Intersecting Convex Sets by Rays

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November 24, 2008

Abstract

What is the smallest number $\tau = \tau(n)$ such that for any collection of n pairwise disjoint convex sets in d-dimensional Euclidean space, there is a point such that any ray (half-line) emanating from it meets at most τ sets of the collection? This question of Urrutia is closely related to the notion of regression depth introduced by Rousseeuw and Hubert (1996). We show the following:

Given any collection $\mathcal{C}$ of n pairwise disjoint compact convex sets in d-dimensional Euclidean space, there exists a point p such that any ray emanating from p meets at most $\frac{dn+1}{d+1}$ members of $\mathcal{C}$.

There exist collections of n pairwise disjoint *(i)* equal length segments or *(ii)* disks in the Euclidean plane such that from any point there is a ray that meets at least $\frac{2n}{3} - 2$ of them.

We also determine the asymptotic behavior of $\tau(n)$ when the convex bodies are fat and of roughly equal size.

*Supported by the Norwegian research council project number: 166618, and by Brain Korea 21 Project, the School of Information Technology, KAIST in 2008.

†Supported by NSF Grant CCF-05-14079, and by grants from NSA, PSC-CUNY, the Hungarian Research Foundation OTKA, and BSF.

1 Introduction

Suppose we have a field scattered with obstacles, and we want to set up a mobile wireless sensor network to monitor this environment. The base station has to be placed so that it can easily communicate with the sensor nodes, no matter where they are situated. According to a model recently patented by Liu and Hung [8], the signal transmitted by a sensor can penetrate only at most a certain number, τ, of obstacles and will not be received by the base station if more than τ obstacles block the visibility between the sensor and the base station. They call this predetermined threshold the *obstacle number* of the network.

Based on this model, Jorge Urrutia [11] asked the following question: What is the smallest number $\tau = \tau(n)$ such that for any system of $n > 0$ pairwise disjoint segments in the plane, there is a point such that any ray (half-line) emanating from it meets at most τ segments? It has been conjectured (by Urrutia and others) that the right order of magnitude of this function should be around $\frac{n}{2}$. The aim of this note is to show that $\tau(n)$ is roughly $\frac{2n}{3}$. In fact, we will consider the more general case of arbitrary convex bodies in $\mathbb{R}^d$.

Definitions and statement of results. Let $\mathcal{C}$ be a collection of n pairwise disjoint convex sets in d-dimensional Euclidean space ($\mathbb{R}^d$). For any point $p \in \mathbb{R}^d$, denote by $\mathcal{R}(p)$ the set of all rays emanating from p. Let $h(r, \mathcal{C})$ denote the number of sets intersected by the ray r. We define

$$\tau(p, \mathcal{C}) = \max_{r \in \mathcal{R}(p)} h(r, \mathcal{C}),$$

that is, the maximal number of sets from $\mathcal{C}$ that can be intersected by a ray emanating from p. The *obstacle number* of a given collection $\mathcal{C}$ of pairwise disjoint convex sets is defined as

$$\tau(\mathcal{C}) = \min_{p \in \mathbb{R}^d} \tau(p, \mathcal{C}).$$

Finally, we define

$$\tau_d(n) = \max_{|\mathcal{C}|=n} \tau(\mathcal{C}),$$

that is, the maximum value of $\tau(\mathcal{C})$ as $\mathcal{C}$ varies over all collections of n pairwise disjoint convex sets in $\mathbb{R}^d$.

We are interested in studying the asymptotic growth of the functions $\tau_d(n)$. Our most general bound on $\tau_d(n)$, $d \geq 2$, is given by the following.

Theorem 1. *For $d \geq 2$, $\tau_d(n) \leq \frac{dn+1}{d+1}$.*

For the particular case $d = 2$ we were able to show that the bound of Theorem 1 is asymptotically tight. We will give constructions that show the following.

Theorem 2. *For every $k > 0$ there exists*

1. *a collection $\mathcal{C}_S$ of $3k$ pairwise disjoint equal length segments in the Euclidean plane such that $\tau(\mathcal{C}_S) = 2k-1$.*

2. *a collection $\mathcal{C}_D$ of $3k$ pairwise disjoint disks in the Euclidean plane such that $\tau(\mathcal{C}_D) = 2k-2$.*

We were not able to find an example showing that the bound in Theorem 1 is tight for dimensions greater than 2, but it would be surprising if this is not the correct bound.

In view of Theorem 2, the value of $\tau_2(n)$ is not affected by bounding the ratio of the diameters of the sets (Theorem 2.1), or by bounding the *fatness*[1] of the sets (Theorem 2.2). However, if we bound the ratio of the diameter of the sets *and* the fatness, i.e. if the convex bodies are γ-fat of roughly equal size, then the asymptotic behavior of the obstacle number is quite different. For $0 < \gamma < 1$, we call a convex body γ-*round* if it is contained in a disk of unit radius and it contains a disk of radius γ. We then have:

Theorem 3. *For any collection $\mathcal{C}$ of $n > 0$ pairwise disjoint γ-round convex bodies in the plane there exists a point p such that $\tau(\mathcal{C}) = O(\sqrt{n \log n})$.*

Theorem 4. *For every $n > 0$ there exists a collection $\mathcal{C}$ of n pairwise disjoint unit disks in the plane such that $\tau(\mathcal{C}) = \Omega(\sqrt{n \log n})$ for all $p \in \mathbb{R}^2$.*

Our proof of Theorem 1, which resembles Chakerian's proof of Helly's theorem, relies on two classical theorems: Brouwer's fixed point theorem and Carathéodory's theorem. Brouwer's fixed point theorem states that any continuous function from the d-dimensional ball to itself must have a fixed point. Carathéodory's theorem claims that a point p is contained in the convex hull of a set S in d-dimensional Euclidean space if and only if p is contained in a simplex spanned by points of S. The proofs of Theorems 3 and 4 are based on results by Alon *et al.* [1] and Besicovitch [3].

It should be noted that the obstacle number is closely related to the notions of *regression depth* and, dually, *depth in hyperplane arrangements*, which have been extensively studied the last decade (see for instance [2], [7], [10]). We will comment further on this connection, and address some computational aspects of these problems in Section 4.

Theorems 1 and 2 will be proved in Sections 2 and 3, respectively. Theorems 3 and 4 will be proved in Section 4.

[1] One can measure "fatness" by the ratio of the diameters of smallest circumscribed circle and largest inscribed circle.

2 Proof of Theorem 1

Preliminaries. A *ray* R from the point p is the set of points

$$R = \{p + t(s - p) \,:\, t > 0\}$$

where p and s are distinct points of $\mathbb{R}^d$. The point p will be referred to as the *starting point* of R, and the point $s - p$ will be referred to as the *direction* of R. There are of course infinitely many directions that correspond to the same ray. We say that the ray R *meets* the convex *body* A if $R \cap A \neq \emptyset$. Here a convex body means a compact convex set with non-empty interior.

In what follows let $\mathcal{C}$ be a given collection of n convex bodies in $\mathbb{R}^d$. After rescaling (if necessary) we may assume that $0 < \mu(A) < 1$ for every $A \in \mathcal{C}$ where μ is the Lebesgue measure on $\mathbb{R}^d$. We let $|\cdot|$ denote the Euclidean distance. When we speak of convergence of sequences of convex bodies we mean with respect to the Hausdorff metric.

For a given point $p \in \mathbb{R}^d$ and $A \in \mathcal{C}$, let

$$K_A(p) = \operatorname{conv}(\{p\} \cup A)$$

and

$$K_{\mathcal{C}}(p) = \bigcap_{A \in \mathcal{C}} K_A(p).$$

Note that every $K_A(p)$ is a convex body and that $p \in K_{\mathcal{C}}(p)$. It follows that $K_{\mathcal{C}}$ is compact and convex, but the interior of $K_{\mathcal{C}}$ may be empty.

The ray $R = \{p + t(s - p) \,:\, t \geq 0\}$ meets every member of $\mathcal{C}$ if and only if there exist real numbers $0 < t_1 \leq \cdots \leq t_n$ such that each member of $\mathcal{C}$ contains one of the points $p + t_i(s - p)$. By convexity $p + t_1(s - p) \in K_{\mathcal{C}}(p)$. Thus we have established:

Claim 5. *If there exists an $s \in K_{\mathcal{C}}(p)$ such that $s \neq p$, then there exists a ray with direction $s - p$ that meets every member of $\mathcal{C}$. Conversely, if there exists a ray starting at p in the direction $s - p \neq 0$ that meets every member of $\mathcal{C}$, then there exists a $t > 0$ such that the point $p + t(s - p) \in K_{\mathcal{C}}(p)$.*

When $\mu(K_{\mathcal{C}}(p)) > 0$ let $m_{\mathcal{C}}(p) \in \mathbb{R}^d$ denote the center of mass of $K_{\mathcal{C}}(p)$, and when $\mu(K_{\mathcal{C}}(p)) = 0$ let $m_{\mathcal{C}}(p) = p$. We have the following:

Claim 6. *Let $\{p_k\}_{k=1}^{\infty}$ be an infinite sequence of points in $\mathbb{R}^d$ converging to $p \in \mathbb{R}^d$. We then have, as k tends to infinity,*

1. $K_A(p_k) \to K_A(p)$.
2. *If* $\mu(K_{\mathcal{C}}(p)) > 0$*, then* $K_{\mathcal{C}}(p_k) \to K_{\mathcal{C}}(p)$.

3. $\mu(K_C(p_k)) \to \mu(K_C(p))$.
4. *If* $\mu(K_C(p)) > 0$*, then* $m_C(p_k) \to m_C(p)$.

Proof. Claim 6.1 should intuitively be quite clear. Let $|p - p_k| = \varepsilon$, and suppose there exists a point $q \in K_A(p_k)$ such that the ball B_ε of radius ε centered at q does not contain any point of $K_A(p)$. Then there exists a hyperplane H that strictly separates $K_A(p)$ from $\mathrm{conv}(\{p_k\} \cup B_\varepsilon)$, which implies that the distance from p_k to H is strictly less than ε. But this is impossible since the translate of H that passes through p_k would strictly separate A from q, which implies that $q \notin K_A(p_k)$. Thus the distance between $K_A(p_k)$ and $K_A(p)$ (in the Hausdorff metric) is bounded from above by $|p - p_k| = \varepsilon$.

Claim 6.2 follows from 6.1 since $K_C(p_k) = \bigcap_{A \in C} K_A(p_k)$, and Claim 6.4 follows immediately from Claim 6.2.

Finally, if $\mu(K_C(p)) > 0$, then Claim 6.3 follows immediately from 6.2. If $\mu(K_C(p)) = 0$, then $K_C(p)$ is empty or contained in some affine hyperplane of $\mathbb{R}^d$, and then it is easily verified that $\mu(K_C(p)) \to 0$ as $k \to \infty$. (We leave the details to the reader). □

Now define the function $g_C : \mathbb{R}^d \to \mathbb{R}^d$ as

$$g_C(p) = \mu(K_C(p))(m_C(p) - p).$$

Consider a sequence $\{p_k\}_{k=1}^{\infty}$ in $\mathbb{R}^d$ converging to $p \in \mathbb{R}^d$. If $\mu(K_C(p)) > 0$, Claims 6.3 and 6.4 imply that $g_C(p_k)$ converges to $g_C(p)$ as k tends to infinity. If $\mu(K_C(p)) = 0$ it is not necessarily true that $m_C(p_k)$ converges to $m_C(p)$, because $\mu(K_C(p_k))$ may be positive for all k. On the other hand, by Claim 6.3 $\mu(K_C(p_k))$ converges to 0 as k tends to infinity. This guarantees that $g_C(p_k) \to 0 = g_C(p)$ as $k \to \infty$. Therefore, Claims 5 and 6 imply:

Claim 7. *The function* g_C *is continuous. If* $g_C(p) \neq 0$*, then there exists a ray from* p *with direction* $m_C(p) - p$ *that meets every member of* C *in an interior point.*

Let $\mathcal{F}$ be a finite set whose elements are finite collections of convex bodies. Note that we allow repetitions, that is, the same body could belong to several different collections. For each $C \in \mathcal{F}$ we can define the function g_C as described above.

Now define the function $G_{\mathcal{F}} : \mathbb{R}^d \to \mathbb{R}^d$ as

$$G_{\mathcal{F}}(p) = \sum_{C \in \mathcal{F}} g_C(p),$$

which is continuous by Claim 7. Since $\mathcal{F}$ is finite and every $C \in \mathcal{F}$ is finite, the union of the convex bodies represented by $\mathcal{F}$ is contained in a d-dimensional ball B. It follows that if B is sufficiently large, and the union of the convex bodies are sufficiently close to the center of B, then $p + \frac{G_{\mathcal{F}}(p)}{|G_{\mathcal{F}}(p)|+1} \in B$ for every $p \in B$. Clearly, the function $p + \frac{G_{\mathcal{F}}(p)}{|G_{\mathcal{F}}(p)|+1}$ is also continuous, so it satisfies the hypothesis of Brouwer's fixed point theorem. Therefore there exists a fixed point, i.e., a point $p \in B \in \mathbb{R}^d$ such that $p + \frac{G_{\mathcal{F}}(p)}{|G_{\mathcal{F}}(p)|+1} = p$. In other words, we have:

Claim 8. *For any finite set $\mathcal{F}$ whose elements are finite collections of convex bodies in $\mathbb{R}^d$, there exists a point $p \in \mathbb{R}^d$ such that $G_{\mathcal{F}}(p) = 0$.*

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. Let $1 \leq \tau \leq n$ be the greatest integer such that from any point there is a ray that meets at least τ members of $\mathcal{C}$.

Using a standard compactness argument, we inflate every $A \in \mathcal{C}$ by some small $\varepsilon > 0$ to a body $(1+\varepsilon)A$ such that $A \subset \operatorname{int}(1+\varepsilon)A$ for all $A \in \mathcal{C}$, and the sets $(1+\varepsilon)A$, $A \in \mathcal{C}$, are pairwise disjoint. Let $\mathcal{C}'$ denote the collection of $(1+\varepsilon)A$, $A \in \mathcal{C}$. It is clear that any ray that meets A will meet the interior of $(1+\varepsilon)A$, so from any point $p \in \mathbb{R}^d$ there is a ray that meets at least τ members of $\mathcal{C}'$ in interior points.

Let $\mathcal{F}$ be the set of all subcollections of $\mathcal{C}'$ on τ elements. By Claim 8, there exists a point $p \in \mathbb{R}^d$ such that the function $G_{\mathcal{F}}(p) = 0$. Let $\mathcal{F}' \subset \mathcal{F}$ be the subset of all $C \in \mathcal{F}$ such that $g_C(p) \neq 0$. As a consequence of inflating the sets (described in the previous paragraph) the subcollection $\mathcal{F}'$ is non-empty. The equation

$$G_{\mathcal{F}}(p) = \sum_{C \in \mathcal{F}} g_C(p) = \sum_{C \in \mathcal{F}'} \mu(K_C(p))(m_C(p) - p) = 0$$

implies that

$$p \in \operatorname{conv}\left(\bigcup_{C \in \mathcal{F}'} \{m_C(p)\} \right),$$

where $m_C(p) \neq p$ for every $C \in \mathcal{F}'$. By Carathéodory's theorem, there exists an $\mathcal{F}'' \subset \mathcal{F}'$, with $|\mathcal{F}''| \leq d+1$, such that

$$p \in \operatorname{conv}\left(\bigcup_{C \in \mathcal{F}''} \{m_C(p)\} \right).$$

By Claim 7, for every $C \in \mathcal{F}''$ there exists a ray from p with direction $m_C(p) - p$ that meets the members of C in interior points. Therefore, if a convex body A belongs to all $C \in \mathcal{F}''$, then $p \in A$. Hence, by using the fact that the members of $\mathcal{C}'$ are pairwise disjoint, at most one member of $\mathcal{C}'$ can have this property.

If no A belongs to all $C \in \mathcal{F}''$, then

$$|\mathcal{F}''|\tau = \sum_{C \in \mathcal{F}''} |C| \leq (|\mathcal{F}''| - 1)n$$

which implies $\tau \leq \frac{(|\mathcal{F}''|-1)n}{|\mathcal{F}''|} \leq \frac{dn}{d+1}$.

If some A belongs to all $\mathcal{C} \in \mathcal{F}''$, then letting $\overline{\mathcal{C}} = \mathcal{C} \setminus \{C\}$, we obtain

$$|\mathcal{F}''|(\tau - 1) = \sum_{\mathcal{C} \in \mathcal{F}''} |\overline{\mathcal{C}}| \leq (|\mathcal{F}''| - 1)(n-1),$$

which implies $\tau \leq \frac{(|\mathcal{F}''|-1)(n-1)}{|\mathcal{F}''|} + 1 \leq \frac{dn+1}{d+1}$, as required. This completes the proof of Theorem 1.

3 Proof of Theorem 2: Constructions

In this section we prove Theorem 2. Before proceeding with the constructions we must clarify some terms. Given a point p in the plane, let R_1 and R_2 be rays with starting point at p. Apart from the two exceptions, $R_1 = R_2$ and $R_1 = -R_2$ [2], the two rays will form a positive angle less than π. The convex hull of $R_1 \cup R_2$ is a wedge, and we will say that R_1 and R_2 bound a *wedge with apex at* p, and that R_1 and R_2 are the *boundary rays* of the wedge. Let A be a convex body. We will say that A is *tangent to the wedge* W if A is contained in W and tangent to both of the boundary rays of W. Two sets in the plane are said to be *separable* if there exists a line L such that the sets are contained in opposite open halfplanes bounded by L. The following observation is simple but crucial for our constructions.

Claim 9. *Let W be a wedge with apex p and C a compact set that contains p and at least one point of the interior of W. We can find*

1. *a segment*
2. *a disk*

that is tangent to W and separable from C.

Proof. It follows from the separation theorem for convex sets that X and Y are separable if and only if conv$X$ and conv$Y$ are separable. We may therefore assume that $C = \text{conv}\, C$.

Since C is compact, it is bounded, therefore there exists a line l that intersects the boundary rays of W and is disjoint from C. The intersection $l \cap W$ gives us the desired segment. This proves Claim 9.1.

For every $r > 0$ there is a unique disk, D_r, of radius r which is tangent to W. The distance from p to D_r is an increasing function in terms of r which tends to infinity as $r \to \infty$, so for a sufficiently large r, D_r is disjoint (and therefore separable) from C. This yields the desired disk, which proves Claim 9.2. □

[2]For a given ray R, we denote by $-R$ the ray with the same starting point and opposite direction as R.

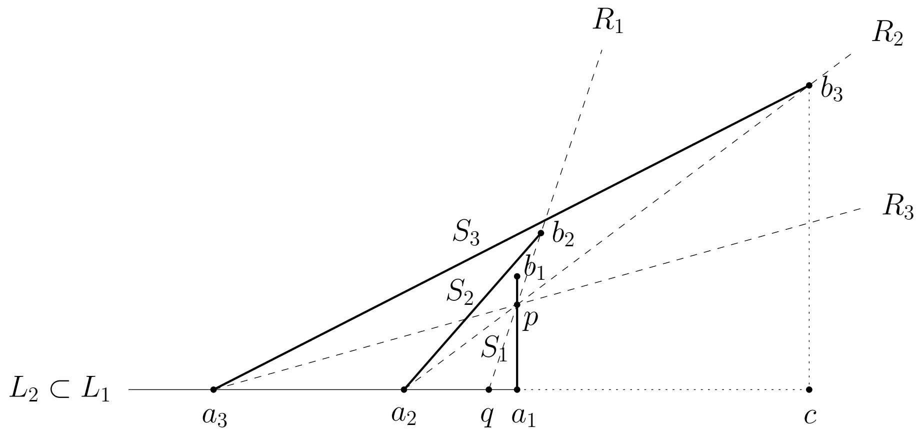

Figure 1: The construction of $\mathcal{A}$ for $k = 3$.

3.1 Construction 1: Equal length segments

We first describe a collection $\mathcal{A} = \{S_1, \ldots, S_k\}$ of $k \geq 1$ pairwise disjoint segments in the plane where the segment $S_i = \text{conv}(\{a_i, b_i\})$.

Start with a non-degenerate segment S_1 (which determines the points a_1 and b_1), let p be a point in the interior of S_1, and q a point which is not on the line determined by S_1. Let R_1 be the ray that starts at q and passes through p, and L_1 the ray that starts at q and has direction $q - a_1$. Let W_1 be the wedge bounded by the rays R_1 and L_1. By Claim 9.1 we can find a segment, S_2, which is tangent to W_1 and separable from $S_1 \cup \{q\}$, such that $a_2 \in L_1$ and $b_2 \in R_1$.

For $1 < i < k$, let R_i be the ray that starts at a_i and passes through p, and L_i the ray that starts at a_i and is contained in the ray L_{i-1}. Let W_i be the wedge bounded by the rays R_i and L_i. By Claim 9.1 we can find a segment, S_{i+1}, which is tangent to W_i and separable from $S_1 \cup \cdots \cup S_i$, such that $a_{i+1} \in L_i$ and $b_{i+1} \in R_i$. Finally, let R_k be the ray that starts at a_k and passes through p, and let c be the projection in the direction $a_1 - b_1$ of the point b_k onto the line determined by the a_i. (See Figure 1.)

Claim 10. *Let z be a point which is separable from $S_1 \cup \cdots \cup S_k$. The ray starting at z which passes through p meets all but at most one of the members of $\mathcal{A}$.*

Proof. Let $K = \text{conv}(S_1 \cup \cdots \cup S_k)$. First we observe (allthough not crucial for the proof) that for $1 \leq i \leq k$, starting at the point a_i the ray R_i intersects the members of $\mathcal{A}$ in the order $S_i\, S_{i-1} \cdots S_1\, S_{i+1}\, S_{i+2} \cdots S_k$. If we extend each ray R_i to a line λ_i, $\mathbb{R}^2 \setminus (K \cup \lambda_1 \cup \cdots \cup \lambda_k)$ is a collection of $2k$ open, connected regions.

For $1 \leq i < k$, it follows by the construction of $\mathcal{A}$ that if z is contained in a region bounded by λ_i and λ_{i+1}, the ray starting at z which passes through p will intersect every member of $\mathcal{A} \setminus \{S_{i+1}\}$. If z is in a region bounded by λ_1 and λ_k, the ray starting at z which passes through p will intersect every member $\mathcal{A}$. $\square$

Note that Claim 10 is invariant under affine transformations of the plane, as well as reflections. Let T be the triangle with vertices $u = (-\varepsilon, 0)$, $v = (0, h)$, and $w = (0, 0)$, where $0 < \varepsilon < h < 1$ are to be specified later. After a reflection, if necessary, we apply an affine transformation that maps the points $a_k \mapsto u$, $b_k \mapsto v$, $c \mapsto w$. Let $\mathcal{B}$ denote the resulting collection of segments. Note that the convex hull of the segments of $\mathcal{B}$ is contained in T. Let T_0, T_1, and T_2 be congruent copies of T, where T_i is a $\frac{2i\pi}{3}$ clockwise rotation of T, and let $\mathcal{B}_0$, $\mathcal{B}_1$, and $\mathcal{B}_2$ be the corresponding collections of segments.

Now, let $v_0v_1v_2$ be an equilateral triangle of side length 1. For each $i = 0, 1, 2$ translate the triangle T_i (and thus also the collections $\mathcal{B}_i$) such that the vertex $v \in T_i$ coincides with the vertex v_i and the edge $vw \subset T_i$ is contained in the edge v_iv_{i-1} (indices are taken mod 3; see Figure 2).

By construction it is clear that we may extend the segments in each collection $\mathcal{B}_i$ arbitrarily far while they remain disjoint (the directions which we can extend are uniquely determined). Thus we can make all the segments of equal length. Let M_i be the ray that starts at the center of the triangle $v_0v_1v_2$ and passes through the midpoint of the edge v_iv_{i-1}. Consider a point z belonging to the wedge bounded by M_i and M_{i-1}. If ε and h are chosen sufficiently small (say $h < \frac{1}{10}$ and $\varepsilon \ll h$), and the segments are extended sufficiently far, then any ray starting from z that meets the (translated) triangle T_i will also meet every member of $\mathcal{B}_{i+1}$. Therefore, by Claim 10, $\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$ is a collection of $3k$ equal length, pairwise disjoint segments such that from any point there is a ray that meets at least $2k - 1$ segments. (See Figure 2 for an illustration). This completes the proof of Theorem 2.1.

3.2 Construction 2: Disks

We first describe a collection $\mathcal{C}_1 = \{D_1, \ldots, D_k\}$ of $k \geq 2$ pairwise disjoint closed disks in the plane. (This part of the construction is essentially the same as the construction of $\mathcal{A}$.)

Start with a disk D_1 centered at the point p, and tangent to a line L at the point a_1. Let R_1 be the ray starting at a_1 that passes through p, and L_1 a ray starting at a_1 which is contained in L. The rays R_1 and L_1 bound the wedge W_1. By Claim 9.2 we can find a disk, D_2, which is tangent to W_1 and separable from D_1. Let a_2 be the point of tangency between D_2 and L_1.

For $1 < i < k$, let R_i be the ray starting at a_i that passes through p, and L_i the ray starting at a_i which is contained in L_{i-1}. The rays R_i and L_i bound the wedge W_i. Furthermore, let b_{i-1} be the point where the ray R_i enters the disk D_{i-1}. By Claim 9.2 we can find a disk, D_{i+1}, which is tangent to W_i and separable from $D_1 \cup \cdots \cup D_i$. Let a_{i+1} be the point of tangency between D_{i+1} and L_i. Finally, let R_k be the ray starting at a_k which passes through p, and for $1 < i < k$ let M_i be the ray that starts at a_1 and passes through

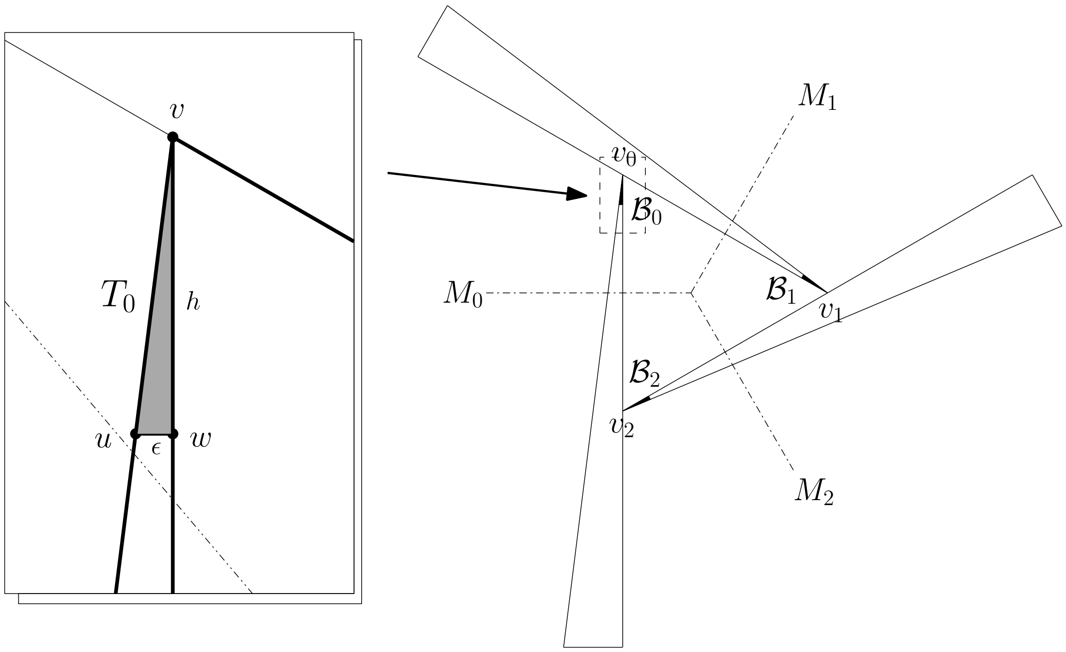

Figure 2: The collection $\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$

b_i. (See Figure 3). The reader will easily verify that the ray M_i is contained in the interior of the wedge bounded by M_{i-1} and M_{i+1} $(2 < i < k-1)$.

Claim 11. *Let z be a point in the plane. We then have the following.*

1. *If z is separable from $D_1 \cup \cdots \cup D_k$ by the line L, then the ray starting at z passing through the point p meets all but at most one member of $\mathcal{C}_1$.*
2. *If z is contained in the wedge bounded by L_1 and M_{k-1} then the ray starting at z passing through the point p meets all but at most two members of $\mathcal{C}_1$.*
3. *If z is contained in the wedge bounded by M_{k-1} and R_1 then one of the rays starting at z passing through a_1 or a_k meets at least half the members of $\mathcal{C}_1$.*
4. *If z is contained in the wedge bounded by R_1 and $-L_1$, then there exists a ray starting at z that intersects the segment with endpoints a_1 and a_k that meets at least half the members of $\mathcal{C}_1$.*

Proof. The proof of Claim 11.1 is basically the same as the proof for Claim 10, so it is omitted.

For the remainder of the proof of Claim 11, the reader may find it helpful to refer to Figure 4. Each arc represents a disk in the family $\mathcal{C}_1$, and it follows by Claim 9.2 that this arrangement is in fact realizable by a collection of pairwise disjoint disks.

To prove Claim 11.2, note that the rays $R_2, \ldots, R_k$ divide the wedge bounded by L_1 and

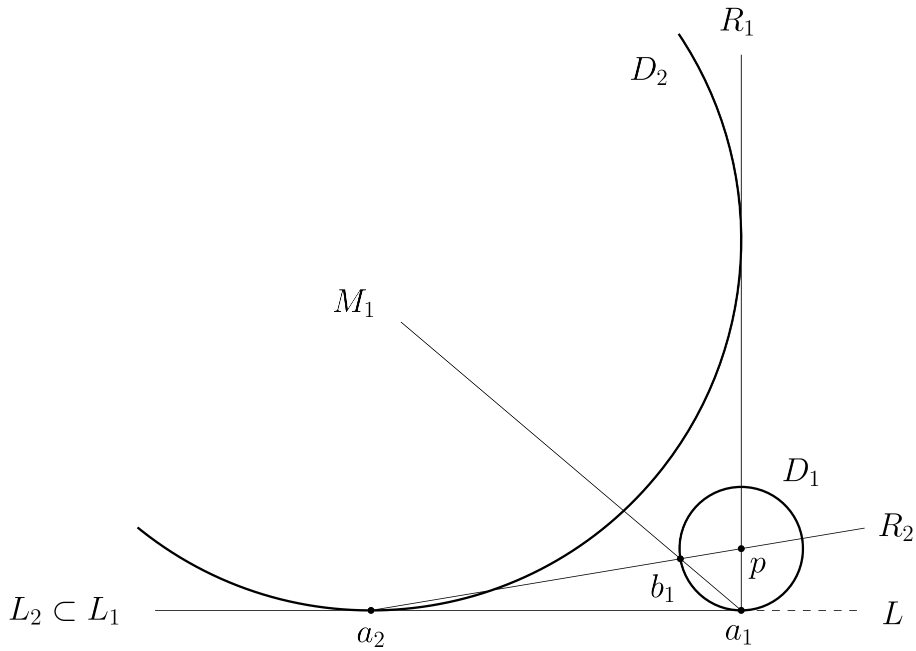

Figure 3: First two disks of $\mathcal{C}_1$.

M_{k-1} into k convex regions. For $1 < i < k$, if z belongs to the region between R_i and R_{i+1}, the ray starting at z passing through p meets every member of $\mathcal{C}_1 \setminus \{D_i, D_{i+1}\}$. If z belongs to the region bounded by the rays M_{k-1}, L_1, and R_2, the ray starting at z passing through p meets every member of $\mathcal{C}_1 \setminus \{D_2\}$. Finally, if z belongs to the region bounded by the rays M_{k-1}, L_1, and R_k, the ray starting at z passing through p meets every member of $\mathcal{C}_1 \setminus \{C_k\}$.

For Claim 11.3, note that the line through z and a_1 meets every member of $\mathcal{C}_1$. Therefore, either the ray starting at z passing through a_1, or its negative, meets at least half the members of $\mathcal{C}_1$. In the latter case, the ray starting at z passing through a_k also meets at least half the members of $\mathcal{C}_1$.

For Claim 11.4, note that the line through z and p meets all but at most one member of $\mathcal{C}_1$. If z lies between R_k and L, then the ray starting at z passing through a_k meets every member of $\mathcal{C}_1$. If z lies between R_k and R_1, then the ray starting at z passing through p, or its negative, meets at least half the members of $\mathcal{C}_1$. In the latter case the ray starting at z passing through a_k also meets at least half the members of $\mathcal{C}_1$. $\square$

Next, we define a collection $\mathcal{C}_2 = \{E_1, \ldots, E_m\}$ of $m \geq 1$ pairwise disjoint closed disks which will be appended to the collection $\mathcal{C}_1$ such that, $\mathcal{C}_1 \cup \mathcal{C}_2$ is a collection of pairwise disjoint disks. Let W be the wedge with apex at a_1, bounded by the rays L_1 and $-M_{k-1}$ (where a_1, L_1, and M_{k-1} are from the construction of $\mathcal{C}_1$). By Claim 9.2 we can find a disk, E_1, which is tangent to W and separable from $\{a_1, a_k\}$. For $1 < i \leq m$, let E_i be a disk which is tangent to W and separable from $\{a_1\} \cup E_1 \cup \cdots \cup E_{i-1}$, which we can

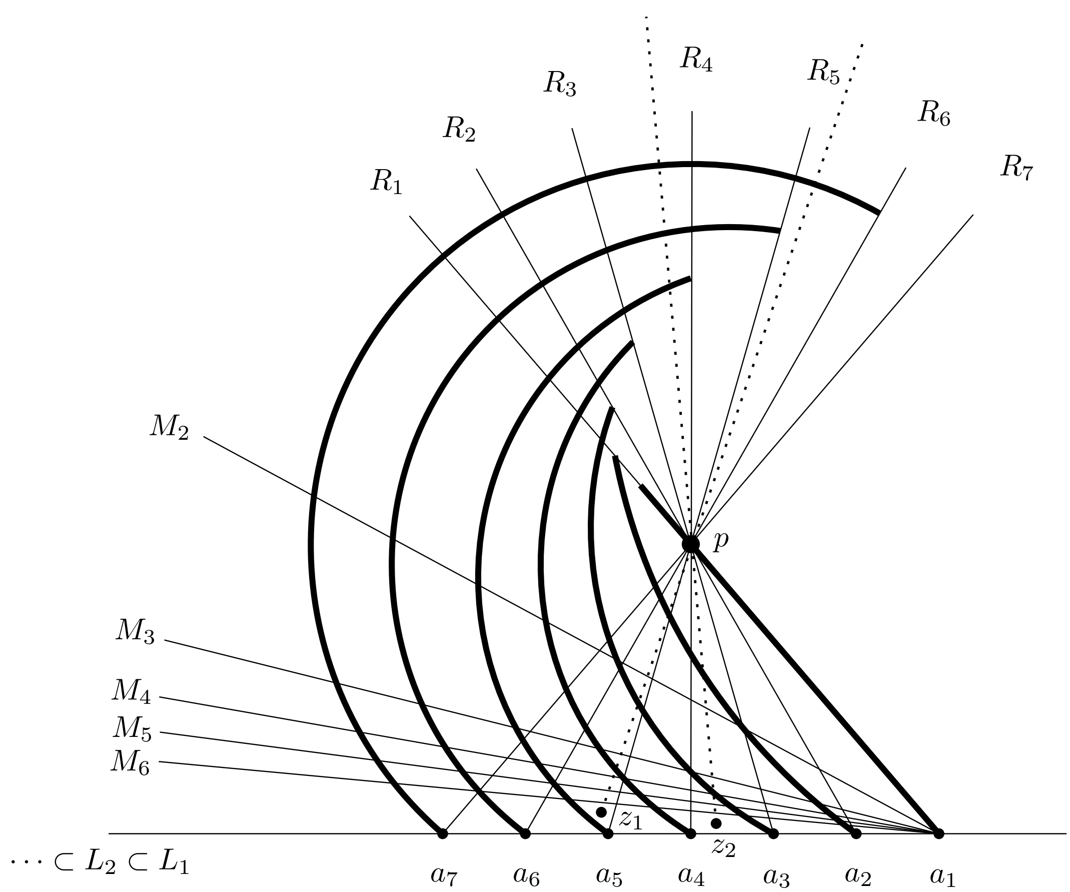

Figure 4: Schematic representation of $\mathcal{C}_1$. The ray from the point z_1 meets every members of $\mathcal{C}_1 \setminus \{D_5, D_6\}$. The ray from the point z_2 meets every members of $\mathcal{C}_1 \setminus \{D_4\}$.

find by Claim 9.2.

Claim 12. *Let z be a point contained in the wedge bounded by M_{k-1} and $-L_1$. Then any ray starting at z that intersects the segment with endpoints a_1 and a_k meets every member of $\mathcal{C}_2$.*

Proof. This follows since every disk of $\mathcal{C}_2$ is tangent to the wedge bounded by $-M_{k-1}$ and L_1, and since a_k lies between a_1 and $E_i \cap L_1$, for every $1 \leq i \leq m$. $\square$

Now set $k = 2m$ and $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Thus, $|\mathcal{C}| = 3m$, and it follows from Claims 11 and 12 that from any point in the plane there is a ray that meets all but at most *two* members of $\mathcal{C}_1$ or a ray that meets at least half the members of $\mathcal{C}_1$ and every member of $\mathcal{C}_2$. This completes the proof of Theorem 2.2.

Remark. Note that Construction 2 can be carried out for other convex sets as well, in particular for segments. The rotational symmetry of Construction 1 is not possible with disks.

It is also worth noting that Construction 2 also implies that there is a convex subdivision of the plane which has obstacle number at least as large as that of Construction 2. This follows by considering the so-called *power diagram*: For a family of disks in the plane the power diagram is an associated convex subdivision of the plane (see e.g. [9] for details). An important property of the power diagram is that if the disks are non-overlapping, then each cell of the power diagram contains exactly one disk.

4 Concluding remarks

We end with some remarks on the obstacle number for some restricted classes of convex sets. We also discuss some computational aspects of this problem.

4.1 Collections of segments and disks

When we restrict our attention to collections of disjoint segments or disks there are much simpler proofs of Theorem 1.

Collections of segments. As pointed out in the introduction the questions we consider here are closely related to the notion of depth in an arrangement of hyperplanes or, dually, regression depth. In the plane, *the depth of a simple arrangement of lines $\mathcal{A}$* is the maximum number $d = d(\mathcal{A})$ for which there exists a point p such that any path from p to infinity crosses at least d lines of $\mathcal{A}$. It was shown by Rousseeuw and Hubert [10] that any simple arrangement of lines has depth at least $\frac{n}{3}$.

Using the result from [10], it is simple to argue that for any collection $\mathcal{S}$ of pairwise disjoint segments in the plane, $\tau(\mathcal{S}) \leq \frac{2}{3}|\mathcal{S}|$. First we note that a slight perturbation of the segments will not affect the value of $\tau(\mathcal{S})$. Thus we can make sure that when we extend each segment to a line, the resulting arrangement of lines is a simple one. By the Rousseeuw-Hubert result there exists a point p such that any path from p to infinity crosses at least $\frac{n}{3}$ of the lines. Therefore any ray from p crosses at most $n - \frac{n}{3} = \frac{2n}{3}$ of the segments of $\mathcal{S}$.

Collections of disks. We now consider collections of pairwise disjoint disks in the plane. In this case there is also a simple argument that shows that $\tau(\mathcal{D}) \leq \frac{2}{3}|\mathcal{D}|$. Our argument relies on Rado's *center point theorem* which states that for any finite set of points P in the plane, there exists a point c (not necessarily in P) such that for any half-plane H that contains c, we have $|H \cap P| \geq \frac{1}{3}|P|$. The point c is called a center point of P.

Now we make a simple geometric observation: If the center of a disk D lies on or below the x-axis and D does not contain the origin, then the positive y-axis does not intersect D. Let $\mathcal{D}$ be a collection of pairwise disjoint disks, and let c be a center point with respect to the centers of the disks in $\mathcal{D}$. Since any line L through c has at least $\frac{n}{3}$ centers on either side, and at most one disk can contain p, a ray from p orthogonal to L meets at most $\frac{2}{3}|\mathcal{D}| + 1$ disks of $\mathcal{D}$.

4.2 Computational questions

Here we address the following computational problems: Given a collection $\mathcal{C}$ of pairwise disjoint compact convex sets in $\mathbb{R}^2$, we want to **(1)** find a point that witnesses the general upper bound of Theorem 1, or **(2)** find a point $p \in \mathbb{R}^2$ for which $\tau(p, \mathcal{C}) = \tau(\mathcal{C})$. We will restrict our attention to restricted classes of convex sets in the plane: segments, disks, and convex polygons of bounded complexity.

Problem (1). Consider a collection $\mathcal{S}$ of pairwise disjoint segments in the plane. We want to find a point p for which $\tau(p, \mathcal{S}) \leq \frac{2}{3}|\mathcal{S}|$, that is, which witnesses the bound of Theorem 1. By the argument in Section 4.1 we can extend the segments to lines and obtain an arrangement $\mathcal{A}$, thus reducing the problem to finding a point that witnesses the depth of the arrangement $\mathcal{A}$. This can be done by a recent result by Langerman and Steiger [7]. They give an optimal $O(n \log n)$ deterministic algorithm for finding the depth and a witness point in an arrangement of lines in the plane.

Next, consider a collection $\mathcal{D}$ of pairwise disjoint disks in the plane. As above, we want to find a point p for which $\tau(p, \mathcal{D}) \leq \frac{2}{3}|\mathcal{D}|$. We first recall the notion of the *Tukey median*: Given a finite set of points in the plane, the Tukey median is a point p which maximizes the minimum number of points of P belonging to a closed half-space that contains p. (Thus the center point theorem states that the Tukey median of a point set P is at least $\frac{1}{3}|P|$). By the argument in Section 4.1, we need only consider the set of centers of the disks in $\mathcal{D}$, and we can reduce our problem to finding a point that witnesses the Tukey median of the set of centers. In this case there is an $O(n \log n)$ randomized algorithm for computing the Tukey median of a collection of points in the plane due to Chan [5].

Remark. For both of the cases considered above there are simple examples $\mathcal{C}$ (of segments or disks) which show that for the given output point p, the value of $\tau(p, \mathcal{C})$ may be quite far from the value of $\tau(\mathcal{C})$.

Problem (2). In this case we will restrict our attention to collections $\mathcal{C}$ of pairwise disjoint convex sets in the plane for which we can efficiently compute the common tangents for each pair of sets of the collection. We also require that we can efficiently compute the intersection points of a given line with an object in $\mathcal{C}$. Examples are collections of polygons or collections of disks. We will outline an algorithm that produces in polynomial time a point $p \in \mathbb{R}^2$ for which $\tau(p, \mathcal{C})$ is minimal. This algorithm is based on the work of the first author and for more details we refer the reader to [6].

Let $\mathcal{A}$ be a collection of closed arcs on the unit circle ($\mathbb{S}^1$). When we orient the unit circle it is natural to speak of the start- and endpoint of an arc of $\mathcal{A}$. Let $\mathcal{A} = \{I_1, \ldots, I_n\}$ and $\mathcal{A}' = \{I'_1, \ldots, I'_n\}$ be collections of closed arcs on $\mathbb{S}^1$. We say that $\mathcal{A}$ and $\mathcal{A}'$ have the same *combinatorial type* if for every subset $J \subset [n]$ we have $\bigcap_{j \in J} I_j \neq \emptyset$ if and only if $\bigcap_{j \in J} I'_j \neq \emptyset$. Note that the combinatorial type depends only on the cyclic order of the start- and endpoints of the arcs of the collection.

For a given collection of arcs $\mathcal{A}$ and a point $x \in \mathbb{S}^1$, let $\pi_{\mathcal{A}}(x)$ denote the number of arcs containing the point x. It is a simple fact that the maximum of $\pi_{\mathcal{A}}(x)$ is attained for an endpoint of some arc of $\mathcal{A}$, and therefore depends only on the cyclic order of the start- and endpoints of the arcs in $\mathcal{A}$. In other words, the maximum of $\pi_{\mathcal{A}}(x)$ is determined by the combinatorial type of $\mathcal{A}$.

Now consider a collection $\mathcal{C} = \{C_1, \ldots, C_n\}$ of pairwise disjoint convex sets in the plane. For a given point $p \in \mathbb{R}^2$ we centrally project each $C_j \in \mathcal{C}$ to obtain an arc I_j on the unit circle centered at p. We denote by $\mathcal{A}_{\mathcal{C}}(p) = \{I_1, \cdots, I_n\}$ the resulting collection of arcs. Clearly we have

$$\tau(p, \mathcal{C}) = \max_{x \in \mathbb{S}^1} \pi(x).$$

Since two disjoint convex sets have exactly *four* common tangents (two outer and two inner tangents), by taking the common tangents of each pair of sets in $\mathcal{C}$, we obtain an arrangement of lines $\mathcal{L}_{\mathcal{C}}$ of size $O(n^2)$. The crucial observation is that for any two points p and p' belonging to the same cell of the arrangement $\mathcal{L}_{\mathcal{C}}$, the collections $\mathcal{A}_{\mathcal{C}}(p)$ and $\mathcal{A}_{\mathcal{C}}(p)$ will have the same combinatorial type. Therefore, if we choose a point p in each cell of the arrangement $\mathcal{L}_{\mathcal{C}}$, it suffices to compute $\tau(p,\mathcal{C})$ and take the minimum which gives us a point which witnesses $\tau(\mathcal{C})$.

Remark. If $\mathcal{C}$ is a collection of convex sets for which we can efficiently compute the common tangents and intersection points between objects and lines, the above argument can be turned into an algorithm that has a total running time of $O(n^4 \log n)$. Here the complexity is expressed in the total complexity of the input, e.g. in the case of polygons: the total number of sides.

4.3 k-wise disjoint objects

A collection of convex sets is called *k-wise disjoint* if no k members have a point in common. It follows immediately from our proof of Theorem 1 that for collections of k-wise disjoint convex sets that $\tau(n) \leq \frac{dn+(k-1)}{d+1}$.

4.4 Helly's theorem

Here we note that our proof of Theorem 1 can be modified to obtain a proof of Helly's theorem. First one must slightly modify our definition of the function $g_{\mathcal{C}}$: rather than considering $K_{\mathcal{C}}(p)$ as defined in the proof of Theorem 1, we consider $K(p) := K_{\mathcal{C}}(p) \setminus \{A\}_{A\in\mathcal{C}}$, thus when p is contained in a member of $\mathcal{C}$ we have $K(p) = \emptyset$. When $\mu(K(p)) > 0$ let $m_{\mathcal{C}}(p)$ be the center of mass of $K(p)$, and when $\mu(K(p)) = 0$ let $m_{\mathcal{C}}(p) = p$ (this definition of $g_{\mathcal{C}}(p)$ would have worked just as well for the proof of Theorem 1). Next, one proceeds as in the proof of Theorem 1 except that instead of considering all subcollections of size τ, we simply consider all subcollections consisting of a single convex body. The key observation is that if $m_A(p) \neq p$ then there exists a hyperplane orthogonal to $m_A(p) - p$ that strictly separates A from p. This yields a proof of Helly's theorem of similar flavor to the proof given by Chakerian [4].

4.5 Collections of γ-round objects

Here we consider collections of convex bodies that can be inscribed in a circle of unit radius and contain a circle of radius γ $(0 < \gamma < 1)$. We will refer to such convex bodies as γ-round. (In other words we are speaking of fat objects of roughly equal size). It turns out that the order of magnitude of $\tau(n)$ for the class of γ-round convex bodies is $\Theta(\sqrt{n \log n})$. Both the upper and lower bound can be derived as simple corollaries of the work by Alon-Katchalski-Pulleyblank [1]. They studied the asymptotic behavior of the minimum integer $f = f(n)$ such that for any collection of n pairwise disjoint γ-round convex bodies, there exists a direction such that any line in this direction intersects at most f of the convex bodies. The proof of the upper bound of $f(n)$ is by a simple counting argument, that could be applied to our case as well. However, in our argument we actually use only the existence of this bound. The construction for the lower bound is more involved and relies heavily on the famous construction due to Besicovitch [3] from his solution of the Kakeya problem.

Proof of Theorem 3. Let D be a disk that contains every member of $\mathcal{C}$. By the above mentioned result by Alon et al. (see Theorem 1.1 and the concluding remarks of [1]) we may assume that any horizontal line intersects $f(n) \in O(\sqrt{n \log n})$ sets of $\mathcal{C}$. Let $\beta = \arctan(\frac{\gamma}{\operatorname{diam} D})$. We choose a point p on the horizontal line through the center of D such that any ray starting at p which intersects D makes an angle with the horizontal direction that is less than β.

Now, it is enough to show that any ray R from p that intersects D, meets $O(\sqrt{n \log n})$ elements in $\mathcal{C}$. Indeed, if R intersects D in the points q_1 and q_2, the distance between the horizontal lines that pass through q_1 and q_2 is at most γ. Therefore any set of $\mathcal{C}$ that is met by R must meet at least one of the horizontal lines through q_1 or q_2. Thus R intersects $O(\sqrt{n \log n})$ sets of $\mathcal{C}$, which finishes the proof.

Proof of Theorem 4. Let $\mathcal{C}$ be the collection of $\lfloor \frac{n}{2} \rfloor$ pairwise disjoint disks of unit radius from the proof of the lower bound in Theorem 1.1 of [1]. The collection $\mathcal{C}$ has the property that for every direction α in the plane, there exists a line with direction α that meets $\Omega(\sqrt{n \log n})$ of the disks of $\mathcal{C}$. Let D be a disk that contains every member of $\mathcal{C}$, and $\beta = \arctan(\frac{1}{\operatorname{diam} D})$. We then have:

Claim 13. *Let z be a point in the plane such that any ray from z that intersect D and the ray from z that passes through the center of D form an angle less than β. There exists a ray from z that meets $\Omega(\sqrt{n \log n})$ of the disks of $\mathcal{C}$.*

Proof. Assume that the direction of the line through z and the center of D is the horizontal one. There exists a horizontal line L that meets $\Omega(\sqrt{n \log n})$ of the disks of $\mathcal{C}$. The line L intersects D in two points q_1 and q_2. For $i = 1, 2$ let R_i be the ray from z that passes through q_i, and let W be the wedge bounded by R_1 and R_2. By our choice of β it follows that the distance from any point in $L \cap D$ to R_i is less than 1 ($i = 1, 2$). This implies that any disk of $\mathcal{C}$ that meets L must also meet R_1 or R_2, and therefore one of these rays must meet at least half of the disks that meet L. $\square$

To complete the proof of Theorem 4 we consider two distinct copies $\mathcal{C}_1$ and $\mathcal{C}_2$ of the collection $\mathcal{C}$ contained in disks D_1 and D_2, respectively. By placing D_1 and D_2 sufficiently far apart, we can ensure that for any point in the plane the hypothesis of Claim 13 is satisfied for at least one of the collections $\mathcal{C}_i$, which concludes the proof.

4.6 Dimensions greater than two

The approaches sketched in Section 4.1 to bound $\tau_d(n)$ also works in dimensions greater than two:

For collections of pairwise disjoint $(d-1)$-dimensional convex sets in $\mathbb{R}^d$, using the notion of hyperplane depth [2] yields the bound $\tau_d(n) \leq \frac{dn}{n+1}$.

For collections of pairwise disjoint balls, using the center point theorem on the centers of the balls yields the bound $\tau_d(n) \leq \frac{dn}{d+1}$.

As pointed out in the introduction, for $d > 2$, we do not have any construction of a collection $\mathcal{C}$ of convex bodies in $\mathbb{R}^d$ for which $\tau(\mathcal{C}) \approx \frac{d}{d+1}|\mathcal{C}|$. There is however a simple construction $\mathcal{C}$ of pairwise disjoint convex bodies in $\mathbb{R}^d$ for which $\tau(\mathcal{C}) \approx \frac{1}{2}|\mathcal{C}|$: Consider a simple arrangement $\mathcal{A} = \{H_1, \ldots, H_k\}$ of hyperplanes through the origin. For the hyperplane H_1 we associate a pair of d-dimensional balls $\{B_1, B_1'\}$ disjoint from H_1, with centers that are antipodal about the origin and whose connecting segment is orthogonal to H_1. By making the distance between B_1 and H_1 (and thus also B_1' and H_1) sufficiently small, we are guaranteed that any line through the origin which does not meet B_1 and B_1' makes a small angle with H_1. This can be repeated for H_2 and balls $\{B_2, B_2'\}$, and by using a higher-dimensional analogue of Claim 9.2, we can make B_2 and B_2' disjoint from B_1 and B_1' and guarantee that any line through the origin that misses B_2 and B_2' makes a small angle with H_2. This process can be repeated for every $H_i \in \mathcal{A}$, resulting in a collection of $2k$ pairwise disjoint balls in $\mathbb{R}^d$ for which any line through the origin misses at most $2(d-1)$ balls, thus from any point in $\mathbb{R}^d$ there is a ray that meets at least $k-d+1$ of the balls.

We did find a collection $\mathcal{T}$ of pairwise disjoint triangles in $\mathbb{R}^3$ such that $\tau(\mathcal{T}) = \frac{2}{3}|\mathcal{T}| - 3$. We do not give the explicit construction here but mention that it comes from 'lifting' Construction 2 (Section 3).

Problem 14. *For every $n > 0$ and $d \geq 3$ provide a lower bound on $\tau_3(n)$ greater than $\frac{2n}{3} + O(1)$, and $\tau_d(n)$ greater than $\frac{n}{2} + O(1)$, or an upper bound less than $\frac{nd}{d+1} + O(1)$.*

Acknowledgment. We would like to thank Gábor Tardos for good advice, useful comments and suggestions.

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