Extremal Combinatorics
Bollobás inequality, Shadow theorem, Sauer Lemma

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An *intersecting set pair system (ISPS)* is a sequence $\{(A_i, B_i)\}_{i=1}^m$ of pairs with $A_i \cap B_i = \emptyset$ for all $i = 1, 2, \ldots, m$ and $A_i \cap B_j \neq \emptyset$ for all $1 \leq i, j \leq m$ with $i \neq j$.

Theorem 1 (Bollobás [?]). *For any ISPS, we have* $\sum_{i=1}^m \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1$.

Proof. Let us assume that $\bigcup_{i=1}^m (A_i \cup B_i) = [M]$. Then let us count the pairs (i, π) such that π is a permutation of $[M]$ and all elements of A_i preceed all elements of B_i in π. Observe that for any permutation π there can be at most one index i with (i, π) having this property. Indeed, if i, j were two such indices and the latest element of $A_i \cup B_i$ according to π belonged to A_i, then A_j cannot intersect B_i that comes later in π. So the number of such pairs is at most $M!$. On the other hand, for fixed i, the number of permutations with which i forms a pair is $\binom{M}{|A_i|+|B_i|}|A_i|! \cdot |B_i|! \cdot (M - |A_i| - |B_i|)!$ (one picks the places of $A_i \cup B_i$ in π and then fix the permutation on the parts A_i, B_i and $[M] \setminus (A_i \cup B_i)$. We obtain

$$\sum_{i=1}^m \binom{M}{|A_i| + |B_i|} |A_i|! \cdot |B_i|! \cdot (M - |A_i| - |B_i|)! \leq M!,$$

and dividing by $M!$ yield the result.

The *lexicographic ordering* of sets is defined by A being smaller than B if and only if $\min A \setminus B < \min B \setminus A$ holds (i.e. the first difference matters). However, the following theorem uses an ordering where the last different element matters. Let us define the *colex ordering* $\prec_k$ on $\binom{\mathbb{Z}^+}{k}$ by setting $A \prec_k B$ if and only if $\max A \setminus B < \max B \setminus A$ holds. The smallest element of this ordering is the set $[k]$ and for every $n \geq k$ the family $\binom{[n]}{k}$ is an initial segment of $(\prec_k, \binom{\mathbb{Z}^+}{k})$. For general m, we denote its initial segment of size m by $\mathcal{I}_m^k$.

Theorem 0.1 (Kruskal, Katona, [5, 3]). *Let $\mathcal{F}$ be a k-uniform family of size m. Then the inequality $|\Delta(\mathcal{F})| \geq |\Delta(\mathcal{I}_m^k)|$ holds.*

Theorem 0.2 (Lovász, [6]). *Let $\mathcal{F}$ be a k-uniform family with $|\mathcal{F}| = \binom{x}{k}$ for some real number $x \geq k$. Then $|\Delta(\mathcal{F})| \geq \binom{x}{k-1}$ holds. Moreover if $\Delta(\mathcal{F}) = \binom{x}{k-1}$, then x is an integer and $\mathcal{F} = \binom{X}{k}$ for a set X of size x.*

We present a proof by Keevash [4].

Proof. We start with stating an equivalent form of the theorem. Let $\mathcal{K}^r_{r+1}$ be the family consisting of all the $r+1$ r-sets on an underlying set of size $r+1$. For an r-uniform family $\mathcal{G}$ we denote by $\mathcal{K}^r_{r+1}(\mathcal{G})$ the set of copies of $\mathcal{K}^r_{r+1}$ in $\mathcal{G}$ and by $\mathcal{K}^r_{r+1}(v) = \mathcal{K}^r_{r+1}(\mathcal{G}, v)$ the set of copies of K^r_{r+1} in $\mathcal{G}$ containing v.

Lemma 2. *If $\mathcal{G}$ is a $(k-1)$-uniform family with $|\mathcal{G}| = \binom{x}{k-1}$, then $|\mathcal{K}^{k-1}_k(\mathcal{G})| \leq \binom{x}{k}$ and equality holds if and only if x is an integer and $\mathcal{G} = \binom{X}{k-1}$ for some set X of size x.*

Before proving the lemma we show that it is indeed equivalent to the theorem. We prove first that the theorem implies the lemma. If $\mathcal{G}$ satisfies the conditions of the lemma, then let us define $\mathcal{F} = \mathcal{K}^{k-1}_k(\mathcal{G})$. If $|\mathcal{F}| > \binom{x}{k}$ held, then by the theorem we would have $\binom{x}{k-1} < |\Delta(\mathcal{F})| \leq |\mathcal{G}|$, a contradiction. Assume now the lemma holds and let $\mathcal{F}$ satisfy the conditions of the theorem. Suppose $|\Delta(\mathcal{F})| = \binom{y}{k-1}$ with $y < x$ and apply the lemma to $\mathcal{G} = \Delta(\mathcal{F})$. We obtain $|\mathcal{F}| \leq |\mathcal{K}^{k-1}_k(\mathcal{G})| \leq \binom{y}{k} < \binom{x}{k}$, a contradiction.

The equivalence of the cases of equality follows similarly.

Proof of Lemma. We proceed by induction on k with the case $k-1=1$ being trivial. We will need the following definition. If $\mathcal{G}$ is a $(k-1)$-uniform family and v is an element of the underlying set, then the *link* $\mathcal{L}(v)$ is a $(k-2)$-uniform family with $S \in \mathcal{L}(v)$ if and only if $S \cup \{v\} \in \mathcal{G}$.

Claim 0.3. **(i)** $|\mathcal{K}^{k-1}_k(v)| \leq |\mathcal{G}| - d_{\mathcal{G}}(v)$,
(ii)$|\mathcal{K}^{k-1}_k(v)| \leq |\mathcal{K}^{k-2}_{k-1}(\mathcal{L}(v))|$,
(iii) $|\mathcal{K}^{k-1}_k(v)| \leq (\frac{x}{k-1}-1)d_{\mathcal{G}}(v)$ for every vertex v and equality holds only if $d_{\mathcal{G}}(v) = \binom{x-1}{k-2}$.

Proof of Claim. Observe that for a k-set S not containing v, $S \cup \{v\}$ spans a copy of $\mathcal{K}^{k-1}_k$ in $\mathcal{G}$ if and only if $S \in \mathcal{G}$ and S spans a copy of $\mathcal{K}^{k-2}_{k-1}$ in $\mathcal{L}(v)$. The first condition implies **(i)**, the second implies **(ii)**.

To see **(iii)**, suppose first that $d_{\mathcal{G}}(v) \geq \binom{x-1}{k-2}$. Then by **(i)**, we have $|\mathcal{K}^{k-1}_k(v)| \leq \binom{x}{k-1} - d_{\mathcal{G}}(v) \leq (\frac{x}{k-1}-1)d_{\mathcal{G}}(v)$. If $d_{\mathcal{G}}(v) \leq \binom{x-1}{k-2}$, then let $x_v \leq x$ be the real number with $d_{\mathcal{G}}(v) = \binom{x_v-1}{k-2}$. Using induction and **(ii)**, we obtain $|\mathcal{K}^{k-1}_k(v)| \leq |\mathcal{K}^{k-2}_{k-1}(\mathcal{L}(v))| \leq \binom{x_v-1}{k-1} = (\frac{x_v}{k-1} - 1)d_{\mathcal{G}}(v) \leq (\frac{x}{k-1}-1)d_{\mathcal{G}}(v)$. In both cases, $d_{\mathcal{G}}(v) = \binom{x-1}{k-2}$ is necessary for all inequalities to hold with equality. □

Using **(iii)** of Claim 0.3 we have

$$k|\mathcal{K}_k^{k-1}(\mathcal{G})| = \sum_v |\mathcal{K}_k^{k-1}(v)| \leq \sum_v (\frac{x}{k-1} - 1) d_{\mathcal{G}}(v)$$
$$= (\frac{x}{k-1} - 1)|\mathcal{G}|(k-1) = k\binom{x}{k}.$$

This finishes the proof of the inductive step for the inequality, and equality holds if all the degrees are equal to $\binom{x-1}{k-2}$. But then writing $n = |\cup_{G\in\mathcal{G}} G|$ we have $n\binom{x-1}{k-2} = \sum_v d_{\mathcal{G}}(v) = (k-1)|\mathcal{G}| = (k-1)\binom{x}{k-1} = x\binom{x-1}{k-2}$. Therefore x equals n and $\mathcal{G} = \binom{\cup_{G\in\mathcal{G}}G}{k-1}$. $\square$

The *trace* of a set F on another set X is $F \cap X$, and is denoted by $F|_X$. The trace of a family $\mathcal{F}$ of sets on X is $\mathcal{F}|_X = \{F|_X : F \in \mathcal{F}\}$. No matter how many sets F have the same trace, it is counted only once in $\mathcal{F}|_X$. We say that a family *traces* a set X (the terminology *shatters* is also used very often), if $\mathcal{F}|_X = 2^X$. The collection of sets that are traced by $\mathcal{F}$ is denoted by $tr(\mathcal{F})$. The *Vapnik-Chervonenkis dimension* (or VC-dimension for short) of $\mathcal{F}$ is the size of the largest set in $tr(\mathcal{F})$, and is denoted by $\dim_{VC}(\mathcal{F})$.

The fundamental result concerning traces of families was proven in the early 1970s independently by Sauer [7], Shelah [8], and Vapnik and Chervonenkis [9]. It is very often referred to as the Sauer Lemma.

Theorem 3 (Sauer, Shelah, Vapnik, Chervonenkis). *If $\mathcal{F} \subseteq 2^{[n]}$ has VC-dimension at most k, then $|\mathcal{F}| \leq \sum_{i=0}^k \binom{n}{i}$.*

The proof we present here was found by Alon [1] and Frankl [2] independently. It uses yet another version of shifting.

Proof. For a family $\mathcal{F} \subseteq 2^{[n]}$ and $i \in [n]$, let us define the *down-shifting* operation D_i by

$$D_i(F) = \begin{cases} F \setminus \{i\} & \text{if } i \in F \text{ and } F \setminus \{i\} \notin \mathcal{F} \\ F & \text{otherwise.} \end{cases} \tag{1}$$

Let us write $D_i(\mathcal{F}) = \{D_i(F) : F \in \mathcal{F}\}$. By definition, we have $|\mathcal{F}| = |D_i(\mathcal{F})|$ for any $\mathcal{F}$ and i. Furthermore, writing $w(\mathcal{G}) = \sum_{G\in\mathcal{G}} |G|$, we have $w(D_i(\mathcal{F})) \leq w(\mathcal{F})$, and if $D_i(\mathcal{F}) \neq \mathcal{F}$, then $w(D_i(\mathcal{F})) < w(\mathcal{F})$. Therefore, after a finite number of applications of some down-shifting operations, any family $\mathcal{F}$ can be transformed to a *down-shifted* family $\mathcal{F}^*$, i.e. one with the property that $\mathcal{F} = D_i(\mathcal{F})$ holds for any i.

Observe that a family is down-shifted if and only if it is a downset. Indeed, let us assume $X \subset Y$, $Y \in \mathcal{F}$ and $X \notin \mathcal{F}$. Then writing $Y \setminus X = \{y_1, y_2, \ldots, y_m\}$ and $X_0 = X$, $X_i = X \cup \{y_1, \ldots, y_i\}$, we obtain a pair X_i, X_{i+1} with $X_i \notin \mathcal{F}$, $X_{i+1} \in \mathcal{F}$. Then $X_i \in D_{y_{i+1}}(\mathcal{F})$ and $X_{i+1} \notin D_{y_{i+1}}(\mathcal{F})$, contradicting the downshifted property of $\mathcal{F}$.

The statement of the theorem is trivially true for downsets; therefore it is enough to show that $tr(D_i(\mathcal{F})) \subseteq tr(\mathcal{F})$ holds for any family $\mathcal{F}$ and $i \in [n]$. If $X \in tr(D_i(\mathcal{F}))$ and $i \notin X$, then for any subset Y of X, there exists a $G \in D_i(\mathcal{F})$ with $Y = G \cap X = (G \cup \{i\}) \cap X$. As at least one of G and $G \cup \{i\}$ belongs to $\mathcal{F}$, we have $X \in tr(\mathcal{F})$, as required. So let us suppose now $i \in X \in tr(D_i(\mathcal{F}))$, and consider subsets of X in pairs $(Y, Y \cup \{i\})$ with $i \notin Y$. As $X \in tr(D_i(\mathcal{F}))$, there exists a $G \in D_i(\mathcal{F})$ with $G \cap X = Y \cup \{i\}$. As D_i does not create sets containing i, we must have $G \in \mathcal{F}$, and the only reason for $G \in D_i(\mathcal{F})$ is $G \setminus \{i\} \in \mathcal{F}$. As obviously $(G \setminus \{i\}) \cap X = Y$, and the argument works for any pair $(Y, Y \cup \{i\})$, we obtain $X \in tr(\mathcal{F})$, as required. ■

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