Extremal Combinatorics

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The set $\{1, 2, \ldots, n\}$ of the first n postitive integers is denoted by $[n]$. For any set X, its power set $\{Y : Y \subseteq X\}$ is denoted by 2^X, and the set of all k-element subsets of X is denoted by $\binom{X}{k}$. Also, we will write $\binom{X}{\le k}$ for $\cup_{i=0}^{k}\binom{X}{i}$ and $\binom{X}{\ge \ell}$ for $\cup_{i=\ell}^{|X|}\binom{X}{i}$.

In a graph G, the neighborhood $N_G(v)$ of a vertex v in G is the set of vertices v is connected to. The degree $d_G(v)$ of v is the size of $N_G(v)$. We will omit the subscript G whenever it is clear from context.

The *shadow* of a k-uniform family $\mathcal{F} \subseteq \binom{X}{k}$ is $\Delta(\mathcal{F}) = \{G : |G| = k-1, \exists\, F \in \mathcal{F}\;\; G \subset F\}$, and its *up-shadow* or *shade* is $\nabla(\mathcal{F}) = \{G : |G| = k+1, \exists\, F \in F\;\; F \subset G \subseteq X\}$.

Theorem 0.1 (Sperner [5]). *If $\mathcal{F} \subseteq 2^{[n]}$ is an antichain, then we have $|\mathcal{F}| \le \binom{n}{\lfloor n/2 \rfloor}$. Moreover $|\mathcal{F}| = \binom{n}{\lfloor n/2 \rfloor}$ holds if and only if $\mathcal{F} = \binom{[n]}{\lfloor n/2 \rfloor}$ or $\mathcal{F} = \binom{[n]}{\lceil n/2 \rceil}$.*

First proof of Theorem 0.1. We start with the following simple lemma.

Lemma 1. *Let $G(A, B, E)$ be a connected bipartite graph such that for every vertex $a \in A$ and $b \in B$ the inequality $d_G(a) \ge d_G(b)$ holds. Then for any subset $A' \subseteq A$ the size of its neighborhood $N(A')$ is at least the size of A'. Moreover if $|N(A')| = |A'|$ holds, then $A' = A$, $N(A') = B$ and G is regular.*

Proof of Lemma. The number of edges between A' and $N(A')$ is exactly $\sum_{a \in A'} d_G(a)$ and is at most $\sum_{b \in N(A')} d_G(b)$. By the assumption on the degrees, the number of summands in the latter sum should be at least the number of summands in the former sum; thus we obtain $|N(A')| \ge |A'|$. If the number of summands is the same in both sums, then we must have $d_G(a) = d_G(b)$ for any $a \in A', b \in N(A')$ and all edges incident to $N(A')$ must be incident to A'. By the connectivity of G it follows that $A' = A$ and $N(A') = B$ hold. $\square$

Let $G_{n,k,k+1}$ be the bipartite graph with parts $\binom{[n]}{k}$ and $\binom{[n]}{k+1}$ in which two sets $S \in \binom{[n]}{k}$ and $T \in \binom{[n]}{k+1}$ are joined by an edge if and only if $S \subset T$. We want to apply Lemma 1 to $G_{n,k,k+1}$. It is easy to see that it is connected. The degree of a set $S \in \binom{[n]}{k}$ is $n - k$, while the degree of $T \in \binom{[n]}{k+1}$ is $k + 1$, therefore as long as $k \le \lfloor n/2 \rfloor$ holds, $\binom{[n]}{k}$ can play the role of A, and as soon as $k \ge \lceil n/2 \rceil$ holds, $\binom{[n]}{k+1}$ can play the role of A. Moreover, if $k < \lfloor n/2 \rfloor$

or $k \geq \lceil n/2 \rceil$, then $G_{n,k,k+1}$ is not regular. Note that if $\mathcal{F} \subseteq \binom{[n]}{k}$, then $N(\mathcal{F}) = \nabla(\mathcal{F})$, while if $\mathcal{F} \subseteq \binom{[n]}{k+1}$, then $N(\mathcal{F}) = \Delta(\mathcal{F})$.

To prove Theorem 0.1 let $\mathcal{F} \subseteq 2^{[n]}$ be an antichain. First we prove that if $\mathcal{F}$ is of maximum size, then it contains sets only of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. Suppose not and, say, there exists a set of size larger than $\lceil n/2 \rceil$. Then let m be the largest set size in $\mathcal{F}$ and let us consider the graph $G_{n,m-1,m}$. Applying Lemma 1 to $\mathcal{F}_m = \{F \in \mathcal{F} : |F| = m\}$ we obtain that $|\mathcal{F} \setminus \mathcal{F}_m \cup \Delta(\mathcal{F}_m)| > |\mathcal{F}|$ holds. To finish the proof we need to show that $\mathcal{F}' = \mathcal{F} \setminus \mathcal{F}_m \cup \Delta(\mathcal{F}_m)$ is an antichain. Sets of $\Delta(\mathcal{F}_m)$ have largest size in $\mathcal{F}'$, therefore they cannot be contained in other sets of $\mathcal{F}'$. No set $F' \in \Delta(\mathcal{F}_m)$ can contain any other set F from $\mathcal{F}' \cap \mathcal{F}$ as there exists $F'' \in \mathcal{F}_m$ with $F' \subset F''$, thus $F \subset F''$ would follow, and that contradicts the antichain property of $\mathcal{F}$.

We showed that $\mathcal{F} \subseteq \binom{[n]}{\lfloor n/2 \rfloor} \cup \binom{[n]}{\lceil n/2 \rceil}$, which proves the theorem if n is even. If n is odd, then suppose $\mathcal{F}$ contains sets of size both $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$. Applying the moreover part of Lemma 1 to $\mathcal{F}_{\lceil n/2 \rceil}$, we again obtain a larger antichain $\mathcal{F} \setminus \mathcal{F}_{\lceil n/2 \rceil} \cup \Delta(\mathcal{F}_{\lceil n/2 \rceil})$ unless $\mathcal{F}_{\lceil n/2 \rceil} = \binom{[n]}{\lceil n/2 \rceil}$. This shows that if $\mathcal{F}$ is a maximum size antichain that contains a set of size $\lceil n/2 \rceil$, then $\mathcal{F} = \binom{[n]}{\lceil n/2 \rceil}$. Similarly, if $\mathcal{F}$ is a maximum size antichain that contains a set of size $\lfloor n/2 \rfloor$, then $\mathcal{F} = \binom{[n]}{\lfloor n/2 \rfloor}$. ∎

We give a second proof of Theorem 0.1 based on the inequality proved independently by Lubell, Yamamoto, and Meshalkin. As Bollobás obtained [1] an even more general inequality, it is sometimes referred to as YBLM-inequality (Miklós Ybl was a famous Hungarian architect in the nineteenth century, who designed, among others, the State Opera House and St. Stephen's Basilica in Budapest), but we will use the more common acronym.

Lemma 2 (LYM-inequality, [3, 6, 4])**.** *If $\mathcal{F} \subseteq 2^{[n]}$ is an antichain, then the following inequality holds:*

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1.$$

Moreover, the above sum equals 1 if and only if $\mathcal{F}$ is a full level.

Before the proof let us introduce a further definition. A chain $\mathcal{C} \subseteq 2^{[n]}$ is called a *maximal chain* if it is of length $n+1$, i.e. it contains a set of size i for every $0 \leq i \leq n$.

Proof. Let $\mathcal{F} \subseteq 2^{[n]}$ be an antichain and let us consider the pairs $(F, \mathcal{C})$ such that $\mathcal{C}$ is a maximal chain in $[n]$ and $F \in \mathcal{F} \cap \mathcal{C}$. There are exactly $|F|!(n-|F|)!$ maximal chains containing F, therefore the number of such pairs is $\sum_{F \in \mathcal{F}} |F|!(n-|F|)!$. On the other hand, by the antichain property of $\mathcal{F}$, every maximal chain contains at most one set from $\mathcal{F}$ and thus the number of pairs is at most $n!$. We obtained

$$\sum_{F \in \mathcal{F}} |F|!(n-|F|)! \leq n! \tag{1}$$

and dividing by $n!$ yields the LYM-inequality.

Let us now prove the moreover part of the lemma. Clearly, if $\mathcal{F}$ is a full level $\binom{[n]}{k}$ for some k, then $\sum_{F\in\mathcal{F}}\frac{1}{\binom{n}{|F|}}=\binom{n}{k}\cdot\frac{1}{\binom{n}{k}}=1$ holds. If $\mathcal{F}$ is not a full level, then there exist two sets $F\in\mathcal{F}, G\notin\mathcal{F}$ with $|F|=|G|=|F\cap G|+1$. We construct maximal chains that do not contain any set from $\mathcal{F}$ and thus (1) cannot hold with equality. Consider any maximal chain $\mathcal{C}$ that contains $F\cap G$, G, and $F\cup G$. Any set C of $\mathcal{C}$ with $C\subseteq F\cap G\subset F$ cannot be in $\mathcal{F}$ by the antichain property of $\mathcal{F}$, G is not in $\mathcal{F}$ by definition, finally all sets $C\in\mathcal{C}$ with $C\supseteq F\cup G\supset F$ are not in $\mathcal{F}$ by the antichain property. By the choice of F and G there are no other sets in $\mathcal{C}$ and thus $\mathcal{F}\cap\mathcal{C}=\emptyset$ holds. ■

The function $\sum_{F\in\mathcal{F}}\frac{1}{\binom{n}{|F|}}$ in the LYM-inequality is called the *Lubell function* of $\mathcal{F}$ and it is often used. It is denoted by $\lambda(\mathcal{F},n)$ and we omit n if it is clear from the context. Having Lemma 2 in hand, the second proof of Theorem 0.1 is immediate.

Second proof of Theorem 0.1. Let $\mathcal{F}\subseteq 2^{[n]}$ be an antichain. Then we have $\sum_{F\in\mathcal{F}}\frac{1}{\binom{n}{\lfloor n/2\rfloor}}\le\sum_{F\in\mathcal{F}}\frac{1}{\binom{n}{|F|}}\le 1$ by Lemma 2. Therefore $|\mathcal{F}|$, that equals the number of summands, is at most $\binom{n}{\lfloor n/2\rfloor}$. The moreover part of the theorem follows from the moreover part of Lemma 2. ■

Let us illustrate the strength of the LYM-inequality by the following generalization of Theorem 0.1. We say that a family $\mathcal{F}$ of sets is *k-Sperner* if all chains in $\mathcal{F}$ have length at most k. We define $\Sigma(n,k)$ to be the sum of the k largest binomial coefficients of order n, i.e. $\Sigma(n,k)=\sum_{i=1}^{k}\binom{n}{\lfloor\frac{n-k}{2}\rfloor+i}$. Let $\Sigma^*(n,k)$ be the collection of families consisting of the corresponding full levels, i.e. if $n+k$ is odd, then $\Sigma^*(n,k)$ contains one family $\cup_{i=1}^{k}\binom{[n]}{\lfloor\frac{n-k}{2}\rfloor+i}$, while if $n+k$ is even, then $\Sigma^*(n,k)$ contains two families of the same size $\cup_{i=0}^{k-1}\binom{[n]}{\frac{n-k}{2}+i}$ and $\cup_{i=1}^{k}\binom{[n]}{\frac{n-k}{2}+i}$.

Theorem 0.2 (Erdős, [2]). *If $\mathcal{F}\subseteq 2^{[n]}$ is a k-Sperner family, then $|\mathcal{F}|\le\Sigma(n,k)$ holds. Moreover, if $|\mathcal{F}|=\Sigma(n,k)$, then $\mathcal{F}\in\Sigma^*(n,k)$.*

Proof. We start with the following simple observation.

Lemma 3. *A family $\mathcal{F}$ of sets is k-Sperner if and only if it is the union of k antichains.*

Proof of Lemma. If $\mathcal{F}$ is the union of k antichains, then any chain in $\mathcal{F}$ has length at most k as any chain can contain at most one set from each antichain. Conversely, if $\mathcal{F}$ is k-Sperner, we define the required k antichains recursively. Let $\mathcal{F}_1$ denote the family of all minimal sets in $\mathcal{F}$ and if $\mathcal{F}_j$ is defined for all $1\le j<i$, then let $\mathcal{F}_i$ denote the family of all minimal sets in $\mathcal{F}\setminus\cup_{j=1}^{i-1}\mathcal{F}_j$. The $\mathcal{F}_i$'s are antichains by definition and for every $F\in\mathcal{F}_i$,

there exists an $F' \in \mathcal{F}_{i-1}$ with $F' \subset F$. Hence, the existence of a set in $\mathcal{F}_{k+1}$ would imply the existence of a $(k+1)$-chain in $\mathcal{F}$. The partition we obtained is often referred to as the *canonical partition* of $\mathcal{F}$. $\square$

Let $\mathcal{F} \subseteq 2^{[n]}$ be a k-Sperner family. By Lemma 3, $\mathcal{F}$ is the union of k antichains $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k$. Adding up the LYM-inequalities for all $\mathcal{F}_i$'s we obtain

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq k.$$

This immediately yields $|\mathcal{F}| \leq \Sigma(n,k)$. If $|\mathcal{F}| = \Sigma(n,k)$, then all these LYM-inequalities must hold with equality. Therefore, by the moreover part of Lemma 2, all $\mathcal{F}_i$'s are full levels and thus $\mathcal{F} \in \Sigma^*(n,k)$. ■

Theorem 0.3. *For every $n \geq 1$ there exists a symmetric chain decomposition of $2^{[n]}$ into $\mathcal{C}_1, \mathcal{C}_2 \ldots, \mathcal{C}_m$ with every $\mathcal{C}_i$ being a chain $C^i_1 \subsetneq C^i_2 \subsetneq \cdots \subsetneq C^i_{\ell(i)}$ such that $|C^i_{j+1}| = |C^i_j| + 1$ for all i and j and $|C^i_1| + |C^i_{\ell(i)}| = n$ for all i.*

Proof. We prove the theorem by induction on n. For $n = 1$, one chain $\emptyset \subsetneq \{1\}$ suffices, and for $n = 2$ one can have $\{1\}$ and $\emptyset \subsetneq \{2\} \subsetneq \{1,2\}$.

If $\mathbf{C}_n$ is a symmetric chain decomposition of $2^{[n]}$, then for every $\mathcal{C} \in \mathbf{C}_n$ with $\mathcal{C} = \{C_k, C_{k+1}, \ldots, C_{n-k}\}$ we consider $\mathcal{C}' = \{C_k, C_{k+1}, \ldots, C_{n-k}, C_{n-k} \cup \{n+1\}\}$ and if $\mathcal{C}$ does not consist only of a single set, then we also consider $\mathcal{C}'' = \{C_k \cup \{n+1\}, C_{k+1} \cup \{n+1\}, \ldots, C_{n-k-1} \cup \{n+1\}\}$. It is not hard to check that $\mathbf{C}_{n+1} := \{\mathcal{C}', \mathcal{C}'' : \mathcal{C} \in \mathbf{C}_n\}$ is a symmetric chain decomposition of $2^{[n+1]}$. ■

Corollary 4. *Yet another proof of Theorem 0.1 and Theorem 0.2.*

Theorem 5 (Dilworth)**.** *For any finite poset P, the maximum size of a chain in P is the same as the minimum number of chains that cover C.*

Proof. Clearly max $\leq$ min as an antichain can contain at most 1 element from every every chain.

To see min $\leq$ max, we apply induction on $|P|$. The statement is trivial is $|P| = 1$. So suppose the theorem is proved for all posets of size smaller than p and let P be a poset of size p with m being the size of a maximum antichain. Let C be a maximal (unextendable) chain in P.

Case I $P \setminus C$ contains antichains only of size at most $m - 1$.

Then, by induction, $P \setminus C$ can be covered with at most $m-1$ chains, so P can be covered with those chains and C.

Case II $P \setminus C$ contains an antichain $A = \{a_1, a_2, \ldots, a_m\}$ of size m.

Then let us introduce $P^+ = \{p \in P : \exists j : a_j \leq p\}$ and $P^- = \{p \in P : \exists j : a_j \geq p\}$. As A is a maximum size antichain, so every $p \in P$ is comparable to at least one element of A and thus $P = P^- \cup P^+$. Also, $P^+ \cap P^- = A$ as if for some $p \in P \setminus A$ we have $p \in P^- \cap P^+$, then $a_i < p < a_j$ contradicting the fact that A is antichain. Finally, the top element of C must be in P^+ and the bottom element of C msut be in P^- as otherwise C would not be maximal.

Applying induction to P^+ and P^- (by the last sentence of the previous paragraph, they are both strictly smaller than P!), we obtain a cover $C_1^-, C_2^-, \ldots, C_m^-$ of P^- and a cover $C_1^+, C_2^+, \ldots, C_m^+$ of P^+. The elements of A are covered in both cases, so we can assume that a_i is the top element of C_i^- and the bottom element of C_i^+. So $\{C_i^- \cup C_i^+ : 1 \leq i \leq m\}$ is a cover of P. ■

References

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