A remark on transversal numbers

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"What does the Hungarian parrot say?"
"Log. Log log log log ... "
(Riddle. Folklore.)

1. Introduction

In his classical monograph published in 1935, Dénes König [K] included one of Paul Erdős's first remarkable results: an infinite version of the Menger theorem. This result (as well as the König–Hall theorem for bipartite graphs, and many related results covered in the book) can be reformulated as a statement about transversals of certain hypergraphs.

Let H be a hypergraph with vertex set $V(H)$ and edge set $E(H)$. A subset $T \subseteq V(H)$ is called a *transversal* of H if it meets every edge $E \in E(H)$. The *transversal number* $\tau(H)$ is defined as the minimum cardinality of a transversal of H. Clearly, $\tau(H) \geq \nu(H)$, where $\nu(H)$ denotes the maximum number of pairwise disjoint edges of H. In the above mentioned examples, $\tau(H) = \nu(H)$ holds for the corresponding hypergraphs. However, in general it is impossible to bound τ from above by any function of ν, without putting some restriction on the structure of H.

One of Erdős's closest friends and collaborators, Tibor Gallai (who is also quoted in König's book) once said: "I don't care for bounds involving log n's and loglog n's. I like exact answers. But Paul has always been most interested in asymptotic results." In fact, this quality of Erdős has contributed a great deal to the discovery and to the development of the "probabilistic method" (see [ES], [AS]).

The search for "exact answers" (e.g. to the perfect graph conjecture of Berge [Be]) has revealed some important connections between transversal problems and linear programming that led to the deeper understanding of the König–Hall–Menger–type theorems. It proved to be useful to introduce another parameter, the *fractional transversal number* of a hypergraph, defined by

$$\tau^\star(H) = \min_t \sum_{x \in V(H)} t(x) \quad ,$$

where the minimum is taken over all non–negative functions $t : V(H) \longrightarrow \boldsymbol{R}$ with the property that

$$\sum_{x \in E} t(x) \geq 1 \qquad \text{for every } E \in E(H).$$

Obviously, $\tau(H) \geq \tau^\star(H) \geq \nu(H)$, and $\tau^\star(H)$ can be easily calculated by linear programming. (See [Lo] and [S].)

At the same time, the probabilistic (or shall we say, asymptotic) approach has also led to many exciting discoveries about extremal problems related to transversals (e.g. Ramsey–Turán–type theorems, property B). It was pointed out by Vapnik and Chervonenkis [VC] that in some important families of hypergraphs a relatively small set of randomly selected vertices will, with high probability, be a transversal. They defined the *dimension* of a hypergraph as the size of the largest subset $A \subseteq V(H)$ with the property that for every $B \subseteq A$ there exists an edge $E \in E(H)$ such that $E \cap A = B$. Adapting the original ideas of [VC] and [HW], it was shown in [KPW] (see also [PA]) that

$$\tau(H) \leq (1 + o(1)) \dim(H) \tau^\star(H) \log \tau^\star(H), \tag{1}$$

as $\tau^\star \longrightarrow \infty$, and that this bound is almost tight.

Ding, Seymour and Winkler [DSW] have introduced another parameter of a hypergraph, closely related to its dimension. They defined $\lambda(H)$ as the size of the largest collection of edges $\{E_1, \ldots, E_k\} \subseteq E(H)$ with the property that for every pair (E_i, E_j), $1 \leq i \neq j \leq k$, there exists a vertex x such that $x \in E_i \cap E_j$ but $x \notin E_h$ for any $h \neq i, j$. Combining (1) with Ramsey's theorem, they showed that

$$\tau(H) \leq \; 6\lambda^2(H) \; (\lambda(H) + \nu(H)) \binom{\lambda(H) + \nu(H)}{\lambda(H)}^2 \tag{2}$$

holds for every hypergraph H.

As far as we know, Haussner and Welzl [HW] were the first to recognize that (1) has a wide range of interesting geometric applications, due to the fact that a large variety of hypergraphs defined by geometric means have low Vapnik–Chervonenkis dimensions.

The aim of this note is to illustrate the power of this approach by two examples. In Section 2 we show that (2) easily implies some far–reaching generalizations of results of Erdős and Szekeres [ES1] [ES2]. In Section 3, we use (2) to extend and to give alternative proofs for some old results of Gyárfás and Lehel (see [G], [GL], [L]) bounding the transversal numbers of box hypergraphs.

2. Covering with boxes

Given two points $p, q, \in \boldsymbol{R}^d$, let $\mathrm{Box}[p, q]$ be defined as the smallest box containing p and q, whose edges are parallel to the axes of the coordinate system. The following theorem settles a conjecture of Bárány and Lehel [BL], who established the first non–trivial result of this kind.

Theorem 2.1. *Any finite (or compact) set $P \subseteq \boldsymbol{R}^d$ contains a subset with at most $2^{2^{d+2}}$ elements, $\{p_i | 1 \le i \le 2^{2^{d+2}}\}$, such that*

$$P \subseteq \bigcup_{i,j=1}^{2^{2^{d+2}}} \mathrm{Box}\ [p_i,\ p_j]\,.$$

Proof: Let H be a hypergraph on the vertex set

$$V(H) = \{\mathrm{Box}[p, q] \mid p, q \in P\}\ ,$$

defined as follows. Associate with each point $r \in P$ the set

$$E_r = \{\mathrm{Box}[p, q] | r \in \mathrm{Box}[p, q]\},$$

and let $E(H) = \{E_r | r \in P\}$.

Clearly, $E_p \cap E_q \neq \emptyset$ for any $p, q, \in P$, because $\mathrm{Box}[p,\ q] \in E_p \cap E_q$. Hence, $\nu(H) = 1$.

According to a well–known lemma of Erdős and Szekeres [ES1], any sequence of k^2+1 real numbers contains a monotone subsequence of length $k+1$. By repeated application of this statement, we obtain that any set of $2^{2^{d-1}}+1$ points in $\boldsymbol{R}^d$ has three elements $p_i,\ p_j,\ p_k$ with $p_k \in \mathrm{Box}[p_i,\ p_j]$. This immediately implies that

$$\lambda(H) \le 2^{2^{d-1}}\ .$$

Indeed, for any family of more than $2^{2^{d-1}}$ edges $E_{p_1}, \ldots E_{p_\lambda}\ \in\ E(H)$, one can choose three distinct indices i, j, k with $p_k \in \mathrm{Box}\ [p_i,\ p_j]$, which yields that

$$E_{p_i} \cap E_{p_j} \subseteq E_{p_k}\ .$$

Thus, we can apply (2) to obtain

$$\tau(H) \le 6\lambda^2(H)\,(\lambda(H)+1)^3 < 2^{2^{d+2}}\ ,$$

and the result follows. □

As was shown in [BL], the bound $2^{2^{d+2}}$ in Theorem 2.1 is nearly optimal.

In fact, the above argument yields a slightly stronger result.

Theorem 2.2. *Let $P \subseteq \boldsymbol{R}^d$ by any compact set, and let $\mathcal{B}$ be any family of boxes in parallel position with the property that for any $(\nu+1)$–element subset $P' \subseteq P$ there is a box $B \in \mathcal{B}$ which covers at least two points of P' $(d, \nu \geq 1)$. Then one can choose at most $\binom{2^{2^d}+\nu}{\nu}^5$ members of $\mathcal{B}$ such that their union will cover P.*

In [ES2], Erdős and Szekeres proved the following.

Lemma 2.3. Every set $P \subseteq \boldsymbol{R}^2$ with at least 2^k elements contains three points p_1, p_2, p_3 such that $\sphericalangle p_1, p_2, p_3 \geq \pi\left(1 - \frac{1}{k}\right)$.

Our next result, which improves a theorem of Bárány [B], can be regarded as a generalization of Lemma 2.3.

Theorem 2.4. *Let d be a positive integer, $\epsilon > 0$. Every finite (or compact) set $P \subseteq \boldsymbol{R}^d$ has a subset of at most $2^{(c/\epsilon)^{d-1}}$ elements, $P' = \{p_1, p_2, \ldots\}$, with the property that for any $p \in P \backslash P'$ there exist $p_i, p_j \in P'$ satisfying*

$$\sphericalangle p_i p p_j \geq \pi - \epsilon \ .$$

(Here $c \leq 8$ is a constant.)

Proof: For $d \geq 2, \epsilon > 0$ fixed, let us cover the unit hemisphere centered at $O \in \boldsymbol{R}^d$ with $(4/\epsilon)^{d-1}$ spherical $(d-1)$–dimensional simplices $S_1, S_2, \ldots$ such that the diameter of each S_i is at most $\epsilon/2$. Let $h_{t1}, \ldots, h_{td}$ denote the hyperplanes induced by O and the ($(d-2)$–dimensional) facets of S_t. A parallelotope whose facets are parallel to $h_{t1}, \ldots, h_{td}$, respectively, is called a *box of type t*. The smallest box of type t containing $p, q \in \boldsymbol{R}^d$ will be denoted by $\mathrm{Box}_t[p, q]$.

For any $p, q \in \boldsymbol{R}^d$, choose an index t such that pq is parallel to Os for some $s \in S_t$, and let $\mathrm{Box}[p, q] = \mathrm{Box}_t[p, q]$. Notice that if $r \in \mathrm{Box}\ [p, q]$ then $\sphericalangle prq \geq \pi - \epsilon$.

Just like in the previous proof, define a hypergraph H by

$$\begin{aligned} V(H) &= \{\mathrm{Box}[p, q] \mid p, q \in P\}, \\ E(H) &= \{E_r | r \in P\}, \end{aligned}$$

where $E_r = \{\mathrm{Box}[p, q] | r \in \mathrm{Box}[p, q]\}$, and observe that it is sufficient to bound the transversal number of H. Clearly, $\nu(H) = 1$.

By the definition of $\lambda(H)$, one can select $\lambda(H) = \lambda$ elements $p_1, \ldots, p_\lambda \in P$ with the property that any two of them is enclosed in a box of some type, which does not cover any other p_k. More precisely, for every $1 \leq i < j \leq \lambda$ there exists $t(i, j) \leq (4/\epsilon)^{d-1}$ such that

$$\{p_1, \ldots, p_\lambda\} \cap \mathrm{Box}_{t(i,j)}\ [p_i, p_j] = \{p_i, p_j\}\ .$$

Obviously, p_i and p_j are two antipodal vertices of $\mathrm{Box}_{t(i,j)}[p_i, p_j]$, and every box has 2^{d-1} pairs of antipodal vertices. Let us color the segments $p_ip_j(1 \le i < j \le \lambda)$ with $(4/\epsilon)^{d-1}2^{d-1}$ colors according to the value of $t(i,j)$ and to the particular position of the diagonal p_ip_j within $\mathrm{Box}_{t(i,j)}[p_i, p_j]$. It is easy to see that the segments of the same fixed color form a bipartite subgraph of the complete graph K_λ on the vertex set $p_1, \dots, p_\lambda$. Hence the chromatic number of K_λ,

$$\lambda(H) \le 2^{(4/\epsilon)^{d-1}2^{d-1}} \quad ,$$

and the result follows from (2). $\square$

It is not hard to see that the bound in Theorem 2.4 is asymptotically tight, apart from the exact value of c (see [B], [EF]).

Combining these observations with an analogue of Turán's theorem for hypergraphs (see e.g. [Sp]), we immediately obtain the following result related to a problem of Conway, Croft Erdős and Guy [CC].

Corollary 2.5. *There exists a constant $c > 0$ such that, for any set of n distinct points $p_1, \dots p_n \in \boldsymbol{R}^d$, the number of triples $i < j < k$ for which $\sphericalangle p_ip_jp_k > \pi - \epsilon$, is at least $\lfloor n^3/2^{(c/\epsilon)^{d-1}} \rfloor$. Moreover, apart from the value of c, this bound cannot be improved.*

Finally, we mention another straightforward generalization of Lemma 2.3.

Theorem 2.6. *Let P be any set of at least $k^{(c/\epsilon)^{d-1}}$ points in $\boldsymbol{R}^d$, where c is a suitable constant. Then one can find $p_0, \dots, p_k \in P$ such that they are "almost collinear", i.e., $\sphericalangle p_{i-1}p_ip_{i+1} > \pi - \epsilon$ for every i $(1 \le i < k)$.*

3. Gallai–type theorems

Many problems in geometric transversal theory were motivated by the following famous question of Gallai. Given a family of pairwise intersecting disks in the plane, what is the smallest number of needles required to pierce all of them? (The answer is three. See [D], [DGK], [GPW], [E].).

First we show that (2) implies the following result of Gyárfás and Lehel.

Theorem 3.1. [GL] *For any positive integers k and ν, there exists a number $f = f(k, \nu)$ with the following property. Let H be any finite family of subsets of $\boldsymbol{R}$ such that each of them can be obtained as the union of at most k intervals. If H has no $\nu + 1$ pairwise disjoint members, then all of its members can be pierced by at most f points.*

Proof: In order to apply (2), we have to bound $\lambda(H)$. Let $E_1, \dots, E_\lambda$ be some members (edges) of

H such that, for any $i < j$, $E_i \cap E_j$ has a point x_{ij} which does not belong to any other $E_h (h \neq i, j)$. Write each E_i $(1 \leq i \leq \lambda)$ as the union of k intervals,

$$E_i = I_{i1} \cup \ldots \cup I_{ik} \ .$$

If $x_{ij} \in I_{ip} \cap I_{jq}$ for some $i < j$, then (E_i, E_j) is called a pair of type (p, q). (A pair may have several different types.)

It is easy to check that there are no four edges E_i such that all $\binom{4}{2} = 6$ pairs determined by them are of the same type. Thus,

$$\lambda < R_{k^2}(4) \ ,$$

where $R_s(t)$ denotes the smallest number R such that any complete graph of R vertices, whose edges are colored with s colors, has a monochromatic complete subgraph of t vertices. Hence, the theorem is true with

$$f(k, \nu) \leq 6 \binom{R_{k^2}(4) + \nu}{\nu}^5 . \qquad \square$$

Theorem 3.1. does not generalize to subsets of the plane that can be obtained as the union of k axis–parallel rectangles. Indeed, let $H = \{E_i | 1 \leq i \leq n\}$, where

$$E_i = \left\{(x, y) \in \boldsymbol{R}^2 | 0 \leq x, y \leq n \ \text{ and } \min(|x - i|, |y - i|) \leq \frac{1}{4}\right\} .$$

Then $\nu(H) = 1$, while $\lambda(H) = \tau(H) = n$.

However, one can easily establish the following.

Theorem 3.2. *Let F be a family of open domains in the plane such that each of them is bounded by a closed Jordan curve, and any two of them share at most two boundary points. Furthermore, let H be a finite set system, whose every element can be obtained by taking the union of at most k members of F. If H has no $\nu + 1$ pairwise disjoint elements, then all of its elements can be pierced by at most $g(k, \nu)$ points (where g depends only on k and ν).*

Proof: Pick λ elements (edges) of H,

$$E_i = I_{i1} \cup \ldots \cup I_{ik} \qquad (I_{ip} \in F \ , 1 \leq i \leq \lambda, 1 \leq p \leq k) \ ,$$

and suitable points

$$x_{ij} \in (E_i \cap E_j) \setminus \cup_{h \neq i,j} \ E_h \ ,$$

as in the previous proof. After defining the type of a pair (E_i, E_j), $i < j$ in exactly the same way as above, now one can argue that there are no 6 edges E_i such that all the $\binom{6}{2} = 15$ pairs determined by them have the same type (p,q). Assume, for contradiction, that e.g. $E_1,\ldots,E_6$ satisfy this condition for some $p \neq q$. Then any I_{ip} $(1 \leq i \leq 3)$ and any I_{jq} $(4 \leq j \leq 6)$ have a common interior point (x_{ij}) which is not covered by any other E_k $(k \neq i,j)$. We can conclude (by tedious case analysis) that there exist pairwise disjoint connected open subsets $I'_{ip} \subseteq I_{ip}$ $(1 \leq i \leq 3)$, $I'_{jq} \subseteq I_{jq}$ $(4 \leq j \leq 6)$ such that every I'_{ip} and I'_{jq} share a common boundary segment. This contradicts Kuratowski's theorem on planar maps. The case $p = q$ can be treated similarly.

Thus, $\lambda < R_{k^2}(6)$ and the result follows. We could also apply Theorem 1.1 of Sharir [Sh] to deduce $\lambda < R_{k^2}(c)$ with a much larger constant $c > 6$. $\square$

Theorem 3.2 can be applied to the family F_C of all homothetic copies of a convex set C in the plane. The special case when C is a convex polygon with a bounded number of sides was settled by Gyárfás [G]. (An easy compactness argument shows that C does not need to be strictly convex.)

For any hypergraph H and for any integer $t \geq 1$, let $\nu_t(H)$ denote the maximum number of (not necessarily distinct) edges of H such that every vertex is contained in at most t of them. Furthermore, let $\lambda_t(H)$ be the size of the largest collection of edges $\{E_i | i \in I\} \subseteq E(H)$ with the property that for any t–tuple $J \subseteq I$ there exists $x_J \in V(H)$ such that

$$x_J \in \left(\bigcap_{i \in J} E_i\right) \setminus \left(\bigcup_{i \notin J} E_i\right).$$

Clearly, $\nu_1(H) = \nu(H)$ and $\lambda_2(H) = \lambda(H)$.

Ding, Seymour and Winkler [DSW] have established an upper bound for $\tau(H)$ in terms of $\nu_t(H)$ and $\lambda_{t+1}(H)$, for any fixed $t \geq 1$. Applying their result with $t = 2$, we obtain the following generalization of Theorem 3.1 for the plane.

Theorem 3.3. *Let H be a finite family of open sets in the plane such that*
(i) every member of H is bounded by at most k closed Jordan curves;
(ii) any two distinct members of H have at most ℓ boundary points in common.
Assume that among any $\nu + 1$ members of H there are three with non–empty intersection. Then all members of H can be pierced by at most $f(k,\ell,\nu)$ points, where f does not depend on H.

In higher dimensions we obtain e.g. the following result.

Theorem 3.4. *Let H be a finite family of not necessarily connected polyhedra in $\boldsymbol{R}^d$ $(d \geq 2)$. Assume that every member of H has at most k vertices, and that among any $\nu + 1$ members of H there are $d+1$ whose intersection is non–empty. Then all members of H can be pierced by at most $g(d,k,\nu)$ points, where g does not depend on H.*

The special case of Theorem 3.4, when every member of H is the union of a bounded number of axis–parallel boxes, was proved by Lehel [L].

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