Extremal Combinatorics
Erdős matching conjecture

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The *matching number* $\nu(\mathcal{F})$ of the family $\mathcal{F}$ is the most number of pairwise disjoint sets that $\mathcal{F}$ contains.

Conjecture 1 (Erdős matching conjecture). *If $\mathcal{F} \subseteq \binom{[n]}{k}$ with $\nu(\mathcal{F}) \leq s$ and $n \geq (s+1)k$, then $|\mathcal{F}| \leq \max\{\binom{(s+1)k-1}{k}, \binom{n}{k} - \binom{n-s}{k}\}$.*

If true the conjecture is sharp as shown by $\binom{[(s+1)k-1]}{k}$ and $\mathcal{A}(n,s,k) = \{F \in \binom{[n]}{k} : F \cap [s] \neq \emptyset\}$.

Proposition 2. *If $\mathcal{F} \subseteq \binom{[(s+1)k]}{k}$ with $\nu(\mathcal{F}) \leq s$, then $|\mathcal{F}| \leq \binom{(s+1)k-1}{k}$.*

Proof. Let $\pi_{j,k}$ denote the number of partitions of $[jk]$ into k-sets, i.e. $\pi_{j,k} = \frac{1}{j!}\prod_{i=0}^{j-1}\binom{(j-i)k}{k}$. Let us count the pairs (F, π) with $F \in \mathcal{F} \cap \pi$, and π being an $(s+1)$-partition of $[(s+1)k]$ into k-sets. Clearly, any $F \in \mathcal{F}$ is contained in exactly $\pi_{s,k}$ partitions, and any π can contain at most s sets from $\mathcal{F}$ as otherwise, we would have $\nu(\mathcal{F}) = s+1$. We obtain

$$|\mathcal{F}|\pi_{s,k} \leq s\pi_{s+1,k}$$

and thus

$$|\mathcal{F}| \leq s\frac{\pi_{s+1,k}}{\pi_{s,k}} = \frac{s}{s+1}\binom{(s+1)k}{k} = \binom{(s+1)k-1}{k}.$$

Theorem 3 (Frankl [1]). *If $\mathcal{F} \subseteq \binom{[n]}{k}$ with $\nu(\mathcal{F}) \leq s$ and $n \geq (2s+1)k - s$, then $|\mathcal{F}| \leq \binom{n}{k} - \binom{n-s}{k}$.*

To prove Theorem 3, we will need the following result from last class.

Theorem 4. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a family with $\nu(\mathcal{F}) \leq s$. Then $|\Delta(\mathcal{F})| \geq \frac{1}{s}|\mathcal{F}|$ holds and equality holds if and only if $\mathcal{F} = \binom{X}{k}$ for some $((s+1)k-1)$-set X.*

The families $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{s+1}$ are called nested if $\mathcal{F}_{Fs+1} \subset \mathcal{F}_s \subset \cdots \subset \mathcal{F}_1$ holds. The families $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{s+1}$ are called cross-dependent if there is no choice of $F_i \in \mathcal{F}_i$ such that $F_1, \ldots, F_{s+1}$ are pairwise disjoint.

Theorem 5. *If $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{s+1} \subseteq \binom{Y}{\ell}$ are nested, cross-dependent families with $|Y| \geq t\ell$ and $t \geq 2s+1$, then*

$$|\mathcal{F}_1| + |\mathcal{F}_2| + \cdots + |\mathcal{F}_s| + (s+1)|\mathcal{F}_{s+1}| \leq s\binom{|Y|}{\ell}.$$

Proof. Let us choose uniformly at random t *pairwise disjoint* sets $B_1, B_2, \ldots, B_t \in \binom{Y}{\ell}$. We write $\mathcal{B} = \{B_1, B_2, \ldots, B_t\}$.

Lemma 6. *For every choice of $\mathcal{B}$, we have $|\mathcal{F}_1 \cap \mathcal{B}| + |\mathcal{F}_2 \cap \mathcal{B}| + \cdots + |\mathcal{F}_s \cap \mathcal{B}| + (s+1)|\mathcal{F}_{s+1} \cap \mathcal{B}| \leq st$.*

Proof. [Proof of Lemma] Let G be the bipartite graph with parts $\mathcal{B}$ and $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{s+1}\}$ such that $B_j\mathcal{F}_i$ is an edge if and only if $B_j \in \mathcal{F}_i$. As the $\mathcal{F}_i$s are cross-dependent, we have $\nu(G) \leq s$ and so by König's theorem, there exists a set T of s vertices meeting all edges of G. Let us write $x = |T \cap \mathcal{B}|$ and $s - x = |T \cap \{F_1, \ldots, \mathcal{F}_{s+1}\}|$. Let us bound the number of edges in G. As the degree of any $\mathcal{F}_i$ is at most t, we have at most $(s-x)t$ edges incident with $T \cap \{\mathcal{F}_1, \ldots, \mathcal{F}_{s+1}\}$. Any $B \in \mathcal{B} \cap T$ can be adjacent with at most $(s+1) - (s-x) = x+1$ $\mathcal{F}_i \notin T$. We obtain

$$|E(G)| \leq (s-x)t + x(x+1) = x^2 - (t-1)x + st. \tag{1}$$

As the $\mathcal{F}_i$s are nested, all $B_i \in \mathcal{F}_{s+1}$ have degree $s+1$. Therefore, all such B_i must lie in T (as otherwise to meet the edges adjacent to B_i, we would need the $s+1$ neighbors of B_i). This implies $s \geq x \geq |\mathcal{B} \cap \mathcal{F}_{s+1}| =: b$. As the right hand side of (1) is quadratic in x, it takes its maximum either at b or at s in the interval $b \leq x \leq s$. So

$$|E(G)| = \sum_{i=1}^{s+1} |\mathcal{B} \cap \mathcal{F}_i| \leq \max\{b^2 - (t-1)b + st, s^2 - (t-1)s + st\}.$$

To prove the statement of the lemma, we need to show that the right hand side is at most $st - sb$.

- $b^2 - (t-1)b + st \leq st - sb$ is equivalent to $b(b - (t-1) + s) \leq 0$ which holds by $b \leq s$ and the assumption $2s + 1 \leq t$.
- $s^2 - (t-1)s + st \leq st - sb$ is equivalent to $s(s - (t-1) + s) \leq 0$ which holds by the assumption $2s + 1 \leq t$.

Since the lemma holds for all choices of $\mathcal{B}$, it must hold for the expected values of the sizes. The probability $\mathbb{P}(B_i \in \mathcal{F}_j) = \frac{|\mathcal{F}_i|}{\binom{|Y|}{\ell}}$, so $\mathbb{E}(|\mathcal{B} \cap \mathcal{F}_i|) = t\frac{|\mathcal{F}_i|}{\binom{||Y|}{\ell}}$. Therefore, we have

$$t\sum_{i=1}^{s} \frac{|\mathcal{F}_i|}{\binom{|Y|}{\ell}} + t(s+1)\frac{|\mathcal{F}_{s+1}|}{\binom{|Y|}{\ell}} \leq ts.$$

Rearranging finishes the proof.

Proof. [Proof of Theorem 3] Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a family with $\nu(\mathcal{F}) \leq s$, $n \geq (2s+1)k - s$. We can assume $\mathcal{F}$ is left-shifted. We want to compare $|\mathcal{F}|$ to $|\mathcal{A}(n,k,s)|$. We partition both $\mathcal{F}$ and $\mathcal{A} = \mathcal{A}(n,k,s)$ according to how their sets intersect $[s+1]$: for $Q \subseteq [s+1]$, we write $\mathcal{F}(Q) = \{F \in \mathcal{F} : F \cap [s+1] = Q\}$ and $\mathcal{A}(Q) = \{A \in \mathcal{A} : A \cap [s+1] = Q\}$. For any such set $|Q| \geq 2$, by definition, we have $|\mathcal{A}(Q)| = \binom{n-(s+1)}{k-|Q|}$ and so $|\mathcal{F}(Q)| \leq |\mathcal{A}(Q)|$. As $\mathcal{A}(\emptyset) = \mathcal{A}(\{s+1\}) = \emptyset$, $\mathcal{A}(\{i\}) = \binom{n-s-1}{k-1}$ for all $1 \leq i \leq s$, we need to show

$$|\mathcal{F}(\emptyset)| + \sum_{i=1}^{s+1} |\mathcal{F}(\{i\})| \leq s\binom{n-s-1}{k-1} \tag{2}$$

First we claim that $|\mathcal{F}(\emptyset)| \leq s|\mathcal{F}(\{s+1\})|$. Indeed, as $\mathcal{F}$ is left-shifted, for any $H \in \Delta(\mathcal{F}(\emptyset))$, we have $H \cup \{s+1\} \in \mathcal{F}(\{s+1\}$. So, by Theorem 4, $|\mathcal{F}(\{s+1\})| \geq |\Delta(\mathcal{F}(\emptyset)| \geq \frac{1}{s}|\mathcal{F}(\emptyset)|$ as claimed. So the left hand side of (2) is upper bounded by

$$(s+1)|\mathcal{F}(\{s+1\})| + \sum_{i=1}^{s} |\mathcal{F}(\{i\})|.$$

But introducing $\mathcal{F}_i := \{F \setminus \{i\} : F \in \mathcal{F}(\{i\})\}$, we have $|\mathcal{F}_i| = |\mathcal{F}(\{i\})|$. As $\mathcal{F}$ is left-shifted, we obtain that the $\mathcal{F}_i$s are nested. Also, $\nu(\mathcal{F}) \leq s$ implies that the $\mathcal{F}_i$s are cross-dependent. So Theorem 5 can be applied with $Y = [s+2, n]$, $\ell = k-1$, $t = 2s+1$ and it yields exactly the inequality needed.

References

[1] Frankl, P. Improved bounds for Erdős' matching conjecture. *Journal of Combinatorial Theory, Series A 120*, 5 (2013), 1068–1072.