \binom{n}{|G_i|} \le n \binom{n}{\lceil \frac{n+1}{2} \rceil}.$$

Proof of Lemma. By being an antichain, we see that all the G_i's have distinct left endpoints and therefore $r \le n$ holds. This finishes the proof if n is odd (as in that case $\binom{n}{\lceil \frac{n+1}{2} \rceil}$ is the largest binomial coefficient). If n is even, we distinguish two cases.

Case I: $r = n$.

We can assume that the left endpoint of G_i is $\sigma(i)$. Then by the Sperner property, we must have $|G_i| \le |G_{i+1}|$. Consequently, we obtain $|G_1| \le |G_2| \le \dots |G_n| \le |G_1|$ and therefore all G_i's must have the same size. By the intersecting property this size is at least $\lceil \frac{n+1}{2} \rceil$.

Case II: $r < n$.

By the intersecting property and the "cycle lemma for the EKR Theorem proof", at most $n/2$ intervals have size $n/2$. Therefore we have

$$\sum_{i=1}^{r} \binom{n}{|G_i|} \le \frac{n}{2}\binom{n}{\frac{n}{2}} + \left(\frac{n}{2} - 1\right)\binom{n}{\frac{n}{2}+1} = n\binom{n}{\frac{n}{2}+1} = n\binom{n}{\lceil \frac{n+1}{2} \rceil}.$$

$\square$

Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting Sperner family and let us consider the sum

$$\sum_{\sigma,F} \binom{n}{|F|},$$

where the summation is over all cyclic permutations σ of $[n]$ and sets $F \in \mathcal{F}$ that are intervals of σ. For a fixed set F, the number of cyclic permutations of which F is an interval is $|F|!(n-|F|)!$, therefore the above sum equals $|\mathcal{F}| \cdot n!$. On the other hand, by Lemma 2, the sum is at most $(n-1)! \cdot n\binom{n}{\lceil \frac{n+1}{2} \rceil}$. This yields $|\mathcal{F}| \cdot n! \leq (n-1)! \cdot n\binom{n}{\lceil \frac{n+1}{2} \rceil}$ and the theorem follows. $\blacksquare$

Theorem 3 (Bollobás, [2]). *If $\mathcal{F} \subseteq 2^{[n]}$ is an intersecting Sperner family such that all sets of $\mathcal{F}$ have size at most $n/2$, then the following inequality holds:*

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n-1}{|F|-1}} \leq 1.$$

Note that this inequality is a strengthening of EKR theorem as any family containing sets of the same size is an antichain.

Proof. We start with a generalization of "EKR cycle lemma".

Lemma 4. *Let σ be a cyclic permutation of $[n]$ and let $G_1, G_2, \ldots, G_r$ be intervals of σ of size at most $n/2$ that form an intersecting Sperner family. Then $r \leq \min\{|G_i| : 1 \leq i \leq r\}$ holds.*

Proof of Lemma. By symmetry, it is enough to show that $r \leq |G_1| =: j$. If $G_1 = \{\sigma(i), \sigma(i+1), \ldots, \sigma(i+j-1)\}$, then by the intersecting Sperner property, all G_k's must have one of their endpoints in G_1 and $\sigma(i)$ cannot be a left endpoint and $\sigma(i+j-1)$ cannot be a right endpoint. This would give $2j-2$ possible other intervals in the family. Notice that as all G_k's have size at most $n/2$, if $\sigma(i+h)$ is a right endpoint, then $\sigma(i+h+1)$ cannot be a left endpoint and vice versa. This leaves at most $j-1$ possible other G_k's. $\square$

Let us consider the sum

$$\sum_{\sigma,F} \frac{1}{|F|},$$

where the summation is over all cyclic permutations σ of $[n]$ and sets $F \in \mathcal{F}$ that are intervals of σ. For fixed σ, by Lemma 4, we have that if $F_1, F_2, \ldots, F_r$ are the intervals of σ belonging to $\mathcal{F}$, then $\sum_{i=1}^{r} \frac{1}{|F_i|} \leq 1$. For fixed $F \in \mathcal{F}$ the number of cyclic permutations of which F is an interval is $|F|!(n-|F|)!$, therefore $\sum_{\sigma} \frac{1}{|F|} = (|F|-1)!(n-|F|)!$. We obtained

$$\sum_{F \in \mathcal{F}} (|F|-1)!(n-|F|)! \leq (n-1)!,$$

and dividing by $(n-1)!$ finishes the proof. $\blacksquare$

Theorem 5 (Greene, Katona, Kleitman, [6]). *If $\mathcal{F} \subseteq 2^{[n]}$ is an intersecting Sperner family, then the following inequality holds:*

$$\sum_{F\in\mathcal{F},|F|\le n/2} \frac{1}{\binom{n}{|F|-1}} + \sum_{F\in\mathcal{F},|F|>n/2} \frac{1}{\binom{n}{|F|}} \le 1.$$

Proof. The core of the proof is again an inequality for intersecting Sperner families of intervals equipped with an appropriate weight function.

Lemma 6. *Let σ be a cyclic permutation of $[n]$ and let $G_1, G_2, \dots, G_r$, $G'_1, G'_2, \dots, G'_s$ be intervals of σ that form an intersecting Sperner family such that $|G_i| \le n/2$ for all $1 \le i \le r$ and $|G'_j| > n/2$ for all $1 \le j \le s$. Then the following inequality holds:*

$$\sum_{i=1}^{r} \frac{n-|G_i|+1}{|G_i|} + \sum_{j=1}^{s} 1 \le n.$$

Proof. We distinguish two cases.

Case I: $r = 0$.

Then the first sum of the left hand side of the inequality is empty and all we have to prove is $s \le n$. This follows from the Sperner property as the left endpoints of the intervals must be distinct.

Case II: $r > 0$.

We may assume that $k = |G_1|$ is the smallest size among all G_i's and that $G_1 = \{\sigma(1), \sigma(2), \dots, \sigma(k)\}$. Note that the weight $\frac{n-w+1}{w}$ is monotone decreasing in w, therefore all weights are at most $\frac{n-k+1}{k}$. Because of the intersecting Sperner property every G_i and G'_j has either left endpoint $\sigma(u)$ for some $u = 2, 3, \dots, k$ or right endpoint $\sigma(u')$ for some $u' = 1, 2, \dots, k-1$. Therefore apart from G_1 the possible remaining sets can be partitioned into at most $k-1$ pairs (not all pairs and not both sets of a pair are necessarily present): the $(u-1)$st such pair consists of the set with left endpoint $\sigma(u)$ and the set with right endpoint $\sigma(u-1)$. Observe that because of the intersecting property every pair contains at most one set with size smaller than $n/2$. Therefore the sum of all weights is at most

$$\frac{n-k+1}{k} + (k-1)\Big(\frac{n-k+1}{k} + 1\Big) = n.$$

□

Having Lemma 6 in hand we prove the theorem by considering the sum

$$\sum_{\sigma,F,|F|\le n/2}^{r} \frac{n-|F|+1}{|F|} + \sum_{\sigma,F,|F|>n/2} 1,$$

where the summation in both sums is over all cyclic permutations σ and all sets F of $\mathcal{F}$ that are intervals of σ. By Lemma 6, for fixed σ the sum is at most n. Recall that for fixed F the number of cyclic permutations of which F is an interval is $|F|!(n-|F|)!$. We obtain

$$\sum_{F\in\mathcal{F},|F|\le n/2}^{r} \frac{n-|F|+1}{|F|}|F|!(n-|F|)! + \sum_{F\in\mathcal{F}|F|>n/2} |F|!(n-|F|)! \le$$
$$\le (n-1)!\cdot n = n!.$$

Note that $\frac{n-|F|+1}{|F|}|F|!(n-|F|)! = (|F|-1)!(n-|F|+1)!$, therefore dividing by $n!$ finishes the proof of the theorem. ■

0.1 Known results

Theorem 0.1 (Milner, [8]). *If $\mathcal{F}\subseteq 2^{[n]}$ is a t-intersecting antichain, then*

$$|\mathcal{F}| \le \binom{n}{\lfloor \frac{n+t+1}{2} \rfloor}.$$

In a different direction, Frankl [3] determined the maximum size of an intersecting k-Sperner family. Different proofs were given by Gerbner [4] and by Gerbner, Methuku and Tompkins [5].

Theorem 0.2 (Frankl). *Let $\mathcal{F}\subseteq 2^{[n]}$ be an intersecting, k-Sperner family. Then,*

$$|\mathcal{F}| \le \begin{cases} \sum_{i=\frac{n+1}{2}}^{\frac{n+1}{2}+k-1} \binom{n}{i}, & \text{if } n \text{ is odd,} \\ \binom{n-1}{\frac{n}{2}-1} + \sum_{i=\frac{n}{2}+1}^{\frac{n}{2}+k-1} \binom{n}{i} + \binom{n-1}{\frac{n}{2}+k}, & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, if n is odd, equality holds only if

$$\mathcal{F} = \binom{[n]}{\lfloor \frac{n}{2} \rfloor + 1} \cup \binom{[n]}{\lfloor \frac{n}{2} \rfloor + 2} \cup \ldots \cup \binom{[n]}{\lfloor \frac{n}{2} \rfloor + k},$$

while if n is even and $k>1$, equality holds only if for some $x\in[n]$,

$$\mathcal{F} = \left\{F \in \binom{[n]}{\frac{n}{2}} : x\in F\right\} \cup \binom{[n]}{\frac{n}{2}+1} \cup \ldots \cup \binom{[n]}{\frac{n}{2}+k-1} \cup \left\{F\in \binom{[n]}{\frac{n}{2}+k} : x \notin F\right\}.$$

Theorem 0.3 (Balogh, Linz, Patkós [1]). *Let t and k be positive integers. There exists $n_0 = n_0(t,k)$ such that if $n \ge n_0$ and $n+t$ is even, then the following holds: if $\mathcal{F}\subseteq 2^{[n]}$ is a t-intersecting k-Sperner family, then*

$$|\mathcal{F}| \le \binom{[n]}{\frac{n+t}{2}} + \ldots + \binom{[n]}{\frac{n+t}{2}+k-1}.$$

Furthermore equality holds if and only if $\mathcal{F} = \bigcup_{i=0}^{k-1} \binom{[n]}{\frac{n+t}{2}+i}$.

The conjectured extremal families do not have such a simple structure when $n+t$ is odd. We construct two plausible candidates for the maximum size t-intersecting, k-Sperner family:

$$\mathcal{A}(t,k) = \left\{F \in \binom{[n]}{\frac{n+t-1}{2}} : n \notin F\right\} \cup \left\{A : \frac{n+t-1}{2} + 1 \leq |A| \leq \frac{n+t-1}{2} + (k-1)\right\}.$$

$$\mathcal{B}(t,k) = \left\{F \in \binom{[n]}{\frac{n+t-1}{2}} : [1,t] \in F\right\} \cup \left\{A : \frac{n+t-1}{2} + 1 \leq |A| \leq \frac{n+t-1}{2} + (k-1)\right\}$$
$$\cup \left(\left\{B : |B| = \frac{n+t-1}{2} + k\right\} \setminus \left\{B : |B| = \frac{n+t-1}{2} + k, [1,t] \in B\right\}\right).$$

It is not hard to show that $|\mathcal{B}(t,k)| \gg |\mathcal{A}(t,k)|$ for n sufficiently large (in terms of k and t). However, it may be checked by computer that $\mathcal{A}(t,k)$ is optimal for small values of n and specific choices of t and k, for example $t=2$ and $k=2$. It is conjectured that $\mathcal{B}(t,k)$ is the largest such family when n is sufficiently large.

Conjecture 7. *There exists a positive integer $n_0 = n_0(k,t)$ such that if $n+t$ is odd, $n > n_0$, and $\mathcal{F} \subseteq 2^{[n]}$ is a t-intersecting, k-Sperner family, then*

$$|\mathcal{F}| \leq |\mathcal{B}(t,k)| = \binom{n-t}{\frac{n-t-1}{2}} + \sum_{i=1}^{k} \binom{n}{\frac{n+t-1}{2}+i} - \binom{n-t}{\frac{n-t-1}{2}+k}.$$

0.2 Some Proofs

Push to the middle

Lemma 8. *For any $\mathcal{F} \subseteq 2^{[n]}$ t-intersecting k-Sperner family there exists another one $\mathcal{G}$ such that $|\mathcal{F}| \leq |\mathcal{G}|$ and $\mathcal{G}$ contains sets only of sizes from $\lceil\frac{n+t}{2}\rceil - k + 1, \lceil\frac{n+t}{2}\rceil + 2(k-1)$.*

Observe that Lemma 8 implies Theorem 0.1.

Proof. (Sketch) Consider the canonical decomposition of $\mathcal{F}$ into k antichains $\mathcal{F}^1, \mathcal{F}^2, \ldots, \mathcal{F}^k$.
Pushing $\mathcal{F}$ up:

- First we push up $\mathcal{F}^k$. We imitate Sperner's original proof and replace the subfamily of smallest sets $\mathcal{M} \subset \mathcal{F}^k$ with $\nabla(\mathcal{M})$ as long as $|\nabla(\mathcal{M})| \geq |\mathcal{M}|$. The latter is equivalent to $|\mathcal{M}^c| \leq \Delta(\mathcal{M}^c)$ where $\mathcal{M}^c := \{[n] \setminus M : M \in \mathcal{M}\}$. If $\mathcal{M}^c$ is *intersecting*, then the inequality follows from Katona's intersecting shadow theorem. Observe (prove) that as long as $\mathcal{M}$ consists of sets of size smaller than $\frac{n+t}{2}$, then $\mathcal{M}^c$ is intersecting. (Here one needs to use that $\mathcal{F}$ and thus $\mathcal{F}^k$ and all pushed up versions are t-intersecting). So after the last change $\mathcal{F}^k$ contains only sets of size at least $\frac{n+t}{2}$.

- We repeat the procedure for $\mathcal{F}^{k-1}$, but in order to keep $\mathcal{F}^k, \mathcal{F}^{k-1}$ disjoint, we stop at one level before, then we push up $\mathcal{F}^{k-2}, \mathcal{F}^{k-3}, \dots, \mathcal{F}^1$ and finally obtain that all sets have size at least $\frac{n+t}{2} - k + 1$.

Pushing $\mathcal{F}$ down:

- This time we start by pushing $\mathcal{F}^1$ down. As we stay above level $\lceil n/2 \rceil$, there is no problem with having $|\Delta(\mathcal{M})| \geq |\mathcal{M}|$ from Sperner's original proof where $\mathcal{M}$ this is the subfamily of maximum sized sets in $\mathcal{F}^1$. But we need to keep the $\mathcal{F}$ t-intersecting. As we have already pushed sets up, all sets have size at least $\frac{n+t}{2} - (k-1)$, so if we push down only till the level $\frac{n+t}{2} + (k-1)$, the intersection property will be satisfied.
- Repeat procedure for $\mathcal{F}^2, \mathcal{F}^3, \dots, \mathcal{F}^k$ stopping always one level before than the last time.

Reduction to cycle.

Lemma 9. *Suppose that for a family $\mathcal{G} \subseteq 2^{[n]}$ and **any** cyclic permutation σ, the family $\mathcal{G}_\sigma$ of sets in $\mathcal{G}$ that form intervals in σ satisfy*

$$w(\mathcal{G}_\sigma) := \sum_{G \in \mathcal{G}_\sigma} \binom{n}{|G|} \leq nB.$$

Then we have $|\mathcal{G}| \leq B$.

Proof. The assumption of the lemma implies

$$\sum_{\sigma} \sum_{G \in \mathcal{G}_\sigma} \binom{n}{|G|} \leq (n-1)! \cdot nB = n!B.$$

On the other hand, we have

$$\sum_{G \in \mathcal{G}} \sum_{\sigma : G \in \mathcal{G}_\sigma} \binom{n}{|G|} = \sum_{G \in \mathcal{G}} |G|! \cdot (n - |G|)! \binom{n}{|G|} = \sum_{G \in \mathcal{G}} n! = |\mathcal{G}| n!.$$

References

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