How Many Unit Equilateral Triangles Can Be Induced by n Points in Convex Position?

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Abstract

Any set of n points in strictly convex position in the plane has at most $\lfloor \frac{2(n-1)}{3} \rfloor$ triples that induce equilateral triangles of side length *one*. This bound cannot be improved. The case of general triangles is also discussed.

1 Introduction

What is the maximum number of times that the unit distance can occur among n points in the plane? This more than fifty years old question of Paul Erdős, published in the *American Mathematical Monthly* [E46], opened a whole new area of research in combinatorial geometry [PA95].

An important variant of this problem, raised by Erdős and Leo Moser [EM59], is the following. At most how many times can the unit distance occur among the vertices of a convex n-gon, i.e., among n points in the plane in *strictly convex position*? Denote this maximum by $u^{\text{conv}}(n)$. Erdős and Moser noticed that $u^{\text{conv}}(n) \geq \lfloor 5(n-1)/3 \rfloor$, as is shown by a configuration containing $\lfloor (n-1)/3 \rfloor$ congruent copies of a rhombus with side length *one* and angle $2\pi/3$, rotated by small angles around one of its vertices belonging to such an angle (see Fig. 1). They also suggested that this bound may be tight.

Thirty years later, Herbert Edelsbrunner and Péter Hajnal [EH91] came up with a better construction showing that $u^{\text{conv}}(n) \geq 2n-7$. This is the best currently known lower bound. On the other hand, Füredi [F90] proved that $u^{\text{conv}}(n) = O(n \log n)$ (see [BP01] for a very simple inductional argument).

An equilateral triangle of side length *one* is called a *unit triangle*. The aim of this note is to show that if, instead of unit segments, we count the number of unit *triangles* determined

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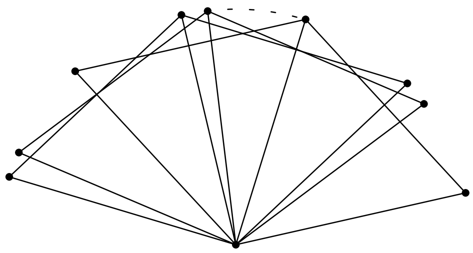

Figure 1: The Erdős-Moser configuration

by n points in strictly convex position in the plane, then the maximum is attained by the Erdős-Moser configuration depicted in Fig. 1. More precisely, we have

Theorem 1. *Any set of n points in strictly convex position in the plane has at most $\lfloor\frac{2(n-1)}{3}\rfloor$ triples which induce unit triangles. This bound is tight for all $n > 0$.*

If we do not require *strict* convexity, we obtain a somewhat different answer. A set of n points in the plane is said to be in *convex position*, if none of its elements is contained in the *interior* of a triangle induced by three others.

Theorem 2. *Any set of n points in convex position in the plane has at most $n-2$ triples which induce unit triangles. This bound is tight for all $n > 1$.*

The proofs of Theorems 1 and 2 are given in Sections 2 and 3, resp. In Section 4, we discuss the analogous questions for non-unit triangles. Our methods yield the following bounds.

Theorem 3. *Any set of n points in strictly convex position in the plane has at most $2n$ triples which induce congruent copies of a fixed triangle with a given orientation.*

The investigation of repeated triangles (or, more generally, simplices) in various point sets was initiated by Erdős and Purdy [EP71, EP76]. For some other results of this kind and their higher dimensional analogues, see Ábrego and Fernández-Merchant [AF00] and Agarwal-Sharir [AS01], resp.

2 Points in strictly convex position – Proof of Theorem 1

Throughout this section, let S be a fixed set of n points in the plane in strictly convex position. Let conv$S$ denote the convex hull of S. Connect two points $x, y \in S$ by a straight-line segment (or *edge*), if S induces a unit triangle, one of whose sides is xy. If xyz is a clockwise oriented unit triangle induced by S, then xy is said to be a *left edge with respect to x* and a *right edge with respect to y*. It is called a *rightmost left* edge with respect x, if there is no left edge that can be obtained from xy by a clockwise rotation around x with an angle smaller than π. Similarly, xy is called a *leftmost right* edge with respect to y, if there is no right edge that can be obtained from xy by a counter-clockwise rotation around y with an angle smaller than π. Obviously,

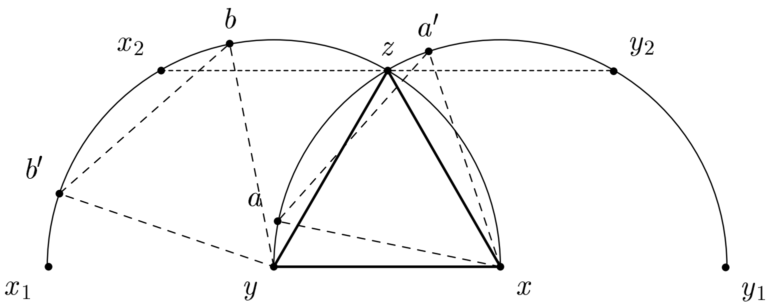

Figure 2: Illustration to the proof of Lemma 2.1

there is at most one rightmost left and at most one leftmost right edge with respect to each vertex.

We need the following observation.

Lemma 2.1. *Let xy be a left edge with respect to x (and hence a right edge with respect to y). Then xy is either the rightmost left edge with respect to x or the leftmost right edge with respect to y.*

Proof: Let xyz be a clockwise oriented unit triangle. Assume, in order to obtain a contradiction, that xa is the rightmost left edge with respect to x and yb is the leftmost right edge with respect to y, for some $a \neq y$ and $b \neq x$. Then, there exist $a', b' \in S$ such that xaa' and $yb'b$ are clockwise oriented unit triangles.

Let C_x and C_y denote the semi-circles obtained by intersecting the unit circles centered at x and y, resp., with the half-plane containing z bounded by the line xy. Let y_1 be the point of C_x opposite to y, and let x_1 be the point of C_y opposite to x. Finally, let y_2 denote the midpoint of the arc of C_x between y_1 and z, and let x_2 denote the midpoint of the arc of C_y between x_1 and z. (See Fig. 2.)

The point a cannot belong to the closed arc $\widehat{[y_1, y_2]}$ of C_x, because then we would have $x \in \mathrm{conv}\{y, a, a'\}$, contradicting our assumption that S is in strictly convex position. Thus, a must lie on the open arc $\widehat{(y, y_2)}$ of C_x. This yields that either a or a' must belong to the arc $\widehat{(z, y_2]} \subset C_x$. Similarly, either b or b' must belong to the arc $\widehat{(z, x_2]} \subset C_y$. So we have $z \in \mathrm{conv}\{x, y, a, a', b, b'\}$, the desired contradiction.

Notice that if we make the somewhat weaker assumption that the elements of S are in convex (but not necessarily strictly convex) position, i.e., no element of S lies in the *interior* of a triangle induced by three others, then there is one possibility: $z = a = b$. ▮

Assume without loss of generality that every element $x \in S$ belongs to at least one unit triangle induced by S. Otherwise, we can discard x and prove Theorem 1 by induction. Let P_1 (resp. P_2) denote the set of ordered pairs (x, e), where $x \in S$ and e is the rightmost left (resp. leftmost right) edge with respect to x.

According to our assumption, for each $x \in S$ there is precisely one rightmost left edge and precisely one leftmost right edge with respect to x. Therefore, we have $|P_1| + |P_2| = 2n$. On the other hand, Lemma 2.1 implies that each side of a unit triangle contributes at least one element to P_1 or P_2. Denoting the set of unit triangles by U, we obtain $|P_1| + |P_2| \geq 3|U|$, whence

$$|U| \leq \frac{|P_1| + |P_2|}{3} = \frac{2n}{3}.$$

This bound is only slightly weaker than the statement of Theorem 1.

To establish Theorem 1, it is sufficient to prove the following.

Lemma 2.2. *There exist at least two ordered pairs (x, y) such that $e = xy$ is a rightmost left edge with respect to x and a leftmost right edge with respect to y, i.e., $(x, e) \in P_1$, and $(y, e) \in P_2$.*

Proof: We may suppose again that every point of S belongs to at least one unit triangle. For any $x \in S$, let $R(x)$ denote the intersection of conv$S$ with the open half-plane to the right of the directed line $\overrightarrow{xy}$ supporting the rightmost left edge at x. Similarly, let $L(x)$ denote the intersection of conv$S$ with the open half-plane to the left of the directed line $\overrightarrow{xz}$ supporting the leftmost right edge at x. Let $\mathcal{R} = \{R(x) \mid x \in S\}$ and $\mathcal{L} = \{L(x) \mid x \in S\}$.

Notice that any *minimal* element of $\mathcal{R} \cup \mathcal{L}$ under containment belongs to $\mathcal{R} \cap \mathcal{L}$. Indeed, assume that, say, $R(x_0)$ is such a minimal element, and let x_0y_0 be the rightmost left edge with respect to x_0. Then x_0y_0 is a right edge with respect y_0. Moreover, it must be the *leftmost* right edge with respect to y_0, otherwise $L(y_0)$ would be a proper subset of $R(x_0)$, contradicting the minimality of $R(x_0)$. Thus, we have $L(y_0) = R(x_0)$, which yields that the pair (x_0, y_0) meets the requirements of the lemma.

Moreover, we can find another ordered pair (x_1, y_1) with this property, by choosing a minimal element among the sets in $\mathcal{R} \cap \mathcal{L}$ entirely contained in $L(x_0)$. It is not hard to verify that $(x_0, y_0) \neq (x_1, y_1)$. (However, it is possible that $x_1 = y_0$ and $y_1 = x_0$.) ∎

In view of Lemma 2.2, we now obtain that $2n = |P_1| + |P_2| \geq 3|U| + 2$, and Theorem 1 follows.

3 Relaxing the condition – Proof of Theorem 2

Here we consider the case when the points are in convex, but not necessarily strictly convex, position. Consider a set of five points, x, y, z, y', z', with the following property: xyz is a clockwise oriented, $xy'z$ and xyz' are counter-clockwise oriented unit triangles. We refer to such a set as a *special configuration.*

The proof is by induction. For $n = 3$, the assertion is trivial. Let S be a set of $n > 3$ points in convex position, and assume that Theorem 2 has already been proved for sets with fewer than n elements.

If S has no five points that form a special configuration, then, according to the remark at the end of the proof of Lemma 2.1, we can apply the argument in the last section to conclude that $|S| \leq \lfloor \frac{2(n-1)}{3} \rfloor \leq n - 2$.

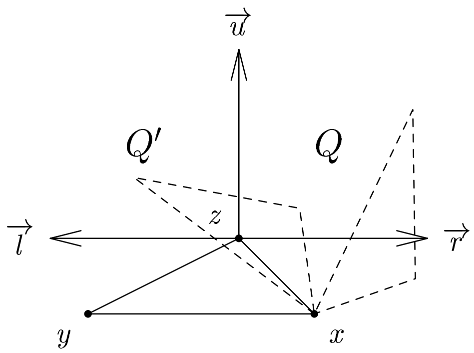

Figure 3: Definition of three auxiliary rays

Thus, we may suppose that S has five points, x, y, z, y', z', forming a special configuration. It is enough to show that S has a point incident to at most one unit triangle. Indeed, by removing such a point and applying the induction hypothesis to the remaining set, the result follows. If y' is incident to only one unit triangle, we are done. Suppose that there is another unit triangle, different from $xy'z$, which is incident to y'. It is easy to verify that the point x_1 obtained by reflecting x about $y'z$ must belong to S. In exactly the same way, we can argue that either x_1 is incident to only one unit triangle or the point z_2 obtained by the reflection of z about x_1y' also belongs to S. This procedure must end in finitely many steps, and produce a point incident to only one unit triangle.

4 General triangles – Proof of Theorem 3

We now extend the arguments in the previous sections to general triangles. Throughout this section, let $T_0 = x_0y_0z_0$ be a fixed clockwise oriented triangle such that x_0y_0 is one of its longest sides, and let S be a fixed set of n points in convex position in the plane. Consider a triangle $T = xyz$ congruent to T_0, whose vertices belong to S and correspond to x_0, y_0, and z_0, resp. Just like before, xy is said to be a left edge with respect to x and a right edge with respect to y. We say that xy is a rightmost left edge with respect to x (resp. a leftmost right edge with respect to y), if there is no triangle T' congruent to T, induced by S, that can be obtained from T by a clockwise rotation around x (resp. by a counter-clockwise rotation around y) with an angle smaller than π.

Lemma 2.1 generalizes as follows.

Lemma 4.1. *Let $T = xyz$ be a clockwise oriented triangle congruent to T_0, which is induced by a set of points in strictly convex position, and let xy be a* longest *edge of T. Then xy is either a rightmost left edge with respect to x or a leftmost right edge with respect to y.*

To prove this, we need a little preparation. Suppose without loss of generality that xy induces a horizontal line and x is to the right of y (see Fig. 3). Denote by $\overrightarrow{u}$, $\overrightarrow{l}$, and $\overrightarrow{r}$ the rays emanating from z, pointing *upwards, to the left* and *to the right*, resp. Let Q be the convex cone (quadrant) bounded by $\overrightarrow{u}$ and $\overrightarrow{r}$, and let Q' be the convex cone bounded by $\overrightarrow{u}$ and $\overrightarrow{l}$.

Claim 4.2. *Assume that xy is not a rightmost left edge with respect to x, i.e., there is a triangle $T' = xy'z'$ that can be obtained from T by a clockwise rotation around x with an angle $0 < \varphi < \pi$. Then at least one vertex $u \in T'$ must belong to the set $Q \setminus \{z\}$.*

Proof: Let α, β, and γ denote the angles of T at x, y, and z, resp. By the maximality of xy, we have that $\gamma \geq \alpha, \beta$. If $0 < \varphi \leq \pi - 2\alpha$, then we have $z' \in Q \setminus \{z\}$ and we are done. If $\pi - 2\gamma < \varphi < \alpha$, then z is in the interior of $\operatorname{conv} T'$, contradicting our assumption that the points are in convex position. Hence, using the fact that $\pi - 2\gamma \leq \pi - 2\alpha$, we can suppose that $\varphi \geq \alpha$. In fact, we can also assume that $\varphi > \alpha$. Indeed, $\varphi = \alpha$ implies $y' = z$, $\beta = \gamma$, whence $\varphi = \alpha = \pi - 2\gamma \leq \pi - 2\alpha$, and this case has been handled before.

If $\alpha < \varphi < \pi - 2\beta$, then z is in the interior of the convex hull of x, y, and y', contradiction. If $\pi - 2\beta \leq \varphi \leq \pi - \alpha$, we have $y' \in D \setminus \{z\}$. To see this, it is enough to note that the height of T belonging to xz is at least as large as the height belonging to the longest side, xy.

Finally, if $\pi - \alpha < \varphi < \pi$, then x must belong to $\operatorname{conv}\{y, y', z'\}$, again a contradiction. ∎

By symmetry, we obtain that if xy is not a leftmost right edge with respect to y, then at least one element $u' \in S$ must belong to the set $Q' \setminus \{z\}$. Therefore, if xy is neither a rightmost left edge with respect to x, nor a leftmost right one with respect to y, then we have $z \in \operatorname{conv}\{x, y, u, u'\}$, which completes the proof of Lemma 4.1.

Now we are in a position to establish Theorem 3.

Count the number of pairs (e, x), where $x \in S$ and e is either a rightmost left edge or a leftmost right edge with respect to x, corresponding to the edge $e_0 = x_0y_0$ of T_0. According to Lemma 4.1, the number of these pairs is at least the number of congruent copies of T_0 induced by S, which have the same orientation. On the other hand, every $x \in G$ belongs to at most 2 such pairs, and the theorem follows. ∎

Remark 4.3. A more careful analysis of the extreme cases shows that, if T_0 is not an isosceles triangle, Lemma 4.1 and Theorem 3 remain true under the weaker assumption that the points of S are in convex (but not necessarily in *strictly* convex) position.

Remark 4.4. In some special cases it is not hard to improve Theorem 3. Suppose, for example, that T_0 is *not* an *acute* triangle , i.e., using the same notation as in the proof of Claim 4.2, we have $\gamma \geq \pi/2$. Assume further that $\alpha \geq 2\beta$. Then, under the assumptions in Lemma 4.1, we can argue that xy is necessarily the rightmost left edge with respect to x. Now every congruent copy $T = xyz$ of T_0 with a given orientation gives rise to a *unique* rightmost left edge at x. On the other hand, there is at most one such rightmost left edge incident to each vertex $v \in S$. Thus, we obtain

Theorem 4. *Assume T_0 is a non-acute triangle, one of whose acute angles is at least twice as large as the other. Then any set of n points in convex position in the plane has at most n copies of T_0 with a given orientation.*

To see that this bound can be attained, let $v_1, \ldots, v_n$ denote the vertices of a regular n-gon ($n \geq 6$) listed in clockwise order, and set $T_0 = x_0y_0z_0$, where $x_0 = v_{k+1}, y_0 = v_1$, and $z_0 = v_k$ for some $3 \leq k \leq n/2$.

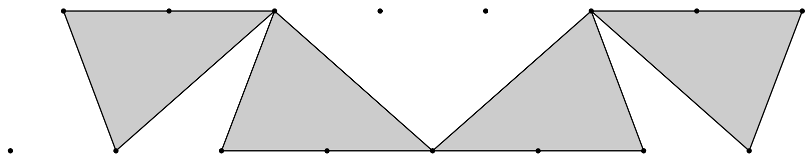

Figure 4: $n = 16$ points with $2n - 8$ congruent triangles

We conjecture that the upper bound $2n$ in Theorem 3 can be replaced by n, without making any assumption on the angles of the triangle. Moreover, for non-isosceles triangles, this conjecture may remain true for point sets in convex (but not necessarily strictly convex) position (cp. Remark 4.3). However, it is not hard to construct a set of n points lying on two parallel lines, which has $2n - 8$ copies of a suitable isosceles triangle.

References

[AF00] B. M. Ábrego and S. Fernández-Merchant, On the maximum number of equilateral triangles I, *Discrete and Computational Geometry* **23** (2000), 129–135.

[AS01] P. K. Agarwal and M. Sharir, On the number of congruent simplices in a point set, in: *Proc. ACM Symposium on Computational Geometry,* ACM Press, 2001, ???

[BP01] P. Brass and J. Pach, The maximum number of times the same distance can occur among the vertices of a convex n-gon is $O(n \log n)$, *J. Combin. Theory Ser. A* **94** (2001), 178–179.

[EH91] H. Edelsbrunner and P. Hajnal, A lower bound on the number of unit distances between the points of a convex polygon, *J. Combin. Theory Ser. A* **56** (1991), 312–316.

[E46] P. Erdős, On sets of distances of n points, *Amer. Mathematical Monthly* **53** (1946), 248–250.

[EM59] P. Erdős and L. Moser, Problem 11, *Canadian Mathematical Bulletin* **2** (1959), 43.

[EP71] P. Erdős and G. Purdy, Some extremal problems in geometry, *J. Combin. Theory Ser. A* **10** (1971), 246–252.

[EP76] P. Erdős and G. Purdy, Some extremal problems in geometry IV, *Congressus Numerantium* **17** (Proc. 7th South-Eastern Conf. Combinatorics, Graph Theory, and Computing, 1976), 307–322.

[F90] Z. Füredi, The maximum number of unit distances in a convex n-gon, *J. Combin. Theory Ser. A* **55** (1990), 316–320.

[PA95] J. Pach and P.K. Agarwal, *Combinatorial Geometry,* J. Wiley & Sons, New York.