The number of (maximal) intersecting families

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Let us define

$$M(k) = \max\{|\mathcal{F}| : \mathcal{F} \text{ is } k\text{-uniform intersecting with } \tau(\mathcal{F}) = k\}$$

and

$$N(k) = \max\{|\bigcup_{F\in\mathcal{F}} F| : \mathcal{F} \text{ is } k\text{-uniform intersecting with } \tau(\mathcal{F}) = k\}.$$

Erdős and Lovász proved $M(k) \le k^k$ and gave an inductive construction of size $\lfloor k!(e-1)\rfloor$ based on the following lemma.

Lemma 1. *$M(k) \ge 1 + kM(k-1)$ holds for all $k \ge 1$.*

Proof. Let $\mathcal{F}$ be a $(k-1)$-uniform intersecting family with $\tau(\mathcal{F}) = k-1$ and $|\mathcal{F}| = M(k-1)$. If G is a k-set disjoint from $\cup_{F\in\mathcal{F}}F$, then $\mathcal{F}' = \{G\} \cup \{F \cup \{x\} : F \in \mathcal{F}, x \in G\}$ is a k-uniform intersecting family with $\tau(\mathcal{F}') = k$. Indeed, if T covers $\mathcal{F}'$, then either $G \subseteq T$ and thus $|T| \ge k$ or $G \cap T \ne \emptyset$ and $G \setminus T \ne \emptyset$. From the latter it follows that $T \cap (\cup_{F\in\mathcal{F}}F)$ covers $\mathcal{F}$ and therefore $|T| \ge |T \cap G| + |T \cap (\cup_{F\in\mathcal{F}}F)| \ge 1 + k - 1 = k$. ■

As $M(1) = 1$, Lemma 1 indeed yields $M(k) \ge \lfloor k!(e-1)\rfloor$. Lovász conjectured [4] that this bound is tight, but it was disproved by Frankl, Ota, and Tokushige [3]. Here we present a recent construction of Majumder [5] that is also larger than the one of Erdős and Lovász. It has the same asymptotic size as the construction by Frankl, Ota and Tokushige, but is much simpler than that.

Construction 2. *For an even number k and $0 \le i \le k-2$ let X_i be pairwise disjoint sets of size $1+k/2$. Let $y \notin \cup_{i=0}^{k-2}X_i$ and define $\mathcal{F}_{1,i} = \{X_i \cup \{x_{i+1}, x_{i+2}, \dots, x_{i+k/2-1}\} : x_{i+j} \in X_{i+j}\}$ where addition in the suffix from here on is modulo $k-1$. Let $\mathcal{F}_2 = \{y\} \times X_0 \times X_1 \times \cdots \times X_{k-2}$ and $\mathcal{F} = \cup_{i=0}^{k-2}\mathcal{F}_{1,i} \cup \mathcal{F}_2$.*

We show that $\mathcal{F}$ is a k-uniform intersecting family with $\tau(\mathcal{F}) = k$. Clearly any $F \in \mathcal{F}_2$ meets all $F' \in \mathcal{F}$. Also, if $F, F' \in \mathcal{F}_{1,i}$, then $X_i \subset F \cap F' \ne \emptyset$. Let $F \in \mathcal{F}_{1,i}$ and $F' \in \mathcal{F}_{1,j}$ with $0 \le i < j \le k-2$. Note that either $j \in \{i+1, i+2, \dots, i+k/2-1\}$ or

$i \in \{j+1, j+2, \ldots, j+k/2-1\}$ (modulo $k-1$), as the distance between i and j on the circle is at most $(k-1)/2$. In the former case we have $X_j \subset F'$ and F contains an element from X_j. Similarly in the latter case we have $X_i \subset F$ and F' contains an element from X_i. This proves that $\mathcal{F}$ is intersecting.

To prove $\tau(\mathcal{F}) = k$ let C be a cover of $\mathcal{F}$ of size at most k. We distinguish two cases.

Case I: $y \notin C$.

Then C must contain one of the X_i's to intersect all sets in $\mathcal{F}_2 \subseteq \mathcal{F}$. As the X_i's have size $1+k/2$ there exists a unique j such that $X_j \subset C$ holds. For any $i \neq j$ let $a_i \in X_i \setminus C$. For any $h = j+l$ with $1 \le l \le k/2-1$ we must have $C \cap X_h \neq \emptyset$ as otherwise C does not meet $X_h \cup \{a_{h+i} : 1 \le i \le k/2-1\} \in \mathcal{F}_1$. This shows that $|C| \ge k$; moreover $C \in \mathcal{F}_{1,j}$.

Case II: $y \in C$.

If C contains some X_i, then C is disjoint from at least $k/2$ many X_j's; in particular C is disjoint from X_{i+h} with some $h \le k/2-1$. But then as in the previous case we could find a set $F \in \mathcal{F}_{1,i+h}$ disjoint from C. Therefore for *every* i there exists $a_i \in X_i \setminus C_i$. This allows us to use the reasoning of the previous case to conclude that C must meet all X_i's and thus $C \in \mathcal{F}_2$, in particular $|C| \ge k$.

The size of $\mathcal{F}$ is $(k-1)(k/2+1)^{k/2-1} + (k/2+1)^{k-1} = (e^2 + o(1))(k/2)^{k-1}$, while the construction of Erdős and Lovász has size $\lfloor k!(e-1) \rfloor = O((k/e)^k)$. To obtain the same asymptotics for odd values of k, one may modify the construction by letting some of the X_i's have size $\frac{k+1}{2}$ and the others have size $\frac{k+1}{2}+1$, and adjusting the definition of sets in $\mathcal{F}_{1,i}$ so that they have size k. The proof that the resulting family is intersecting with covering number k is similar, but a bit more complicated. An easier way to get the same order of magnitude is to use the even case result and apply Lemma 1.

Let us continue with $N(k)$. The following example by Erdős and Lovász gives a lower bound.

Construction 3. *Let $|Y| = 2k-2$. For each 2-equipartition Y as $E \cup E' = Y$, $|E| = |E'| = k-1$, we take a new point $x_{E,E'}$, and set $E \cup \{x_{E,E'}\}$, $E' \cup \{x_{E,E'}\}$. This way we obtain $\binom{2k-2}{k-1}$ k-element sets, forming an intersecting family with covering number k, such that the union of these sets consists of $2k-2+\frac{1}{2}\binom{2k-2}{k-1}$ points.*

Let us recall the definition of an ISPS and introduce some related notions. A family of pairs $\{A_i, B_i\}_{i=1}^m$ forms an intersecting set pair system (ISPS) if

1. $A_i \cap B_i = \emptyset$ for all $1 \le i \le m$,
2. $A_i \cap B_j \neq \emptyset$ for all $1 \le i \neq j \le m$.

If $|A_i| \le a$ and $|B_i| \le b$ hold for all $1 \le i \le m$, then we call $\{A_i, B_i\}_{i=1}^m$ an (a,b)-ISPS. The Bollobás inequality states that if $\{A_i, B_i\}_{i=1}^m$ is an (a,b)-ISPS, then $m \le \binom{a+b}{a}$. Let us introduce the functions

$$n(a,b) = \max\{|\cup_{i=1}^m A_i \cup B_i| : \{(A_i,B_i)\}_{i=1}^m \text{ is } (a,b)\text{-ISPS}\}$$

and

$$n'(a,b) = \max\{|\cup_{i=1}^m A_i| : \{(A_i,B_i)\}_{i=1}^m \text{ is } (a,b)\text{-ISPS}\}.$$

By definition, we have $n'(a,b) \leq n(a,b) = n(b,a)$.

Theorem 4 (Tuza [7])**.** *If* $a \geq b$, *then* $\frac{1}{4}\binom{a+b-1}{a-1} < n'(a,b) \leq n(a,b) \leq \sum_{i=1}^{2b-2}\binom{i}{\lfloor i/2 \rfloor} + \sum_{i=2b-1}^{a+b-1}\binom{i}{b} < \binom{a+b-1}{a-1}$.

Proof. The lower bound is given by the following construction. Let $a' = \lfloor \frac{ab}{b+1} \rfloor < a$. Consider an $a'+b$-element set M and let

$$\{B_1, \ldots, B_{\binom{a'+b}{b}}\} = \binom{M}{b}.$$

Let us consider $\binom{a'+b}{b}$ pairwise disjoint sets C_i of size $a-a'$ $(1 \leq i \leq \binom{a'+b}{b})$ such that they are also disjoint from M. Let $A_i = (M \setminus B_i) \cup C_i$ for $1 \leq i \leq \binom{a'+b}{b}$. It is obvious that, for every i, A_i is disjoint from B_i, and intersects B_j $(j \neq i)$ inside M. A straightforward calculation gives the lower bound.

For the upper bound let $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ form an (a,b)-ISPS with $n(a,b) = |\bigcup_{i=1}^m (A_i \cup B_i)|$. For $j = 0, 1, \ldots, a+b-1$ we define an index set M_j and an ISPS $\mathcal{S}_j = \{(A_i^j, B_i^j) : i \in M_j\}$ in the following way: let $M_0 = [m]$ and $\mathcal{S}_0 = \{(A_i^0, B_i^0) : i \in M_0\}$ with $A_i^0 = A_i, B_i^0 = B_i$. If for $j \leq a+b-2$ the index set M_j and the ISPS $\mathcal{S}_j$ are defined, then let M_{j+1} be a minimal subset of M_j such that $\bigcup_{i \in M_{j+1}} (A_i \cup B_i) = \bigcup_{i \in M_j} (A_i \cup B_i)$ holds. The minimality of M_{j+1} implies, that for every $i \in M_{j+1}$, there exists an element $x_i \in A_i^j \cup B_i^j$ such that $x_i \notin A_l^j \cup B_l^j$ for any $l \in M_{j+1} \setminus \{i\}$. Therefore setting $A_i^{j+1} = A_i^j \setminus \{x_i\}$ and $B_i^{j+1} = B_i^j \setminus \{x_i\}$ we obtain that $\mathcal{S}_{j+1}$ is indeed an ISPS. Furthermore, we have $n(a,b) = |\bigcup_{i=1}^m (A_i \cup B_i)| = \sum_{j=0}^{a+b-1} |M_j|$.

For any $j = 0, 1 \ldots, a+b-1$ and $i \in M_j$ we have $|A_i^j \cup B_i^j| \leq a+b-j$ and therefore by the Bollobás inequality we obtain $|M_j| \leq \binom{a+b-j}{\lfloor \frac{a+b-j}{2} \rfloor}$ for arbitrary j and $|M_j| \leq \binom{a+b-j}{b}$ if $j \leq a-b$. This gives the upper bound on $n(a,b)$. The last inequality can be easily seen by induction. ■

Let $M(n,k)$ denote the number of maximal intersecting families in $\binom{[n]}{k}$. We will use intersecting set pair systems to give bounds on $M(n,k)$.

$$g(k) = \max\{|\cup_{i=1}^s A_i| : \{(A_i,B_i)\}_{i=1}^s \text{ is } (k,k)\text{-ISPS and } \{A_i\}_{i=1}^s \text{ is intersecting}\}.$$

Recall that the similar notion without the additional property that $\{A_i\}_{i=1}^s$ is intersecting was denoted by $n'(k,k)$. Thus by definition we have $g(k) \leq n'(k,k)$. We continue with a couple simple statements about $M(n,k)$ due to Balogh, Das, Delcourt, Liu and Sharifzadeh [1] and Nagy and Patkós [6].

Lemma 5. *[1] **(i)***

$$M(n,k) \le \sum_{i=1}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \le \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

*[6] **(ii)***

$$M(n,k) \le 2^{2^{g(k)}} \binom{n}{g(k)}.$$

Proof of Lemma. For a family $\mathcal{F} \subseteq \binom{[n]}{k}$ let us define $\mathcal{I}(\mathcal{F}) = \{G \in \binom{[n]}{k} : \forall F \in \mathcal{F}\ F \cap G \neq \emptyset\}$, the family of k-covers. A family is intersecting if and only if $\mathcal{F} \subseteq \mathcal{I}(\mathcal{F})$ and is maximal intersecting if and only if $\mathcal{I}(\mathcal{F}) = \mathcal{F}$. Let us consider a function f, that maps any maximal intersecting k-uniform family $\mathcal{F}$ to one of its subfamilies $\mathcal{F}_0$, that is minimal with respect to the property that $\mathcal{I}(\mathcal{F}_0) = \mathcal{F}$. Note that the function f is not unique. Clearly, f is injective, $\mathcal{F}_0$ is intersecting and by the minimality of $\mathcal{F}_0$, for every set $F_i \in \mathcal{F}_0$ there exists a set $G_i \in \mathcal{I}(\mathcal{F}_0 \setminus \{F\}) \setminus \mathcal{F}$. This means that the pairs of k-sets $\{(F_i, G_i)\}_{i=1}^{|\mathcal{F}_0|}$ form an ISPS. Thus, by definition, $|\cup_{F \in \mathcal{F}_0} F| \le g(k)$ holds. Therefore, the families, that can be the image of a maximal intersecting k-uniform family with respect to f, are subfamilies of 2^X for some $X \in \binom{[n]}{g(k)}$. The number of such families is not more than $2^{2^{g(k)}}\binom{n}{g(k)}$. This proves **(ii)**.

To see **(i)**, consider the pairs $\{(A_i, B_i)\}_{i=1}^{2|\mathcal{F}_0|}$ with $A_i = F_i$, $B_i = G_i$ for $i = 1, 2, \dots, |\mathcal{F}_0|$ and $A_i = G_{i-|\mathcal{F}_0|}$, $B_i = F_{i-|\mathcal{F}_0|}$ for $i = |\mathcal{F}_0|+1, |\mathcal{F}_0|+2, \dots, 2|\mathcal{F}_0|$. They form a skew ISPS; therefore by the Theorem of Frankl and Kalai, we obtain $|\mathcal{F}_0| \le \frac{1}{2}\binom{2k}{k}$. Therefore $M(n,k)$ is at most the number of all subfamilies of $\binom{[n]}{k}$ of size at most $\frac{1}{2}\binom{2k}{k}$. ■

Note that by Theorem 4 we have $g(k) \le n'(k,k) \le n(k,k) = O(\binom{2k}{k})$; therefore, if n is large enough, then **(ii)** gives a better bound than **(i)**, but for small values of n it is better to use **(i)**.

Theorem 6 (Nagy, Patkós, [6])**.** *For any fixed integer k, as n tends to infinity, the number $M(n,k)$ satisfies*

$$M(n,k) = n^{\Theta(\binom{2k}{k})}.$$

Moreover,

$$\frac{1}{8} \le \limsup_{n} \frac{\log M(n,k)}{\binom{2k}{k}\log n} \le 1.1 \quad \textit{and} \quad \limsup_{k} \limsup_{n} \frac{\log M(n,k)}{\binom{2k}{k}\log n} \le 1$$

holds.

Proof. The upper bound follows from Lemma 5 **(ii)** and Theorem 4. The lower bound follows from Construction 3 and the following proposition.

Proposition 7. *For any positive integers k and n we have $\binom{n}{N(k)} \le M(n,k)$.*

Proof of Proposition. Consider a k-uniform intersecting family $\mathcal{F}$ with $\tau(\mathcal{F}) = k$ and $|\cup_{F\in\mathcal{F}} F| = N(k)$. As adding more sets to $\mathcal{F}$ can only increase the size of the union, we may assume that $\mathcal{F}$ is maximal intersecting. Every set $X \in \binom{[n]}{N(k)}$ contains at least one family $\mathcal{F}_X$ isomorphic to $\mathcal{F}$. As $\mathcal{F}_X \neq \mathcal{F}_Y$ whenever $\cup_{F\in\mathcal{F}_X} F = X \neq Y = \cup_{F\in\mathcal{F}_Y} F$, we have at least $\binom{n}{N(k)}$ different maximal intersecting k-uniform subfamilies of $\binom{[n]}{k}$. $\square$

For the moreover part, observe that the above proposition implies the lower bound. Also, the upper bound in Theorem 4 for $a = b = k$ is $\sum_{i=1}^{k-1}\binom{i}{\lfloor i/2\rfloor}$. A simple induction gives the first upper bound, while a straightforward analysis of the sequence $\sum_{i=1}^{k-1}\binom{i}{\lfloor i/2\rfloor}/\binom{2k}{k}$ gives the second upper bound. $\blacksquare$

We continue with the number of intersecting families and show that if n is large enough, then almost all intersecting families are trivially intersecting. The number $T(n,k)$ of trivially intersecting families is obviously at least $n2^{\binom{n-1}{k-1}} - \binom{n}{2}2^{\binom{n-2}{k-2}} = (n-o(1))2^{\binom{n-1}{k-1}}$ for any n and $k = k(n)$, if $n > 2k$ and n tends to infinity.

Theorem 8 (Balogh, Das, Delcourt, Liu and Sharifzadeh [1])**.** *If n and $k = k(n)$ are integers, such that $n \geq 3k + 8\ln k$ holds, then the number of k-uniform intersecting families in $2^{[n]}$ is $(n + o(1))2^{\binom{n-1}{k-1}}$.*

Proof. The lower bound follows from the above calculations on $T(n,k)$.

By the Hilton-Milner Theorem, the size of a non-trivially intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most $1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$. Every non-trivially intersecting family is a subfamily of a maximal non-trivially intersecting family. Therefore the number $N(n,k)$ of non-trivially intersecting families is at most $2^{1+\binom{n-1}{k-1}-\binom{n-k-1}{k-1}}M(n,k)$.

$$\frac{N(n,k)}{T(n,k)} \leq M(n,k)\cdot 2^{1-\binom{n-k-1}{k-1}}$$

We have to show that $\log_2 M(n,k) - \binom{n-k-1}{k-1} \to -\infty$. By Lemma 5 **(i)**, we have $\log_2 M(n,k) \leq n\binom{2k-1}{k-1}$, so

$$\log_2 M(n,k) - \binom{n-k-1}{k-1} \leq \left(n - \left(\frac{n-k-1}{2k-1}\right)^{k-1}\right)\binom{2k-1}{k-1}$$

and therefore we need $(\frac{n-k-1}{2k-1})^{k-1} = \omega(n)$. As $n \geq 3k + 8\ln k$ holds, we have $n - k - 1 \geq n/2$ and thus $(\frac{n-k-1}{2k-1})^{k-1} \geq \frac{n^2}{4k^2}(\frac{n-k-1}{2k-1})^{k-3}$. Finally, as there exists a constant $c > 1/2$, such that $1 + x \geq \exp(cx)$, we obtain

$$\left(\frac{n-k-1}{2k-1}\right)^{k-3} \geq \left(1 + \frac{8\ln k}{2k-1}\right)^{k-3} \geq \exp\left(c\frac{8(k-3)\ln k}{2k-1}\right) \geq k^2$$

for large enough k. $\blacksquare$

Frankl and Kupavskii [2] improved the bound on n and showed that the statement of Theorem 8 holds if $n \geq 2k + 2 + 2\sqrt{k \log k}$.

References

[1] Balogh, J., Das, S., Delcourt, M., Liu, H., and Sharifzadeh, M. Intersecting families of discrete structures are typically trivial. *Journal of Combinatorial Theory, Series A 132* (2015), 224–245.

[2] Frankl, P., and Kupavskii, A. Counting Intersecting and Pairs of Cross-Intersecting Families. *Combinatorics, Probability and Computing 27*, 1 (2018), 60–68.

[3] Frankl, P., Ota, K., and Tokushige, N. Covers in uniform intersecting families and a counterexample to a conjecture of Lovász. *Journal of Combinatorial Theory, Series A 74*, 1 (1996), 33–42.

[4] Lovász, L. On minimax theorems of combinatorics (in Hungarian). *Matematikai Lapok 26* (1975), 209–264.

[5] Majumder, K. A new construction of non-extendable intersecting families of sets. *Electronic Journal of Combinatorics 23*, 3 (2016), P3–28.

[6] Nagy, Z. L., and Patkós, B. On the number of maximal intersecting k-uniform families and further applications of Tuza's set pair method. *Electronic Journal of Combinatorics 22*, 1 (2015), P1–83.

[7] Tuza, Z. Critical hypergraphs and intersecting set-pair systems. *Journal of Combinatorial Theory, Series B 39*, 2 (1985), 134–145.