Extremal Combinatorics
Hilton-Milner theorem, t-intersecting non-uniform families, Ahlswede-Khatchatrian theorem

Balázs Patkós

For $x, y \in [n]$, let us define

$$S_{x,y}(F) = \begin{cases} F \setminus \{y\} \cup \{x\} & \text{if } y \in F,\, x \notin F \text{ and } F \setminus \{y\} \cup \{x\} \notin \mathcal{F} \\ F & \text{otherwise} \end{cases} \tag{1}$$

and write $S_{x,y}(\mathcal{F}) = \{S_{x,y}(F) : F \in \mathcal{F}\}$.

Lemma 1. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family and $x, y \in [n]$. Then $S_{x,y}(\mathcal{F}) \subseteq \binom{[n]}{k}$ is intersecting with $|\mathcal{F}| = |S_{x,y}(\mathcal{F})|$.*

Proof of Lemma. The statements $S_{x,y}(\mathcal{F}) \subseteq \binom{[n]}{k}$ and $|\mathcal{F}| = |S_{x,y}(\mathcal{F})|$ are clear by definition. To prove the intersecting property of $S_{x,y}(\mathcal{F})$ let us call a set $G \in S_{x,y}(\mathcal{F})$ *new* if $G \notin \mathcal{F}$ and *old* if $G \in \mathcal{F}$. Two old sets intersect by the intersecting property of $\mathcal{F}$ and two new sets intersect as by definition they both contain x. Finally, suppose F is an old, and G is a new set of $S_{x,y}(\mathcal{F})$. By definition $x \in G$, so if $x \in F$, then F and G intersect. Suppose $x \notin F$ and consider $F' := G \setminus \{x\} \cup \{y\}$. As G is a new set of $S_{x,y}(\mathcal{F})$ we have $F' \in \mathcal{F}$ and therefore $F \cap F' \neq \emptyset$. If there exists $z \in F \cap F'$ with $z \neq y$, then $z \in F \cap G$ and we are done. If $F \cap F' = \{y\}$, then consider $F'' = F \setminus \{y\} \cup \{x\}$. As F is an old set of $S_{x,y}(\mathcal{F})$, we must have $F'' \in \mathcal{F}$, but then $F' \cap F'' = \emptyset$ contradicting the intersecting property of $\mathcal{F}$. Therefore $F \cap F' \neq \{y\}$ holds. This finishes the proof of the lemma. □

Let us define the weight of a family $\mathcal{G}$ to be $w(\mathcal{G}) = \sum_{G \in \mathcal{G}} \sum_{i \in G} i$. Observe that if $x < y$ and $S_{x,y}(\mathcal{F}) \neq \mathcal{F}$, then $w(S_{x,y}(\mathcal{F})) < w(\mathcal{F})$ holds. Therefore, as the weight is a non-negative integer, after applying a finite number of such shifting operations to $\mathcal{F}$, we obtain a family $\mathcal{F}'$ with the following property: $S_{x,y}(\mathcal{F}') = \mathcal{F}'$ for all $1 \leq x < y \leq n$. We call such a family *left-shifted*.

Lemma 2. *If $\mathcal{F} \subseteq \binom{[n]}{k}$ is a left-shifted intersecting family, then for any $F_1, F_2 \in \mathcal{F}$ we have $F_1 \cap F_2 \cap [2k-1] \neq \emptyset$.*

Proof of Lemma. Suppose not and let $F_1, F_2 \in \mathcal{F}$ be such that $F_1 \cap F_2 \cap [2k-1] = \emptyset$ holds and $|F_1 \cap F_2|$ is minimal. Let us choose $y \in F_1 \cap F_2$ and $x \in [2k-1] \setminus (F_1 \cup F_2)$. As $\mathcal{F}$ is intersecting, y exists. By the assumption $F_1 \cap F_2 \cap [2k-1] = \emptyset$, we have $y \geq 2k$, and thus $|F_1 \cap [2k-1]|, |F_2 \cap [2k-1]| \leq k-1$, therefore x exists. By definition, we have $x < y$. As $\mathcal{F}$ is left-shifted, we have $F_1' := F_1 \setminus \{y\} \cup \{x\} \in \mathcal{F}$, but then $F_1' \cap F_2 \cap [2k-1] = \emptyset$ and $|F_1' \cap F_2| < |F_1 \cap F_2|$, contradicting the choice of F_1 and F_2. □

A family $\mathcal{F}$ is *non-trivially intersecting* if it is intersecting and $\cap_{F \in \mathcal{F}} F = \emptyset$. The next theorem gives the maximum size of a non-trivially intersecting family. One could rephrase the result as that any intersecting family, whose size is bigger than the bound below, is a subfamily of an extremal family.

The following two families are the candidates to be the largest non-trivially intersecting families.

$$\mathcal{F}^3 = \left\{ F \in \binom{[n]}{k} : |F \cap [3]| \geq 2 \right\},$$

$$\mathcal{F}_{HM} = \{[2, k+1]\} \cup \left\{ F \in \binom{[n]}{k} : 1 \in F, |F \cap [2, k+1]| \geq 1 \right\}.$$

Theorem 0.1 (Hilton, Milner, [5]). *If $\mathcal{F} \subseteq \binom{[n]}{k}$ is a non-trivially intersecting family with $k \geq 2$ and $2k+1 \leq n$, then we have $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. Moreover, equality holds if and only if $\mathcal{F}$ is isomorphic to $\mathcal{F}_{HM}$ or if $k = 3$ and $\mathcal{F}$ is isomorphic to $\mathcal{F}^3$.*

The proof we present here is due to Frankl and Füredi [4]. It uses the shifting technique and is basically an extension of the first uniqueness proof of Theorem **??**. There we stop applying shifting operations when we arrive to a family of size smaller than $\binom{n-1}{k-1}$; here we further examine that family.

Proof. We proceed by induction on k. The base case $k = 2$ is easy as there is only one non-trivial intersecting family: the triangle $\{\{1,2\}, \{1,3\}, \{2,3\}\}$. To prove the inductive step, let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a non-trivial intersecting family. Our goal is to find a non-trivial intersecting family $\mathcal{H} \subseteq \binom{[n]}{k}$ with $|\mathcal{H}| = |\mathcal{F}|$ and a set X of size at most $2k$ such that for any $H, H' \in \mathcal{H}$ we have $H \cap H' \cap X \neq \emptyset$. Let us start applying shifting operations $S_{x,y}$ with $x < y$.

We consider two cases. If we obtain a left-shifted family $\mathcal{H}$ that is not a star, we can find X by Lemma 2. Otherwise at some point we obtain a family $\mathcal{G}$ and elements x_1, y_1 such that $S_{x_1,y_1}(\mathcal{G})$ is a star. Therefore $G \cap \{x_1, y_1\} \neq \emptyset$ for all $G \in \mathcal{G}$. Whenever this occurs, instead of applying S_{x_1,y_1} we let $X_1 = \{x_1, y_1\}$ and from that point we apply only the shifting operations $S_{x',y'}$ with $x' < y'$ and $x', y' \in [n] \setminus X_1$. We continue this procedure; we always have a set X_i, and we only apply shifting operations $S_{x',y'}$ with $x' < y'$ and $x', y' \in [n] \setminus X_i$. Whenever we arrive to a family $\mathcal{G}$ such that $S_{x_{i+1},y_{i+1}}(\mathcal{G})$ is a star for some $x_{i+1} < y_{i+1}$

and $x_{i+1}, y_{i+1} \in [n] \setminus X_i$, instead of applying $S_{x_{i+1},y_{i+1}}$ we let $X_{i+1} = X_i \cup \{x_{i+1}, y_{i+1}\}$. We continue this as long as we can change the family this way, i.e. until we obtain a family $\mathcal{H}$ with $S_{x',y'}(\mathcal{H}) = \mathcal{H}$ for all $x' < y'$ such that $x', y' \in [n] \setminus X_j$.

At this point $X_j = \{x_1, y_1, x_2, y_2, \ldots, x_j, y_j\}$. Clearly, we still have $H \cap \{x_i, y_i\} \neq \emptyset$ for every $H \in \mathcal{H}$ and every $i \leq j$. This implies $j \leq k$. Let X be the union of X_j and the first $2k-2j$ elements of $[n] \setminus X_j$. We shall show that for any $H, H' \in \mathcal{H}$ we have $H \cap H' \cap X \neq \emptyset$. If $j = k$, then for any $H \in \mathcal{H}$ we have $H \subseteq X_k = X$. So we can assume $j < k$; let us suppose towards a contradiction that H, H' are members of $\mathcal{H}$ with $H \cap H' \cap X = \emptyset$ and $|H \cap H'|$ minimal. This means that H and H' meet $\{x_i, y_i\}$ in different elements for every $i \leq j$, and as $\mathcal{H}$ is intersecting there exists $y' \in H \cap H' \setminus X$. Also there exists $x' \in (X \setminus X_j) \setminus (H \cup H')$ as $H \setminus X_j$ and $H' \setminus X_j$ are $(k-j)$-sets both containing $y' \notin X$. Then $H \setminus \{y'\} \cup \{x'\} \in \mathcal{H}$ by $S_{x',y'}(\mathcal{H}) = \mathcal{H}$ which contradicts the choice of H and H'.

We introduce the families $\mathcal{F}_i = \{F \in \mathcal{F} : |F \cap X| = i\}$ and $\mathcal{A}_i = \{F \cap X : F \in \mathcal{F}_i\}$ for $i = 1, 2, \ldots, k$. By the above, $\mathcal{A}_i$ is intersecting for all i.

Lemma 3. *For $i = 1, 2, \ldots, k-1$ we have $|\mathcal{A}_i| \leq \binom{2k-1}{i-1} - \binom{k-1}{i-1}$.*

Also $|\mathcal{A}_k| \leq \binom{2k-1}{k-1} - \binom{k-1}{k-1} + 1 = \binom{2k-1}{k-1} = \frac{1}{2}\binom{2k}{k}$ holds.

Proof of Lemma. The fact that $\mathcal{A}_1 = \emptyset$ follows from the non-triviality of $\mathcal{H}$. The statement on $\mathcal{A}_k$ follows from the fact that a k-subset A and its complement $[2k] \setminus A$ cannot belong to an intersecting family at the same time.

For $i = 2, 3, \ldots, k-1$ if the statement on $|\mathcal{A}_i|$ does not hold, then by the inductive hypothesis of the theorem we obtain that $\mathcal{A}_i$ is trivial, i.e. there exists $x \in X$ with $x \in A$ for all $A \in \mathcal{A}_i$. As $\mathcal{H}$ is non-trivial, there exists $H \in \mathcal{H}$ with $x \notin H$. As $H \cap A \neq \emptyset$, we obtain $|\mathcal{A}_i| \leq \binom{2k-1}{i-1} - \binom{k-1}{i-1}$. □

As any $A \in \mathcal{A}_i$ can be extended to a k-subset of $[n]$ in $\binom{n-2k}{k-i}$ ways, we have $|\mathcal{F}_i| \leq |\mathcal{A}_i|\binom{n-2k}{k-i}$. Therefore, by Lemma 3 we obtain

$$
\begin{aligned}
|\mathcal{F}| = \sum_{i=1}^{k} |\mathcal{F}_i| &\leq \sum_{i=1}^{k} |\mathcal{A}_i| \binom{n-2k}{k-i} \\
&\leq 1 + \sum_{i=1}^{k} \left(\binom{2k-1}{i-1} - \binom{k-1}{i-1} \right) \binom{n-2k}{k-i} \\
&= 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1},
\end{aligned}
$$

which proves the upper bound of the theorem.

If $|\mathcal{F}| = 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$, then for the family $\mathcal{H}$ obtained by shifting, we must have equality in Lemma 3 for all $i = 2, 3, \ldots, k$; in particular $|\mathcal{A}_2| = k$ must hold. $\mathcal{A}_2$ is either the triangle and thus $\mathcal{H} \subseteq \mathcal{F}^3$ and k should be 3, or $\mathcal{A}_2$ is a star with k elements and with center x. As $\mathcal{H}$ is non-trivial, there exists $H \in \mathcal{H}$ with $x \notin H$. But we know that H is a k-set intersecting every $A \in \mathcal{A}_2$, and there is only one such k-set, namely $H = \cup_{A\in\mathcal{A}_2} A \setminus \{x\}$. All other sets in $\mathcal{H}$ must meet H and contain x; therefore $\mathcal{H}$ is isomorphic to $\mathcal{F}_{HM}$.

All that remains to prove is that $\mathcal{F}^3$ and $\mathcal{F}_{HM}$ "cannot be created by shifting", i.e. if $\mathcal{F}$ is an intersecting family and $S_{x,y}(\mathcal{F})$ is isomorphic to $\mathcal{F}^3$ or $\mathcal{F}_{HM}$, then $\mathcal{F}$ is isomorphic to $\mathcal{F}^3$ or $\mathcal{F}_{HM}$, respectively. Suppose first that $k = 3$, $\mathcal{F}$ is not isomorphic to $\mathcal{F}^3$, but $S_{x,y}(\mathcal{F})$ is. We can assume without loss of generality that $S_{3,4}(\mathcal{F}) = \mathcal{F}^3$ holds. Observe first that each set in $\mathcal{F}$ must contain 1 or 2. Take $F \in \mathcal{F} \setminus \mathcal{F}^3$ and observe that as $S_{3,4}(F) \in \mathcal{F}^3$ we must have $3 \notin F$, $4 \in F$, and $|\{1,2\} \cap F| = 1$. Suppose without loss of generality $F = \{2,4,x\}$ for some $x > 4$, and let $\mathcal{G} = \{\{1,3,y\} : y \neq 1,2,3,4,x\}$. Then, as sets in $\mathcal{G}$ are disjoint from F, we have $\mathcal{G} \subseteq \mathcal{F}^3 \setminus \mathcal{F}$; thus $\mathcal{G}' = \{\{1,4,y\} : y \neq 1,2,3,4,x\} \subseteq \mathcal{F}$. Similarly every set of the form $\{2,3,z\}$ is in $\mathcal{F}^3 \setminus \mathcal{F}$; thus $\{2,4,z\} \in \mathcal{F}$ for every $z > 4$. As $\{1,4,z\}, \{2,4,z\} \in \mathcal{F}$ for any $z > 4$, $z \neq x$; therefore if a set in $\mathcal{F}$ does not contain 4, it has to contain both 1 and 2. As every set in $\mathcal{F}$ intersects $\{1,2\}$, this implies that each set in $\mathcal{F}$ intersects $\{1,2,4\}$ in at least two elements, hence $\mathcal{F}$ is a subfamily of a family isomorphic to $\mathcal{F}^3$.

Suppose next (without loss of generality) that $\mathcal{F} \neq \mathcal{F}_{HM}$ but $S_{x,y}(\mathcal{F}) = \mathcal{F}_{HM}$ holds for some $1 \leq x < y \leq n$. Note that $\mathcal{F}_{HM}$ is the only maximal intersecting family such that it contains exactly one set F that does not contain the element with maximum degree. Therefore, $\mathcal{F}$ contains at least two such sets F, F' and by this we know that $x = 1$ should hold. Thus we have $F = [2, k+1] \in \mathcal{F}_{HM} \cap \mathcal{F}$. Suppose that $F' \in \mathcal{F} \setminus \mathcal{F}_{HM}$. By the above, $1 \notin F', y \in F'$.

Let $z \in [2, k+1] \setminus F'$ and $\mathcal{G}_z = \{G \in \binom{[n]}{k} : 1, z \in G, G \cap F' = \emptyset\}$. By the intersecting property of $\mathcal{F}$ we have $\mathcal{G}_z \subset \mathcal{F}_{HM} \setminus \mathcal{F}$ and thus $\mathcal{G}'_z = \{G \setminus \{1\} \cup \{y\} : G \in \mathcal{G}_z\} \subseteq \mathcal{F}$. As for any k-set S with $1 \in S$, $y, z \notin S$, $S \cap [2, k+1] \neq \emptyset$ there exists $G \in \mathcal{G}'_z$ with $G \cap S = \emptyset$, we must have $S \setminus \{1\} \cup \{y\} \in \mathcal{F}$. Let us denote the family of these sets by $\mathcal{G}^+$. Finally, for any k-set T with $1, z \in T, y \notin T$ and $T \notin \mathcal{G}_z$ there exists $G \in \mathcal{G}^+$ with $T \cap G = \emptyset$. Therefore $T \setminus \{1\} \cup \{y\} \in \mathcal{F}$. Let us denote the family of these sets by $\mathcal{G}^{++}$. We obtained that $\mathcal{G}'_z \cup \mathcal{G}^+ \cup \mathcal{G}^{++} = \{G \in \binom{[n]}{k} : y \in G, G \cap [2, k+1] \neq \emptyset\} \subseteq \mathcal{F}$, i.e. $\mathcal{F}$ contains a family isomorphic to $\mathcal{F}_{HM}$. ■

Lemma 4. *If $\mathcal{F} \subseteq 2^{[n]}$ is t-intersecting, then so is $S_{x,y}(\mathcal{F})$ for any $x, y \in [n]$.*

Theorem 5 (Katona, [6])**.** *Let $\mathcal{F} \subset 2^{[n]}$ be a t-intersecting family with $1 \leq t \leq n$. Then*

$$|\mathcal{F}| \leq \begin{cases} \sum_{i=\frac{n+t}{2}}^{n} \binom{n}{i} & \text{if } n+t \text{ is even} \\ \binom{n-1}{\frac{n+t-1}{2}} + \sum_{i=\frac{n+t+1}{2}}^{n} \binom{n}{i} & \text{if } n+t \text{ is odd} \end{cases} \tag{2}$$

Proof. We proceed by induction on n the case $n = 1, 2$ being trivial. Let us assume that the statement of the theorem has already been proved for $n-1$ and arbitrary values of t. By

Lemma 4 we can restrict our attention to left-shifted families $\mathcal{F}$. Let $\mathcal{F}_0 = \{F \in \mathcal{F} : 1 \notin \mathcal{F}\}$ and $\mathcal{F}_1 = \{F \setminus \{1\} : 1 \in F \in \mathcal{F}\}$. Clearly, we have $|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1|$, $\mathcal{F}_0, \mathcal{F}_1 \subseteq 2^{[2,n]}$ and $\mathcal{F}_1$ is $(t-1)$-intersecting by definition.

Lemma 6. *The family $\mathcal{F}_0$ is $(t+1)$-intersecting.*

Proof of Lemma. As $\mathcal{F}_0 \subseteq \mathcal{F}$, we know that $\mathcal{F}_0$ is t-intersecting. So suppose that for some sets $F, F' \in \mathcal{F}_0$ we have $|F \cap F'| = t$ and let $x \in F \cap F'$. By the left-shiftedness of $\mathcal{F}$, the set $F'' := F' \setminus \{x\} \cup \{1\}$ belongs to $\mathcal{F}$. This contradicts the t-intersecting property of $\mathcal{F}$ as $|F \cap F''| = t - 1$. ■

We can apply the induction hypothesis to $\mathcal{F}_0$ and $\mathcal{F}_1$. We have to distinguish two cases depending on the parity of $n+t$. We consider here $n+t$ being even, the other case is similar. Note that the parity of $n+t$ is the same as that of $(n-1)+(t-1)$ and $(n-1)+(t+1)$. Therefore, we obtain

$$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| \leq \sum_{i=\frac{(n-1)+(t-1)}{2}}^{n-1} \binom{n-1}{i} + \sum_{i=\frac{(n-1)+(t+1)}{2}}^{n-1} \binom{n-1}{i} = \sum_{i=\frac{n+t}{2}}^{n} \binom{n}{i}.$$

□

Observe that the bound of Theorem 5 is sharp, as shown by the families $\cup_{i=\frac{n+t}{2}}^{n} \binom{[n]}{i}$ if $n+t$ is even and $\{G \cup \{n\} : G \in \binom{[n-1]}{\frac{n+t-1}{2}}\} \cup \bigcup_{i=\frac{n+t+1}{2}}^{n} \binom{[n]}{i}$ if $n+t$ is odd. The extremal family is not unique if $t = 1$, but it is unique if $t \geq 2$.

Let us briefly summarize what is known about uniform t-intersecting families. A family $\mathcal{F}$ is said to be *trivially t-intersecting* if there exists a t-set T such that $T \subseteq F$ for all $F \in \mathcal{F}$, and non-trivially t-intersecting if it is t-intersecting but $|\cap_{F \in \mathcal{F}} F| < t$. It was proved by Erdős, Ko, and Rado in their seminal paper [2] that for any $1 \leq t < k$, the largest t-intersecting families $\mathcal{F} \subseteq \binom{[n]}{k}$ are the maximal trivially t-intersecting ones if n is large enough. Therefore, $|\mathcal{F}| \leq \binom{n-t}{k-t}$ holds for large enough n. The threshold function $n_0(k,t)$ of this property was determined by Wilson [7] after Frankl [3] settled the cases $t \geq 15$. They proved $n_0 = (t+1)(k-t+1)$. Frankl in his paper [3] introduced the following t-intersecting families for $0 \leq i \leq k-t$:

$$\mathcal{A}(n,k,t,i) = \left\{ A \in \binom{[n]}{k} : |A \cap [2i+t]| \geq i+t \right\}.$$

Note that the family $\mathcal{A}(n,k,t,0)$ is maximal trivially t-intersecting and $|\mathcal{A}(n,k,t,0)| \geq |\mathcal{A}(n,k,t,1)|$ if and only if $n \geq (t+1)(k-t+1)$. Frankl conjectured that the largest possible size of a t-intersecting family is always attained at one of the $\mathcal{A}(n,k,t,i)$'s. This was proved almost twenty years later by Ahlswede and Khachatrian.

Theorem 7 (The Complete Intersection Theorem [1]). *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be t-intersecting with $t \geq 2$.*

(i) *If for some non-negative integer i we have $(k-t+1)(2+\frac{t-1}{i+1}) < n < (k-t+1)(2+\frac{t-1}{i})$, then $|\mathcal{F}| \leq |\mathcal{A}(n,k,t,i)|$ and equality holds if and only if $\mathcal{F}$ is isomorphic to $\mathcal{A}(n,k,t,i)$,*

(ii) *If for some non-negative integer i we have $(k-t+1)(2+\frac{t-1}{i+1}) = n$, then $|\mathcal{F}| \leq |\mathcal{A}(n,k,t,i)| = |\mathcal{A}(n,k,t,i+1)|$ and equality holds if and only if $\mathcal{F}$ is isomorphic to $\mathcal{A}(n,k,t,i)$ or $\mathcal{A}(,n,k,t,i+1)$.*

References

[1] Ahlswede, R., and Khachatrian, L. H. The complete intersection theorem for systems of finite sets. *European Journal of Combinatorics 18*, 2 (1997), 125–136.

[2] Erdős, P., Ko, C., and Rado, R. Intersection theorems for systems of finite sets. *The Quarterly Journal of Mathematics 12*, 1 (1961), 313–320.

[3] Frankl, P. The Erdős-Ko-Rado theorem is true for $n = ckt$. In *Combinatorics (Proceedings of the Fifth Hungarian Colloquium, Keszthely, 1976)* (1978), vol. 1, pp. 365–375.

[4] Frankl, P., and Füredi, Z. Non-trivial intersecting families. *Journal of Combinatorial Theory, Series A 41*, 1 (1986), 150–153.

[5] Hilton, A. J., and Milner, E. C. Some intersection theorems for systems of finite sets. *The Quarterly Journal of Mathematics 18*, 1 (1967), 369–384.

[6] Katona, G. O. H. Intersection theorems for systems of finite sets. *Acta Mathematica Hungarica 15*, 3-4 (1964), 329–337.

[7] Wilson, R. M. The exact bound in the Erdős-Ko-Rado theorem. *Combinatorica 4*, 2-3 (1984), 247–257.