Piercing quasi-rectangles—
On a problem of Danzer and Rogers

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Abstract

It is an old problem of Danzer and Rogers to decide whether it is possible to arrange $O(\frac{1}{\varepsilon})$ points in the unit square so that every rectangle of area $\varepsilon > 0$ within the unit square contains at least one of them. We show that the answer to this question is in the negative if we slightly relax the notion of rectangles, as follows.

Let δ be a fixed small positive number. A *quasi-rectangle* is a region swept out by a continuously moving segment s, with no rotation, so that throughout the motion the angle between the trajectory of the center of s and its normal vector remains at most δ. We show that the smallest number of points needed to pierce all quasi-rectangles of area $\varepsilon > 0$ within the unit square is $\Theta\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$.

1 Introduction

An old problem of Danzer and Rogers [Mo85, BeC87, CrFG91, BrC97] is the following: What is the area of the largest convex region not containing in its interior any one of n given points in a unit square? Vertical lines through the points partition the square into $n+1$ rectangles. At least one of these rectangles has area at least $\frac{1}{n+1}$, so this is clearly a lower bound. Can the order of magnitude of this bound be improved for all point sets, as n tends to infinity? We do not know. In 1982, Moser [Mo85] reported only a fairly weak upper bound, $O\left(\frac{\sqrt{\log n}}{n^{3/4}}\right)$, due to Fan Chung. Since then, the problem has been analyzed a little better. To explain the new developments, we need some preparation.

It is more convenient to rephrase the question as follows. Given $\varepsilon > 0$, what is the size of the smallest set of points with the property that every convex set of area ε

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within the unit square contains at least one of them. Denoting this minimum by $f(\varepsilon)$, we clearly have $f(\varepsilon) = \Omega(1/\varepsilon)$. The question is whether $f(\varepsilon) = O(1/\varepsilon)$ holds.

This problem can be regarded as a continuous version of the ε-net problem in an infinite *range space* $(N, \mathcal{R})$, where the ground set N is the unit square, the *ranges* $R \in \mathcal{R}$ are convex subsets of N, and we want to "hit" every range R with $|R \cap N| = |R| \geq \varepsilon|N| = \varepsilon$, where $|.|$ stands for the Lebesgue measure (area). A subset of N that intersects every such range is said to be an *ε-net* for the range space $(N, \mathcal{R})$.

A subset A of a ground set X is called *shattered* by the family $\mathcal{R}$ if for every subset $B \subseteq A$, one can find a range $R_B \in \mathcal{R}$ with $R_B \cap A = B$. The size of the largest shattered subset $A \subseteq X$, is said to be the *Vapnik-Chervonenkis dimension* (or *VC-dimension*) of the range space $(X, \mathcal{R})$ (see [VaC71, PaA95, Ch00]). It follows from the celebrated results of Haussler and Welzl [HaW87] that every range space of VC-dimension at most Δ admits an ε-net of size $O\left(\frac{\Delta}{\varepsilon} \log \frac{\Delta}{\varepsilon}\right)$.

We apply these ideas to our original problem. The area of the largest rectangle contained in a plane convex set R is at least half of the area of R [Ra52]. Thus, in order to hit (pierce) all plane convex sets of area ε in the unit square, it is sufficient to find an $\varepsilon/2$-net for all rectangles. The family of rectangles has bounded VC-dimension $\Delta < 10$. Therefore, the theorem of Haussler and Welzl implies that $f(\varepsilon) = O\left(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}\right)$.

It has been known for a long time that, in the "abstract" combinatorial setting, the logarithmic factor in the Haussler-Welzl theorem cannot be removed [PaW90, KoPW92]. More recently, following the work of Alon [Al10], the present authors constructed a variety of geometric range spaces with the same property [PaT11].

Nevertheless, it is perfectly possible that $f(\varepsilon) = O\left(\frac{1}{\varepsilon}\right)$, that is, all rectangles of area at least $\varepsilon > 0$ in the unit square can be pierced by $O\left(\frac{1}{\varepsilon}\right)$ points.

The aim of the present note is to show that, if we slightly enlarge the family of rectangles, by including "quasi-rectangles," then $O\left(\frac{1}{\varepsilon}\right)$ points do not suffice.

A rectangle is a region swept out by a line segment s moving orthogonally to itself. If we continuously translate s almost orthogonally to itself, without rotating it, so that the angle between s and the trajectory of its center always remains between $90-\delta$ and $90+\delta$ degrees for a fixed small $\delta > 0$, then we call the resulting region a *quasi-rectangle*. To be concrete, set $\delta = 1°$. The motion of the segment s is supposed to be monotone in the direction orthogonal to it, so that the segment is not allowed to turn back. Therefore, the area of a quasi-rectangle is equal to the length of s multiplied by the distance it traveled in the direction orthogonal to s.

A quasi-rectangle is not necessarily convex, but it is "almost" convex. Although the VC-dimension of the family of quasi-rectangles is unbounded, it is not hard to see that all quasi-rectangles of area ε inside the unit square can be stabbed by $O\left(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}\right)$ points. (See Lemma 2.) Our main theorem shows that this bound is tight up to a constant factor.

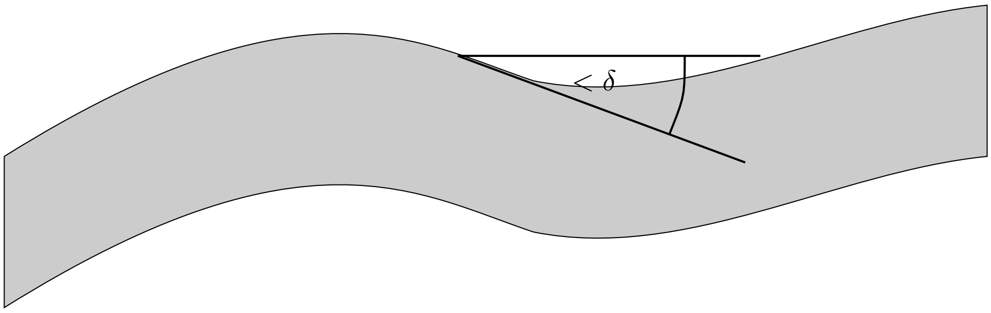

Figure 1: A quasi-rectangle.

Theorem 1 *For any $\varepsilon > 0$, let $F(\varepsilon)$ denote the smallest number of points in a set with the property that every quasi-rectangle of area ε inside the unit square contains at least one of them. We have $F(\varepsilon) = \Theta\left(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}\right)$.*

2 Quasi-rectangles—Proof of Theorem 1

Let $N = [0,1] \times [0,1]$ denote the unit square. For any integer $k \geq 6$, define a set of $O(k2^k)$ points in N, as follows. Let

$$S_k = \left\{(a/2^i, b/2^{k-i}) \mid 0 \leq i \leq k,\ 0 \leq a \leq 2^i,\ 0 \leq b \leq 2^{k-i}\right\},$$

where i, a, and b are integers.

Lemma 2 *Every quasi-rectangle $Q \subset N$ of area 2^{9-k} contains at least one point of S_k. Setting $k = \lceil \log \frac{1}{\varepsilon} \rceil + 9$, this yields*

$$F(\varepsilon) \leq |S_k| = O\left(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}\right).$$

Proof. Let $Q \subset N$ be a quasi-rectangle of area 2^{9-k}. We can choose a quasi-rectangle $Q' \subseteq Q$ of area at least 2^{6-k} such that the length of the segment s generating it is at most $2^{2-k/2}$. Indeed, if the original segment s_0 generating Q satisfies $|s_0| \leq 2^{2-k/2}$, then $Q' = Q$ will do. If $2^{2-k/2} < |s_0| \leq 2^{5-k/2}$, then a subsegment $s \subset s_0$ of length $2^{2-k/2}$ will sweep at least 1/8th of the area of Q, forming a suitable quasi-rectangle Q'. Finally, if $|s_0| > 2^{5-k/2}$, then we choose Q' to be a rectangle. In this case, Q contains a symmetric trapezoid on the base s_0 with altitude $2^{9-k}/|s_0|$ and with angles 89° at s_0. Within this trapezoid, we can find a rectangle R with one side, s_1, of length $2^{9-k}/|s_0|$ and area at least 2^{8-k}. In case $|s_1| \leq 2^{2-k/2}$, we can pick $Q' = R$ with $s = s_1$. Otherwise, let s be a subinterval of s_1 with length $2^{2-k/2}$ and let Q' be the part of R generated by s. Obviously, the area of Q' is at least one quarter of the area of R.

Suppose by symmetry that the angle between s and the x-axis is at most $45°$. Let i be the smallest integer with $2^{1-i} \leq |s|$. Clearly, we have $|s| < 2^{2-i}$. The distence between the starting and ending positions of the interval s sweeping Q' is at least $2^{6-k}/|s|$, thus the vertical component of the motion is at least

$$\cos 46° \cdot 2^{6-k}/|s| > 2|s| + 1/2^{k-i}.$$

This implies that during its motion the segment s must pass from one side of a horizontal line of the form $y = b/2^{k-i}$ to the other side. According to the definition of i, we have $|s| \geq 2/2^i$. Since the motion of s was almost orthogonal to itself, during the process s must have passed through a point of the form $(a/2^i, b/2^{k-i}) \in S_k$. This point belongs to $Q' \subseteq Q$. $\Box$

The lower bound on $F(\varepsilon)$ stated in Theorem 1 is an easy corollary of the following result.

Lemma 3 *Let $k > 1$, and let S be a set of points in the unit square, with $|S| \leq k2^k/320$. Then there exist an i ($k/2 \leq i \leq k$) and a sequence of axis-parallel closed squares $N_1, N_2, \ldots, N_{2^{2i-k}} \subset N$ of side length 2^{-i}, satisfying the following conditions.*

1. *$N_j \cap S = \emptyset$ for every j ($1 \leq j \leq 2^{2i-k}$).*
2. *For every j ($1 \leq j < 2^{2i-k}$), the square N_{j+1} can be obtained from N_j by translating it by a distance 2^{-i} in the positive direction parallel to one of the coordinate axes.*

Before turning to the proof of Lemma 3, we show how it implies Theorem 1.

Proof of Theorem 1 (using Lemma 3). A δ-quasi-rectangle is a set swept out by a segment s moving without rotation almost orthogonally to itself, in the sense that the angle between s and the trajectory of its center remains between $90 - \delta$ and $90 + \delta$ degrees.

Let $N^* = N_1 \cup N_2 \cup \ldots \cup N_{2^{2i-k}}$, where i and N_j denote the same objects as in Lemma3. We claim that N^* contains a $45°$-quasi-rectangle generated by a segment parallel to the $x + y = 0$ line, whose area is at least half of the area of N^*. Indeed, the whole area of N^*, with the exception of the lower left corner of N_1 and the upper right corner of $N_{2^{2i-k}}$, can be swept out by the diagonal of N_1 moving either vertically or horizontally. This region is a $45°$-quasi-rectangle and contains at least half of N^* if $i > k/2$. In the extremal $i = k/2$ case we have $N^* = N_1$ and the square determined by the midpoints of the edges of N_1 will do.

Thus, Lemma 3 implies that $F(\varepsilon) = \Omega\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$ holds for this relaxed definition of quasi-rectangles.

To complete the proof for any $\delta > 0$, choose an affine transformation $\tau : N \to N$ which maps N into a rhombus with smaller angle 2δ. If $|S| \leq k2^k/320$, then applying Lemma 2 to the set $S' = \tau^{-1}(S)$, we obtain a "path" $N^* \subseteq N$ which evades all elements of S' and has area $\Omega(2^{-k})$. However, this means that the set $\tau(N^*) \subseteq N$ is disjoint from S and contains a δ-quasi-rectangle whose area is at least half of the area of $\tau(N^*)$. The area of this quasi-rectangle is $\Omega(2^{-k})$, where the constant factor hidden in the Ω-notation depends on the mapping τ (and hence on δ). Therefore, we have $F(n) = \Omega\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$, for any fixed $\delta > 0$, as required. $\square$

For the proof of Lemma 3, we need some preparation. Let $S \subset N$ be a finite set of points.

Fix a positive integer i, and place a square grid on the plane, parallel to the coordinate axes, so that every elementary square (cell) has side length 2^{-i}. For every cell T and any integers a and b, let $T + (a, b)$ denote the cell obtained from T by a translation by the vector $(a/2^i, b/2^i)$. A sequence of cells $T_1, T_2, \ldots, T_k$ is called a *path* if $T_{j+1} = T_j + (1, 0)$ or $T_{j+1} = T_j + (0, 1)$ for every $1 \leq j < k$. The *length* of a path is the number k of cells in it. In notation, a path will often be identified with the union of its cells and it will be called *empty* if it is contained in the unit square N but contains no point from S. Note that we treat all points outside N the same way as the points in S leading to the notion that a path is not empty unless it is contained in N.

A *detour for a cell* T is a path, which consists of the cells $T + (-a, j)$ and $T + (j, a)$, where a is a fixed nonnegative integer and j runs through the integers with $-a \leq j \leq a$. Notice that, by this definition, a single cell T is also considered a detour for itself (with $a = 0$). We call this detour *trivial.* Every other detour is *nontrivial.* A path consisting of the cells $T_0 + (j, j)$ for $0 \leq j \leq c$ and $T_0 + (j+1, j)$ for $0 \leq j < c$ is called a *staircase* starting at T_0. Note that the length of a staircase and the length of a detour are always odd.

Lemma 4 *Let S be a set of points in the unit square N. If each cell of a staircase Σ admits an empty detour, then there exists an empty path whose length is the same as that of Σ.*

Proof. Let us choose an empty detour for each cell in Σ. We construct the empty path as the upper envelope (in a diagonal direction) of all these detours. More precisely, we include a cell T if T is contained in one of the detours we chose, but $T + (-a, a)$ is not included in any of them for any positive integer a. We leave the simple proof that the cells selected indeed form an empty path of the required length or longer to the reader. $\square$

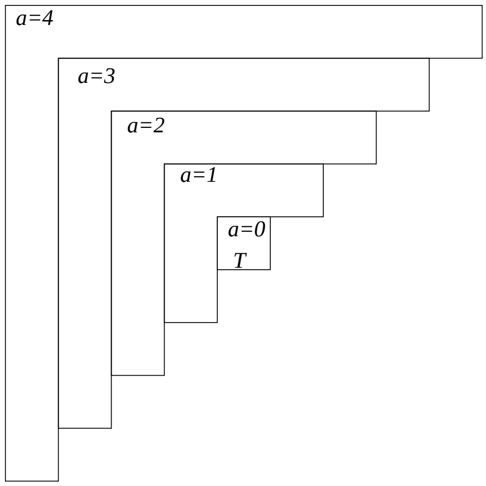

Figure 2: Detours for a cell T.

Proof of Lemma 3. Let S be a set of at most $k2^k/320$ points in the plane. For every i, $k/2 \leq i \leq k$, place a randomly and uniformly shifted axis-parallel grid G_i on the plane, with side length 2^{-i}. Assume for contradiction that there is no i for which G_i has an empty path of length at least 2^{2i-k}. By Lemma 4, this implies that for every i $(k/2 \leq i \leq k)$, every staircase of length at least 2^{2i-k} in G_i has a cell that does not admit an empty detour. Recall that we have defined "empty" to imply "contained in N", so any cell that does not lie entirely in N admits no empty detour. We call a cell *dead* if it lies entirely in N but still admits no empty detour.

Note that every dead cell T contains at least one element of S, otherwise T would serve as a (trivial) empty detour for itself. In every dead cell $T \subset N$ of side length 2^{-i}, assign to each point $p \in T \cap S$ the weight

$$w_i(p) = \frac{1}{|T \cap S|}.$$

We assign no weight to those elements of S that do not belong to a dead cell. Obviously, the total weight w_i we have distributed among the points of S is equal to the number of dead cells of G_i. For simplicity we assume no point in S lies on the boundary of a cell, an event of probability one.

It is easy to see that there are at least 2^{k-3} internally pairwise disjoint staircases of length at least 2^{2i-k} in G_i, which are entirely contained in N. According to our assumption, each of them has at least one dead cell. Thus, the total weight $\sum_{p\in S} w_i(p)$

distributed at level i is at least 2^{k-3}. Denoting the sum of these values over all i by W, we have

$$W = \sum_{i=\lceil k/2 \rceil}^{k} \sum_{p \in S} w_i(p) \geq k2^{k-4}.$$

Next, we give an upper bound on the expected total weight assigned to a single point of $p = (x, y) \in S$. Let us choose the grids G_i for all values of i in reverse order, starting with $i = k$. Let i' be the first (largest) integer, for which $w_{i'}(p) > 0$.

For any $t \geq 1$, let $N_t(p)$ denote the $2t - 1$ by $2t - 1$ square of grid cells of $G_{i'}$ such that p is contained in its central cell. Notice that if $N_t(p)$ lies entirely in N, then its part above the diagonal is the union of t detours for the cell containing p. Since the cell of $G_{i'}$ containing p is dead, $N_t(p)$ must contain at least t elements from S.

For any $i < i'$, the probability that the cell of G_i that contains p does not cover the whole square $N_t(p)$, is at most $2(2t-1)2^{i-i'} < 4t2^{i-i'}$. If this cell does cover $N_t(p)$, then $w_i(p) \leq 1/t$. (It is also possible that $w_i(p) = 0$ in this case, provided that its cell sticks out of the unit square N.) Thus, the expected weight of p given at level i satisfies

$$\mathrm{Exp}[w_i(p)] < 4t2^{i-i'} + 1/t.$$

Setting $t := 2^{\lceil (i'-i)/2 \rceil}$, the right-hand side becomes smaller than $5/2^{(i'-i)/2}$. Summing over all i, we obtain that for every $p \in S$

$$\mathrm{Exp}\left[\sum_{i=\lceil k/2 \rceil}^{k} w_i(p)\right] < \sum_{i=0}^{i'} 5 \cdot 2^{(i-i')/2} < 20.$$

Hence, the expected value of W, that is, the expected total weight assigned to all points of S summed over all levels i, is smaller than $20|S|$. Comparing this estimate to the lower bound $W \geq k2^{k-4}$, we obtain that $|S| > k2^k/320$, contradicting our assumption. This completes the proof of the lemma and hence Theorem 1. □

3 Concluding remarks

1. Recall the definition of δ-quasi-rectangles: A region is called a *δ-quasi-rectangle* if it swept out by a segment s translated almost orthogonally to itself with a possibly changing velocity vector that encloses an angle of absolute value at most δ with the positive normal vector of s. As $\delta \to 0$, a δ-quasi-rectangle resembles more and more a real rectangle.

It is well known that there is a set of $O(1/\varepsilon)$ points in the unit square $N = [0, 1] \times [0, 1]$ such that every axis-parallel rectangle $R \subset N$ with area at least $\varepsilon > 0$ contains at least one of them. It follows from the proof of Theorem 1 that this statement does not remain true for δ-quasi-rectangles, for any fixed $\delta > 0$. We have the following result.

Theorem 1'. *There exists an absolute constant $C > 0$ such that for any $\delta, \varepsilon > 0$ with $\delta > 2\varepsilon$, and for any set of points $S \subset \mathbb{R}^2$ with $|S| < C\frac{1}{\varepsilon}\log\frac{\delta}{\varepsilon}$, there is a δ-quasi-rectangle with vertical sides that does not contain any element of S. This bound is tight up to the value of the constant C.* $\square$

2. The theory of "weak" ε-nets allows us to handle a number of other piercing questions, related to the Danzer-Rogers problem. An interesting example discussed by Chazelle, Edelsbrunner et al. [ChEG95] is the following. Let γ denote the circle of radius $1/2$ centered at the point $(1/2, 1/2) \in [0,1] \times [0,1]$. At least how many points are needed to hit all convex sets $C \subset [0,1] \times [0,1]$ such that the total length of the part of γ covered by C is at least ε? Using a beautiful construction from hyperbolic geometry, it was shown in [ChEG95] that $O(1/\varepsilon)$ points suffice.

It is tempting to conjecture that a similar result holds when, instead of measuring the total length of the part of a *circle* covered by C, we measure the total length of the pieces of any other closed convex curve γ' lying within C. Unfortunately, in this case only a slightly weaker result is known. Alon, Kaplan, Nivasch et al. [AlKN08] proved that it is sufficient to pick $O((1/\varepsilon)\alpha(1/\varepsilon))$ points, where α denotes the inverse Ackermann function.

3. The notion of quasi-rectangles can be generalized to higher dimensions in more than one way. We consider two possible extensions of our results.

A. A set of points in d-dimensional Euclidean space is called a *δ-ball-trajectory* if it is the set of points swept by a ball of arbitrary radius that is continuously moved in an "almost straight" direction. By almost straight we mean that the direction of the motion must remain within an angle of $\delta < 90°$ to a fixed (but arbitrary) direction. Note that 2-dimensional δ-ball-trajectories are not exactly the same as δ-quasi-rectangles, but for our purposes they are equivalent. More precisely, any 2-dimensional δ-ball-trajectory T contains a δ-quasi-rectangle R with $|R| = c_\delta |T|$, and conversely, any δ-quasi-rectangle R contains a δ-ball-trajectory T with $|T| = c_\delta |R|$, where $c_\delta > 0$ is a constant. Again, for concreteness, set $\delta = 1°$, and call a $1°$-ball-trajectory simply a *ball-trajectory.*

Our methods naturally extend to ball-trajectories in any fixed dimension. To construct a hitting set in dimension d consider the point set $S^i_{a,b}$ consisting of points $(x_1, \ldots, x_d)$, where $0 \le x_j < 1$ for all j, $2^b x_i$ is an integer and $2^a x_j$ is an integer for all $j \neq i$. For a positive integer k let

$$S_k = \bigcup_{\substack{1 \le i \le d,\\ (d-1)a+b=k}} S^i_{a,b}.$$

We have $|S_k| = O_d(k2^k)$ and S_k hits all δ-ball-trajectories of volume at least $C_d/2^k$ that are within the unit cube $[0,1]^d$, where C_d is a constant depending on the dimension d. Thus, hitting sets for ball-trajectories of volume ε (or ε-nets for these sets) of size $O_d(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon})$ do exist.

To see that smaller hitting sets do not exist, it is enough to consider ball-trajectories where the ball is dragged parallel to the 2-dimensional plane determined by the first two coordinate axes. An argument very similar to the one we presented for quasi-rectangles shows that, even if we want to hit only ball-trajectories of volume ε with this special property, we need $\Omega_d(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon})$ points. Thus, we obtain

Theorem 5 *Let $d \geq 2$ be fixed. For any $\varepsilon > 0$, let $F_d(\varepsilon)$ denote the smallest number of points with the property that every ball-trajectory of volume ε inside the d-dimensional unit cube contains at least one of them. We have $F_d(\varepsilon) = \Theta\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$.*

B. In some sense, δ-ball-trajectories are reminiscent of boxes with $d-1$ out of d sides being equal. One can generalize this notion by getting rid of the last condition, as follows.

Let c be a fixed positive number. A function $f : \mathbb{R}^d \to \mathbb{R}$ is called *c-Lipschitz* if $|f(x) - f(y)| \leq c \cdot d(x,y)$ for any points x, y in the domain of f. Here $d(\cdot,\cdot)$ stands for the Euclidean distance. An interval is regarded as a 1-dimensional *c-quasi-box*. For $d \geq 2$, a subset B of the d-dimensional Euclidean space is called a c-quasi-box if the space can be written as the direct (orthogonal) product of a hyperplane H and the real line, and using these coordinates we have

$$B = \{(x,z) \mid x \in B_0, f(x) \leq z \leq f(x) + h\},$$

for a suitable $(d-1)$-dimensional c-quasi-box B_0, for a c-Lipschitz function f, and for a constant (height) $h > 0$.

Notice that in the plane c-quasi-boxes and δ-quasi-rectangles are exactly the same if $c = \tan\delta$. Moreover, our two-dimensional lower bound for the size of hitting sets for quasi-rectangles trivially yields a similar lower bound for the size of hitting sets for quasi-boxes in any dimension $d \geq 2$. This follows from the fact that the direct product of a quasi-rectangle with a (real) box is a quasi-box.

Corollary. *Let $d \geq 2, c > 0$ be fixed. For any $\varepsilon > 0$, let $G_{d,c}(\varepsilon)$ denote the smallest number of points with the property that every c-quasi-box of volume ε inside the d-dimensional unit cube contains at least one of them. We have $G_{d,c}(\varepsilon) = \Omega\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$.*

On the other hand, the obvious generalization of the construction of hitting sets for quasi-rectangles implies that $G_{d,c}(\varepsilon) = O\left(\frac{1}{\varepsilon}(\log\frac{1}{\varepsilon})^{d-1}\right)$, for any fixed $d \geq 3, c > 0$. It would be interesting to close this gap.

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