The number of simplices embracing the origin

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Abstract

Using the Upper Bound Theorem for polytopes and Gale transforms, Uli Wagner and Emo Welzl have recently proved the following remarkable theorem. For any absolutely continuous probability distribution in d-space, the probability that the convex hull of $d+1$ randomly and independently selected points contains the origin is at most $1/2^d$, and this bound is tight. We present two very short proofs for the planar version of this result, and discuss some related questions.

1 Introduction

Pick three points on the perimeter of the unit circle around the origin O, independently with uniform distribution. What is the probability that their convex hull contains O? There is a short and sweet argument that goes back at least to the sixties (see Wendel [3]), which

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shows that the answer is $1/4$. For any point x on the circle, let $-x$ denote the point diametrically opposite to x. For any distinct points x_1, x_2, and x_3 on the circle, consider the unordered triples $T = \{\varepsilon_1 x_1, \varepsilon_2 x_2, \varepsilon_3 x_3\}$, where each $\varepsilon_i = +1$ or -1. Observe that out of these 8 triples precisely 2 induce a triangle which contains the origin in its interior, and the claim readily follows.

In exactly the same way one can argue that the probability that the simplex induced by $d+1$ randomly, independently, and uniformly selected points on the surface of the unit sphere in d-space contains the origin in its interior is $1/2^d$. Moreover, this statement remains true for any absolutely continuous distribution symmetric about the origin, i.e., when the measure of any half-space bounded by a hyperplane through the origin is precisely $1/2$.

By introducing an ingenious continuous analogue of the Upper Bound Theorem (cf. [4]), Uli Wagner and Emo Welzl have proved that for *every* absolutely continuous distribution in d-space, not necessarily symmetric to the origin, the probability defined above cannot exceed $1/2^d$. They raised the question whether their theorem can be established by a simpler and "more illuminating" direct argument, at least for $d = 2$. In the following two sections, we describe two such arguments. Both methods solve some discrete variants of the problem, from where the Wagner-Welzl result follows by passing to the limit.

2 Discrete distributions on a regular n-gon

Let $n \geq 3$ be an odd integer, and let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of a regular n-gon in the plane, centered at O, where the indices are taken modulo n. Assume that the elements of V are numbered in such a way that the angle $v_i O v_{i+1}$ is equal to $(1 - 1/n)\pi$ for every $1 \leq i \leq n$.

Let P be a discrete probability distribution on V, for which $P[v_i] = p_i \geq 0$ and $\sum_i p_i = 1$. The set of those indices i for which $p_i \neq 0$ is called the *support* of P and is denoted by $\mathrm{supp}(P)$.

Theorem 1. *The probability that the triangle determined by three randomly and independently selected points of V contains O in its interior is at most $\frac{1}{4}\left(1-\frac{1}{n^2}\right)$. The maximum is attained if and only if P is the uniform distribution P_0.*

Proof: The probability we have to maximize is 6 times $S(P) = \sum_{\{i,j,k\}} p_i p_j p_k$, where the sum is taken over all triples $\{i,j,k\}$ such that the triangle $v_i v_j v_k$ contains O in its interior.

Fix a distribution P for which $S(P)$ attains its maximum, and assume that $\mathrm{supp}(P)$ is also maximal under this condition. We are going to show that $S(P)$ does not exceed $S(P_0)$.

Observe that there are no two consecutive indices $i+1, i+2 \notin \mathrm{supp}(P)$. Indeed, if $i \in \mathrm{supp}(P)$ and $i+1, i+2 \notin \mathrm{supp}(P)$, then choose a small constant $\varepsilon > 0$. For the distribution P' defined by

$$P'[v_j] = p'_j = \begin{cases} p_j & \text{if } j \neq i, i+2, \\ p_i - \varepsilon & \text{if } j = i, \\ p_{i+2} + \varepsilon & \text{if } j = i+2, \end{cases} \tag{1}$$

we have $S(P') = S(P)$, and the support of P' is larger than that of P. This contradicts the maximality of P.

Since n is odd, it follows that there are two consecutive indices $i, i+1 \in \mathrm{supp}(P)$. We separate two cases depending on whether the support of P is full or not.

Case A: $\mathrm{supp}(P) \neq \{1, 2, \ldots, n\}$.

There exist (not necessarily disjoint) indices i and k such that

$$\{i\} \cup \{i+1, i+3, i+5, \ldots, k\} \cup \{k+1\} \subseteq \mathrm{supp}(P),$$

and $\{i+2, i+4, \ldots, k-1\} \notin \mathrm{supp}(P)$.
For the distribution P' defined in (1) we have

$$S(P) - S(P') = p_{i+1}\varepsilon[(p_{k+1} + p_{k+3} + \ldots + p_{i-1}) -$$

$$(p_{i+3} + p_{i+5} + p_{i+7} + \ldots + p_i - \varepsilon)] \geq 0,$$

whenever $\varepsilon > 0$ is sufficiently small. This yields

$$p_{k+1} + p_{k+3} + \ldots + p_{i-1} \geq p_{i+3} + p_{i+5} + p_{i+7} + \ldots + p_i.$$

Exchanging the roles of i and $k+1$, we obtain, by symmetry, that

$$p_{k+2} + p_{k+4} + \ldots + p_i \geq p_{k+1} + p_{k+3} + p_{k+5} + \ldots + p_{k-2}.$$

Comparing the last two inequalities, it follows that

$$p_{i+1} = p_{i+3} = p_{i+5} = \ldots = p_k = 0,$$

contradicting our assumption that e.g. $i+1 \in \mathrm{supp}(P)$.

Case B: $\mathrm{supp}(P) = \{1, 2, \ldots, n\}$.
Using the same argument as before, now we obtain that for every i,

$$p_{i+2} + p_{i+4} + \ldots + p_{i-1} = p_{i+3} + p_{i+5} + \ldots + p_i.$$

Substituting i with $i+1$, we obtain

$$p_{i+3} + p_{i+5} + \ldots + p_i = p_{i+4} + p_{i+6} + \ldots + p_{i+1}.$$

Therefore, $p_{i+1} = p_{i+2}$ holds for every i.

The discussion of the case of equality is left to the reader. □

3 Counting alternating subsequences of length 3

It was just a matter of convenience that we assumed that $V = \{v_1, v_2, \ldots, v_n\}$ is the vertex set of a regular n-gon centered at O. The proof of Theorem 1 applies to every set V with the property that any line connecting O with an element $v_i \in V$ has precisely $(n-1)/2$ points of V on both of its sides. In other words, V has the *antipodality property* with respect to O: if we draw n rays from O through all elements of V, for any two consecutive rays there will be a third one lying in the cone induced by their reflections about O.

Theorem 2. *Let n be an odd positive integer, and let V be a set of n points in the plane such that $V \cup \{O\}$ is in general position and the number of triangles induced by V that contain O in their interiors, is as large as possible. Then V has the antipodality property with respect to O. Conversely, every set V which has the antipodality property* *with respect to O, maximizes the number of triangles containing O in their interiors.*

Proof: Fix an x–y coordinate system in the plane with O as the origin, and assume without loss of generality that no point of $V = \{v_1, v_2, \ldots, v_n\}$ lies on the x-axis. Let $0 < \alpha_i < \pi$ be the counter-clockwise angle from the x-axis to the line Ov_i. Suppose without loss of generality $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. For every $i = 1, 2, \ldots, n$, let $\text{sign}(i) = +$ if the y-coordinate of v_i is positive, and let $\text{sign}(i) = -$ otherwise.

Notice that, for every $i < j < k$, the triangle $v_i v_j v_k$ contains O in its interior if and only if $(\text{sign}(i), \text{sign}(j), \text{sign}(k))$ is an alternating sequence, i.e., $(+, -, +)$ or $(-, +, -)$.

If we rotate our coordinate system until after the x-axis passes through the first point of V, the sequence $(\text{sign}(1), \text{sign}(2), \ldots, \text{sign}(n))$ changes according to the following rule: the first or the last element will change its sign and move to the other end of sequence. We call this operation *shifting*.

Obviously, the number of alternating subsequences of length 3 does not change during shifting.

If $S = (\text{sign}(1), \text{sign}(2), \ldots, \text{sign}(n))$ itself is an alternating sequence and n is odd, then every other sequence obtained from S by shifting is also alternating. Note that S is an alternating sequence if and only if V has the antipodality property with respect to O.

Thus, it is sufficient to verify the following

Lemma. *Let n be a (not necessarily odd) positive integer, and let $S = (s_1, s_2, \ldots, s_n)$ be a sequence of $+$ and $-$ signs, for which the number of alternating subsequences of length 3 is as large as possible.*

Then the maximum is

$$f(n) = \begin{cases} \frac{n(n^2-1)}{24} & \text{if } n \text{ is odd,} \\ \frac{n(n^2-4)}{24} & \text{if } n \text{ is even.} \end{cases}$$

For odd n, this maximum is attained if and only if S is an alternating sequence. For even n, the maximum is attained if and only if in every string of consecutive members of S the number of plus signs and the number of minus signs differ by at most 2.

It remains to prove the Lemma.

Let $n \geq 3$, and assume that we have already established the assertion for all sequences whose length is smaller than n. By shifting, if necessary, we can achieve that $s_1 = s_n = +$. Deleting s_1 and s_n from S, we are left with a sequence S' consisting of p plus and m minus signs, $p+m = n-2$. Clearly, $f(S)$, the number of alternating subsequences of length 3 in S, satisfies

$$f(S) = f(S') + m + pm \leq f(n-2) + (p+1)m \leq$$

$$f(n-2) + \lfloor \frac{n-1}{2} \rfloor \cdot \lceil \frac{n-1}{2} \rceil = f(n),$$

as required.

If n is odd, then equality can hold only if S' is an alternating subsequence of length $n-2$, and $p+1 = m$. Therefore, S' must start and end with minus signs, and S must be alternating, too. One can also check that the cases when equality holds for even n are precisely those characterized in the theorem. $\Box$

4 Open problem

Both Theorem 1 and Theorem 2 immediately imply the planar case of the result of Wagner and Welzl mentioned in the abstract.

Corollary. [1] *For any absolutely continuous probability distribution in the plane, the probability that a triangle induced by* 3 *randomly and independently selected points contains* O *in its interior is at most* 1/4. *Equality holds here if the measure of any half-plane bounded by a line passing through* O *is* 1/2.

Problem 1. (Unicity) Is it possible to argue, based on the discrete variants of the result, that all distributions for which the bound 1/4 is attained in the Corollary satisfy the condition that the measure of every half-plane bounded by a line passing through O is 1/2?

Problem 2. Can one extend the above arguments to higher dimensions?

In order to generalize our proofs to 3-space, one should solve the following planar problem. Given n points in general position in the plane, colored red and blue. We want to maximize the number of multi-colored 4-tuples with the property that the convex hull of its red elements and the convex hull of its blue elements have at least one point in common. In particular, we want to show that when the maximum is attained, the number of red and blue elements are roughly the same.

References

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