Extremal Combinatorics
Katona's intersecting shadow theorem

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Theorem 1 (Katona). *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family. Then $|\Delta(\mathcal{F})| \geq |\mathcal{F}|$ holds and equality holds if and only if $\mathcal{F} = \binom{X}{k}$ for some $(2k-1)$-set X.*

The *matching number* $\nu(\mathcal{F})$ of the family $\mathcal{F}$ is the most number of pairwise disjoint sets that $\mathcal{F}$ contains.

Theorem 2 (Frankl [1]). *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a family with $\nu(\mathcal{F}) \leq s$. Then $|\Delta(\mathcal{F})| \geq \frac{1}{s}|\mathcal{F}|$ holds and equality holds if and only if $\mathcal{F} = \binom{X}{k}$ for some $((s+1)k-1)$-set X.*

Proof. Let $\mathcal{F}$ be a k-uniform family with $\nu(\mathcal{F}) \leq s$. By previous lemmas on shifting, we can assume that $\mathcal{F}$ is left-shifted, as shifting can only decrease the value of ν and the size of the shadow. Let us first prove the statement for all k and s with $(s+1)k-1 \geq n$. Let us construct a bipartite graph with partite sets $\mathcal{F}$ and $\Delta(\mathcal{F})$F where we put an edge connecting F and G if and only if G is a subset of F. It is immediate that each $F \in \mathcal{F}$ has degree k, and each $G \in \Delta(\mathcal{F})$ has degree at most $n - |G| = n-k+1$. Since $sk \geq n-k+1$ for $n \leq (s+1)k-1$, the bound of the theorem holds in the above range. Moreover, equality can hold only if $n = (s+1)k-1$ and each $G \in \Delta(F)$ has degree ks, so $G \cup \{y\} \in \mathcal{F}$ for $y \notin G \in \Delta(\mathcal{F})$. It follows that $G \setminus \{x\} \cup \{y\}$ also should be a member of $\Delta(\mathcal{F})$ (for $x \in G, y \notin G$) so $\Delta(\mathcal{F})$ is the complete $(k-1)$-uniform hypergraph on $[(s+1)k-1]$ and $\mathcal{F} = \binom{[(s+1)k-1]}{k}$ follows.

From now on, we suppose that $n \geq (s+1)k$, $k \geq 2$ and the statement of the theorem holds for $n-1$ for both k and $k-1$. Let us use the notation $\mathcal{F}(\overline{n}) := \{F \in F : n \notin F\}$, $\mathcal{F}(n) := \{F \setminus \{n\} : F \in F, n \in F\}$. These are the two families for which we want to use the induction hypothesis. Here $\nu(\mathcal{F}(\overline{n})) \leq s$ is obvious. The inequality $\nu((\mathcal{F}(n))) \leq s$ uses the left-shifted property of $\mathcal{F}$. If one has $s+1$ disjoint sets $F_i \setminus \{n\} \in \mathcal{F}(n)$ (where $F_i \in F, 1 \leq i \leq s+1$), then $n-1 \geq (s+1)(k-1)+s$ implies that there are elements $1 \leq x_1 < \cdots < x_s \leq n-1$ disjoint with each F_i. Then $\mathcal{F}$ being shifted implies that the sets $F_i \setminus \{n\} \cup \{x_i\} \in \mathcal{F}$ (here $1 \leq i \leq s$) together with F_{s+1} form a matching of size $s+1$ in $\mathcal{F}$, a contradiction.

Note that $\Delta(\mathcal{F}(\overline{n}))$ provides us with sets in $\Delta(\mathcal{F})$ which do not contain n. At the same time, adjoining n to any member of $\Delta(\mathcal{F}(n))$ provides us with a member of $\Delta(\mathcal{F})$ which contains n. This proves $|\Delta(\mathcal{F})| \geq |\Delta(\mathcal{F}(\overline{n}))| + |\Delta(\mathcal{F}(n))|$. Using the induction hypothesis yields

$$s|\Delta(\mathcal{F})| = s|\Delta(\mathcal{F}(\overline{n}))| + s|\Delta(\mathcal{F}(n))| \geq \mathcal{F}(\overline{n})| + |\mathcal{F}(n)| = |\mathcal{F}|$$

as desired.

References

[1] Frankl, P. Improved bounds for Erdős' matching conjecture. *Journal of Combinatorial Theory, Series A 120*, 5 (2013), 1068–1072.