Overlap properties of geometric expanders

Extended abstract*

Jacob Fox† Mikhail Gromov Vincent Lafforgue Assaf Naor‡ János Pach§

Abstract

The *overlap number* of a finite $(d+1)$-uniform hypergraph H is the largest constant $c(H) \in (0,1]$ such that no matter how we map the vertices of H into $\mathbb{R}^d$, there is a point covered by at least a $c(H)$-fraction of the simplices induced by the images of its hyperedges. In [18], motivated by the search for an analogue of the notion of graph expansion for higher dimensional simplicial complexes, it was asked whether or not there exists a sequence $\{H_n\}_{n=1}^{\infty}$ of arbitrarily large $(d+1)$-uniform hypergraphs with bounded degree, for which $\inf_{n\geqslant 1} c(H_n) > 0$. Using both random methods and explicit constructions, we answer this question positively by constructing infinite families of $(d+1)$-uniform hypergraphs with bounded degree such that their overlap numbers are bounded from below by a positive constant $c = c(d)$. We also show that, for every d, the best value of the constant $c = c(d)$ that can be achieved by such a construction is asymptotically equal to the limit of the overlap numbers of the complete $(d+1)$-uniform hypergraphs with n vertices, as $n \to \infty$. For the proof of the latter statement, we establish the following geometric partitioning result of independent interest. For any d and any $\varepsilon > 0$, there exists $K = K(\varepsilon, d) \geqslant d+1$ satisfying the following condition. For any $k \geqslant K$, for any point $q \in \mathbb{R}^d$ and for any finite Borel measure μ on $\mathbb{R}^d$ with respect to which every hyperplane has measure 0, there is a partition $\mathbb{R}^d = A_1 \cup \ldots \cup A_k$ into k measurable parts of equal measure such that all but at most an ε-fraction of the $(d+1)$-tuples $A_{i_1}, \ldots, A_{i_{d+1}}$ have the property that either all simplices with one vertex in each A_{i_j} contain q or none of these simplices contain q.

1 Introduction

Let $G = (V, E)$ be an n-vertex graph. Think of G as a 1-dimensional simplicial complex, i.e., each edge is present in G as an actual interval. Assume that for every subset $S \subseteq V$ of size $\lfloor \frac{n}{2} \rfloor$ the number of edges joining S and $V \setminus S$ is at least $\alpha|E|$, for $\alpha \in (0,1]$. It follows that for every $f : V \to \mathbb{R}$, if we extend f to be a linear (or even just continuous) function defined also on the edges of G, there must necessarily exist a point $x \in \mathbb{R}$ such that $|f^{-1}(x)| \geqslant \alpha|E|$. Indeed, x can be chosen to be a median of the set $f(V) \subseteq \mathbb{R}$. In other words, no matter how we draw G on the line, its edges will heavily overlap.

As illustrated by this simple example, the above expander-like condition[1] on G implies that all of its embeddings in $\mathbb{R}$ satisfy a geometric overlap condition. This condition naturally extends to higher-dimensional simplicial complexes, and can thus serve as a potential definition of a higher-dimensional analogue of edge expansion[2]. Such investigations of high-dimensional geometric analogues of edge expansion were initiated in [18]. The present paper follows this approach.

In 1984, answering a question of Kárteszi, two undergraduates at Eötvös University, Boros and Füredi [8], proved the following theorem.

Theorem 1.1. ([8]) *For every set P of n points in the plane, there is a point (not necessarily in P) that belongs to at least $\left(\frac{2}{9} - o(1)\right)\binom{n}{3}$ closed triangles induced by the elements of P.*

The factor $\frac{2}{9}$ in Theorem 1.1 is asymptotically tight, as shown by Bukh, Matoušek and Nivasch in [10]. A short and elegant "book proof" of Theorem 1.1 was given by Bukh [9]. In Section 2, we present an alternative "topological" argument.

The theorem of Boros and Füredi has been generalized to higher dimensions. Bárány [4] proved that for

*Full version available at `http://arxiv.org/abs/1005.1392`. Dedicated to Endre Szemerédi on his 70th birthday.

†Research supported by an NSF Graduate Research Fellowship and a Princeton Centennial Fellowship.

‡Research supported by NSF grants CCF-0635078 and CCF-0832795, BSF grant 2006009, and the Packard Foundation.

§Supported by NSF Grant CCF-08-30272, and by grants from NSA, PSC-CUNY, the Hungarian Research Foundation OTKA, and BSF.

[1] It isn't quite edge expansion since we do not care about boundaries of small sets.

[2] To be precise, what we are detecting here is only that G contains a large expander, rather than being an expander itself.

every $d \in \mathbb{N}$ there exists a constant $c_d > 0$ such that given any set P of n points in $\mathbb{R}^d$, one can always find a point in at least $c_d n^d$ closed simplices whose vertices belong to P. In fact, the following stronger statement due to Pach [26] holds true.

THEOREM 1.2. *([26]) Every set P of n points in $\mathbb{R}^d$ has $d+1$ disjoint $\lfloor c'_d n \rfloor$-element subsets, $P_1, \dots, P_{d+1}$, such that* all *closed simplices with one vertex from each P_i have a point in common. Here $c'_d > 0$ is a constant depending only on the dimension d.*

Recall that a hypergraph $H = (V, E)$ consists of a set V and a set E of non-empty subsets of V. The elements of V are called vertices and the elements of E are called hyperedges. H is *d-uniform* if every hyperedge $e \in E$ contains exactly d vertices. The *degree* of a vertex $v \in V$ in H is the number of hyperedges containing v. To simplify the presentation, we introduce the following terminology.

DEFINITION 1. *Given a $(d+1)$-uniform hypergraph $H = (V, E)$, its* overlap number *$c(H)$ is the largest constant $c \in (0, 1]$ such that for every embedding $f : V \to \mathbb{R}^d$, there exists a point $p \in \mathbb{R}^d$ which belongs to at least $c|E|$ simplices whose vertex sets are hyperedges of H, i.e., there exists a set of hyperedges $S \subseteq E$ with $|S| \geqslant c|E|$ and $p \in \bigcap_{e \in S} \operatorname{conv}(f(e))$ (where $\operatorname{conv}(A)$ denotes the convex hull of $A \subseteq R^d$). An infinite family $\mathscr{H}$ of $(d+1)$-uniform hypergraphs is* highly overlapping *if there exists an absolute constant $c > 0$ such that $c(H) > c$ for every $H \in \mathscr{H}$. An infinite family of d*-dimensional simplicial complexes *is called* highly overlapping *if the family of $(d+1)$-uniform hypergraphs consisting of the vertex sets of their d-dimensional faces (their d-skeletons) is highly overlapping* [3].

Using this terminology, the Boros-Füredi theorem states that the family of all finite complete 3-uniform hypergraphs (or 2-skeletons of all complete simplicial complexes) is highly overlapping. Bárány's theorem says that the same is true for the family of complete $(d+1)$-uniform hypergraphs (or d-skeletons of complete simplicial complexes). The fact that the family of all finite complete graphs (1-skeletons of complete simplicial complexes) is highly overlapping (with $c = 1/2$) is trivial, but its higher dimensional generalizations are much more subtle.

[3] In [18] such simplicial complexes are called "polyhedra with large cardinalities."

It was a simple but very important graph-theoretic discovery by Pinsker [27] and others that there exist arbitrarily large edge *expanders* of bounded degree [19]. As we have seen at the beginning of this paper, expanders with a fixed rate of expansion are necessarily highly overlapping. This fact motivated the question, asked in [18], whether there exist infinite families of higher dimensional simplicial complexes with bounded degree that are highly overlapping. In other words, the question of [18] for 2-dimensional simplicial complexes asks whether a Boros-Füredi type theorem remains true if instead of *all* triangles determined by n points in the plane, we consider only "sparse" systems of triangles. In particular, do there exist arbitrarily large 3-uniform hypergraphs H, in which every vertex belongs to at most a constant number k of triples, and whose overlap numbers are bounded from below by an absolute positive constant?

In Section 3.1, we answer this question in the affirmative, by proving the following result.

THEOREM 1.3. *For any $\varepsilon > 0$, there exists a positive integer $k = k(\varepsilon)$ satisfying the following condition. There is an infinite sequence of 3-uniform hypergraphs H_n with n vertices and n tending to infinity, each of degree k, such that, for any embedding of the vertex set $V(H_n)$ in $\mathbb{R}^2$, there is a point belonging to at least a $(\frac{2}{9} - \varepsilon)$-fraction of all closed triangles induced by images of hyperedges of H_n. Here the constant $\frac{2}{9}$ cannot be improved.*

We also generalize Theorem 1.3 to $(d+1)$-uniform hypergraphs with $d \geqslant 2$.

THEOREM 1.4. *For every integer $d \geqslant 2$, there exist positive constants c_d and k_d with the following property. There is an infinite sequence of $(d+1)$-uniform hypergraphs H_n with n vertices and n tending to infinity, each of degree k_d, such that, for any embedding of the vertex set $V(H_n)$ in $\mathbb{R}^d$, there is a point in $\mathbb{R}^d$ that belongs to at least a c_d-fraction of all closed simplices induced by images of hyperedges of H_n.*

Among the most natural and powerful methods to construct good expanders is the use of certain Cayley graphs of finitely generated groups (see [22, 25, 14]), via arguments related to Kazhdan's property (T) (see [6]). Such graphs yield explicit constructions of expanders that have extremal spectral properties, namely Ramanujan graphs [22]. Being Cayley graphs of finitely generated groups, these constructions can be viewed as quotients of trees (Cayley graphs of free groups). It is natural to study hypergraph versions of this type of construction, based on quotients buildings (a type of higher dimensional simplicial complexes that extends the notion of a tree [29]). In particular, a notion of Ramanujan complex, which is a simplicial complex with extremal spectral properties analogous to Ramanujan graphs, was introduced and constructed in [3, 11, 20, 24, 23, 28]. Here we show that such constructions can yield highly overlapping bounded degree hypergraph families. Specifically, we show that for every integer $r \geqslant 2$, for a large enough odd prime power q, certain finite quotients of the building of $PGL_r(F)$, where F is a non-archimedian local field with residue field of order q, are highly overlapping r-uniform hypergraphs (with degree and overlap number depending only q, r). Rather than defining the relevant notions in this extended abstract, we refer to the full version of this paper [17] for precise definitions and statements. Instead, we state below the following concrete special case of our result, which follows from our argument in Section [17], in combination with a construction of Lubotzky, Samuels and Vishne [23].

Theorem 1.5. *For every odd prime p and every integer $r \geqslant 3$ there exist $k(p,r) \in \mathbb{N}$ and $c(p,r) > 0$ with the following property. For every $m \in \mathbb{N}$, the finite group $G = PGL_r(\mathbb{F}_{p^m})$, where $\mathbb{F}_{p^m}$ is the field of cardinality p^m, has a symmetric generating set $S \subseteq G$ of size bounded above by $k(p,r)$, such that the following holds. Consider the r-regular hypergraph H whose vertex set is G and whose hyperedges are those r-tuples $\{g_1, \ldots, g_r\} \subseteq G$ with $g_i g_j^{-1} \in S$ for all distinct $1 \leqslant i, j \leqslant r$ (i.e., H is the hypergraph consisting of all cliques of size r in the Cayley graph induced by S). Then there exist arbitrarily large integers m for which the hypergraph H has overlap number at least $c(p,r) > 0$.*

By Theorem 1.3, the best value of the constant c_2 in Theorem 1.4 is close to $\frac{2}{9}$, but in higher dimensions $d > 2$, we do not have very good estimates for c_d. Our goal is to show, roughly speaking, that the best constant in Theorem 1.4 is the same as the best constant in the Boros-Füredi-Bárány theorem (Theorem 1.1). To state this formally, it will be convenient to introduce some notation. Let $c(K_n^{d+1})$ be the overlap number of K_n^{d+1}, the complete $(d+1)$-uniform hypergraph on n vertices, and set

$$c(d) = \lim_{n\to\infty} c\left(K_n^{d+1}\right).$$

It is easy to show, via a straightforward point duplication argument, that the limit defining $c(d)$ exists, and the Boros-Füredi-Bárány theorem shows that $c(d) > 0$, for every d.

One might suspect that if H is a $(d+1)$-uniform hypergraph without isolated vertices, then $c(H) \leqslant c(d) + o(1)$, where the $o(1)$ term goes to 0 as the number of vertices of H tends to infinity. This is not the case. Consider, for example, the $(d+1)$-hypergraph H_n^{d+1} on n vertices, whose hyperedges are those sets of size $d+1$ that contain the first d vertices. In any general position embedding of the vertices of H_n^{d+1} in $\mathbb{R}^d$, any segment joining a pair of points sufficiently close and on opposite sides of the face consisting of the first d vertices stabs all the simplices induced by the images of hyperedges of H_n^{d+1}. Hence, $c(H_n^{d+1}) \geqslant 1/2$. However, $c(d)$ decays to 0 at least exponentially in d (see, e.g., [4, 10]). Despite this example, we show that our suspicion is *correct* for bounded degree hypergraphs.

Theorem 1.6. *For any d, $\Delta \in \mathbb{N}$, and $\varepsilon > 0$, there is $n(d, \Delta, \varepsilon) \in \mathbb{N}$ such that every $(d+1)$-uniform hypergraph H on $n \geqslant n(d, \Delta, \varepsilon)$ nonisolated vertices with maximum degree Δ satisfies $c(H) \leqslant c(d) + \varepsilon$.*

In the other direction, we show that there are regular $(d+1)$-uniform hypergraphs H of bounded degree such that $c(H)$ is at least $c(d) - \varepsilon$ for any given $\varepsilon > 0$.

Theorem 1.7. *For each $d \in \mathbb{N}$ and $\varepsilon > 0$, there is $r(d, \varepsilon) \in \mathbb{N}$ such that for every $r \geqslant r(d, \varepsilon)$ and sufficiently large n which is a multiple of $d+1$, there is a $(d+1)$-uniform, r-regular hypergraph H on n vertices with $c(H) \geqslant c(d) - \varepsilon$.*

The previous two theorems essentially show that $c(d)$ is the largest possible overlap number for bounded degree hypergraphs with sufficiently many nonisolated vertices.

The proof of the last theorem is based on a geometric partitioning result of independent interest. A $(d+1)$-tuple of subsets $S_1, \ldots, S_{d+1} \subseteq \mathbb{R}^d$ is said to be *homogeneous* with respect to a point $q \in \mathbb{R}^d$ if either all simplices with one vertex in each of the sets $S_1, \ldots, S_{d+1}$ contain q, or none of these simplices contain q.

Theorem 1.8. *For a positive integer d and $\varepsilon > 0$, there exists another positive integer $K = K(\varepsilon, d) \geqslant d+1$ such that for any $k \geqslant K$ the following statement is true. For any point $q \in \mathbb{R}^d$ and for any finite Borel measure μ on $\mathbb{R}^d$ with respect to which every hyperplane has measure 0, there is a partition $\mathbb{R}^d = A_1 \cup \ldots \cup A_k$ into k measurable parts of equal measure such that all but at most an ε-fraction of the $(d+1)$-tuples $A_{i_1}, \ldots, A_{i_{d+1}}$ are homogenous with respect to q.*

An *equipartition* of a finite set is a partition of the set into subsets whose sizes differ by at most one. A discrete version of Theorem 1.8 is the following.

Corollary 1.1. *Given a positive integer d and $\varepsilon > 0$, there exists another positive integer $K = K(\varepsilon, d) \geqslant d+1$ such that for any $k \geqslant K$ the following statement is true. For any finite set $P \subseteq \mathbb{R}^d$ and for any point $q \in \mathbb{R}^d$, there is an equipartition $P = P_1 \cup \ldots \cup P_k$ such that all but at most an ε-fraction of the $(d+1)$-tuples $P_{i_1}, \ldots, P_{i_{d+1}}$ are homogenous with respect to q.*

Notice that due to Bárány's result [4] that $c(d) > 0$, by taking $\varepsilon \ll c(d)$, Corollary 1.1 immediately implies Theorem 1.2.

The rest of the paper is organized as follows. Section 2 contains a detailed topological proof of the Boros-Füredi theorem (Theorem 1.1), following the approach in [18]. In the two subsections of Section 3, we present randomized constructions for Theorems 1.3 and 1.4. In the plane, these constructions are nearly optimal; their overlap numbers are close to the value $\frac{2}{9}$. In Section 4, we give a deterministic recipe how to turn certain families of explicitly given expander graphs into families of highly overlapping $(d+1)$-uniform hypergraphs. In the full version of this paper [17] we give a criterion which ensures that certain finite quotients of the building of $PGL_r(F)$ are highly overlapping r-uniform hypergraphs; this criterion implies in particular Theorem 1.5. Also in [17], we establish a Szemerédi-type theorem for infinite hypergraphs with a measure on their vertex sets. This is used in [17] for the proof of the geometric partition result Theorem 1.8. The proofs of Theorem 1.7 and Theorem 1.6 are deferred to the full version [17].

For the sake of clarity of the presentation, in the rest of this paper, we systematically omit floor and ceiling signs whenever they are not crucial. We shall also assume throughout that all embeddings of hypergraphs into $\mathbb{R}^d$ are such that the vertices are mapped to points in general position. Even though the corresponding statements for degenerate embeddings will then follow from standard limiting arguments, it is convenient to make this assumption in order to not deal explicitly with such degeneracies in each of the proofs.

Relevance to theoretical computer science. We are not aware of any direct algorithmic application of our results. However, the combinatorial structures investigated in this paper are intimately related to, and may have an impact on, several important areas of theoretical computer science, in the spirit of the well established applications of expander graphs.

In the context of computational geometry, the problems studied here are closely related to questions of central interest. Bounding the complexity of many classical geometric algorithms for range searching [13], computing higher order Voronoi diagrams [2], line fitting [31], etc., reduces to the investigation of the following extremal problem raised by Erdős, Lovász, Simmons, and Strauss [16] forty years ago. What is the maximum number of so-called *k-sets* determined by a set P of n points in $\mathbb{R}^{d+1}$, that is, the maximum number of k-element subsets $Q \subseteq P$ that can be separated from $P \setminus Q$ by a hyperplane? The first nontrivial upper bound, $n^{d+1-\varepsilon_d}$ for a suitable $\varepsilon_d > 0$, was established by Lovász [21] for $d = 1$, by Bárány, Füredi and Lovász [5] for $d = 2$, and by Živaljević and Vrećica [32] for larger values of d. See [7] for the best known general result of this type, and see [30] for an example of an application of k-sets to computer graphics. These bounds are based on various versions of the Colored Tverberg Theorem: For any r and d, there is a constant $n_{r,d}$ such that, given any $d+1$ point sets, $P_1, \ldots, P_{d_1}$ in $\mathbb{R}^d$, each of size at least $n_{r,d}$, we can always find r simplices, each of which contains precisely one vertex from every P_i, such that they share an interior point. In other words, no matter how we embed the vertices of a large balanced complete $(d+1)$-partite $(d+1)$-uniform hypergraph in d-space, there always exists a point in $\mathbb{R}^d$, which is covered by the image of many hyperedges.

2 A topological proof of the Boros-Füredi theorem

We will prove a somewhat stronger statement. Given a set P of n points in the plane, a *ray* (closed half-line) is said to be *exposed* if it has nonempty intersection with fewer than $n^2/9$ segments connecting point pairs in P. The set of all segments connecting two elements of P forms a *complete geometric graph* $K(P)$ on the vertex set P, and we refer to these segments as the *edges* of $K(P)$.

Proposition 2.1. *Given a set P of n points in the plane, one can always find a point q not necessarily in P such that no ray emanating from q is exposed.*

Suppose that such a point q does not belong to P. For each $p \in P$, ray emanating from q in the direction opposite to p intersects at least $n^2/9$ edges of $K(P)$. Each such edge, together with p, spans a triangle that contains q. Every triangle is counted at most *three* times, therefore the total number of triangles containing q is at least $n(n^2/9)/3 = n^3/27$. If q belongs to P, the number of (closed) triangles containing q is larger than $n^3/27$.

Thus, it is sufficient to prove Proposition 2.1. Suppose for a contradiction that for each point q of the plane, there is an exposed ray emanating from q. Let D denote a large disk around the origin O, which contains all elements of P, and let S^1 denote the boundary of D. For $\sigma \in \mathbb{R}^2 \setminus \{O\}$, we denote by $\mathrm{ray}(q,\sigma)$ the ray emanating from q in the direction parallel to $\overrightarrow{O\sigma}$.

Notice that for any two exposed rays, $\mathrm{ray}(q,\sigma)$ and $\mathrm{ray}(q,\tau)$, emanating from the same point, one of the two closed regions bounded by them contains fewer than $n/3$ points of P. Otherwise, one of the regions has x points of P with $n/3 \leqslant x \leqslant 2n/3$, and the two boundary rays together would intersect at least $x(n-x) \geqslant (n/3)(2n/3) = 2n^2/9$ edges, which implies at least one of them was not exposed.

Let I denote the set of all pairs $(q,\varrho) \in D \times S^1$, for which $\mathrm{ray}(q,\varrho)$ is exposed or belongs to the closed region bounded by two exposed rays, $\mathrm{ray}(q,\sigma)$ and $\mathrm{ray}(q,\tau)$, that contains fewer than $n/3$ points of P.

Claim 1. *The set I has the following properties:*

(a) I is an open *subset of $D \times S^1$,*

(b) $(\varrho,\varrho) \in I$ for all $\varrho \in S^1$,

(c) for every $q \in D$, the set $I_q \stackrel{\text{def}}{=} \{\varrho \in S^1 : (q,\varrho) \in I\}$ is a nonempty proper subinterval of S^1.

Proof. Parts (a) and (b) directly follow from the definition. It is also clear, by our contrapositive assumption, that I_q is a nonempty interval for every $q \in D$.

We have to show only that $I_q \neq S^1$. To see this, let $\mathrm{ray}(q,\varrho)$ be an exposed ray emanating from q, and let $\varrho' \in S^1$ be a direction such that both closed regions bounded by $\mathrm{ray}(q,\varrho)$ and $\mathrm{ray}(q,\varrho')$ contain at least $n/2$ points of P.

We claim that $\varrho' \notin I_q$. Otherwise, we can select two exposed rays, $\mathrm{ray}(q,\sigma)$ and $\mathrm{ray}(q,\tau)$, such that $\mathrm{ray}(q,\varrho')$ belongs to the closed region bounded by them which contains fewer than $n/3$ points. The *three* rays, $\mathrm{ray}(q,\varrho)$, $\mathrm{ray}(q,\sigma)$, and $\mathrm{ray}(q,\tau)$, cut the plane into *three* closed regions, and it is easy to see that each of them must contain fewer than $n/3$ points, which is a contradiction. Indeed, if e.g. the region bounded by $\mathrm{ray}(q,\varrho)$ and $\mathrm{ray}(q,\sigma)$ that does not contain $\mathrm{ray}(q,\tau)$ had at least $n/3$ points, then by the discussion above the closure of its complement had fewer than $n/3$ points, contradicting our assumption that both closed regions bounded $\mathrm{ray}(q,\varrho)$ and $\mathrm{ray}(q,\varrho')$ contain at least $n/2$ points.

Now we can obtain the desired contradiction, thus completing the proof of Proposition 2.1, by applying to $J \stackrel{\text{def}}{=} (D \times S^1) \setminus I$ the following version of the Brouwer fixed point theorem.

Lemma 2.1. *Let J be a closed subset of $D \times S^1$ with the property that for every $q \in D$ we have that $J_q \stackrel{\text{def}}{=} \{\varrho \in S^1 : (q,\varrho) \in J\}$ is a nonempty proper (closed) subinterval of S^1. Then $(\varrho,\varrho) \in J$, for some $\varrho \in S^1$.*

To see why Lemma 2.1 holds true, assume for contradiction that $(\varrho,\varrho) \notin J$, for all $\varrho \in S^1$. Write $J_S \stackrel{\text{def}}{=} J \cap (S^1 \times S^1)$, and let $Proj_1, Proj_2 : J_S \to S^1$ denote the projections onto the first and second coordinates, respectively. The fibers of $Proj_1$ are nonempty proper closed intervals, and therefore $Proj_1$ induces a bijection between $\pi_1(J_S)$ and $\pi_1(S^1) = \mathbb{Z}$. But, the contrapositive assumption implies that $Proj_1$ and $Proj_2$ are homotopic, and therefore $Proj_2$ also induces a bijection between $\pi_1(J_S)$ and $\pi_1(S^1)$. This is a contradiction since $Proj_2$ extends to J, and $\pi_1(J) = 0$ since J is fibered over D with fibers equal to intervals.

Lemma 2.1 contradicts part (b) of Claim 1. $\square$

3 Sparse constructions using the probabilistic method

In this section, we prove Theorems 1.3 and 1.4 using the probabilistic method. Our planar construction is nearly optimal, but in higher dimensions the overlap numbers of our hypergraphs will be far from maximal. We note that our proofs use a non-uniformly random choice of $(d+1)$-uniform hypergraphs of degree k_d, which is designed especially for our purposes. Nevertheless, the argument in Section **??**, which uses Theorem 1.8, shows that assuming the degree r satisfies a large enough lower bound depending on d (which is inferior to the bound on k_d obtained in this section), for a hypergraph H chosen uniformly at random among all $(d+1)$-uniform hypergraphs of degree r, with high probability $c(H)$ will be bounded below by a positive constant depending only on d (which is also inferior to the bound on c_d obtained in this section).

3.1 Highly overlapping triple systems—Proof of Theorem 1.3 The outline of the proof of Theorem 1.3 is the following. We first pick t randomly and independently selected partitions of the set $[n] = \{1,2,\ldots,n\}$ into parts of equal size b. We define H_n to be the 3-uniform hypergraph with vertex set $[n]$, consisting of all triples that lie in the same part in at least one of the t partitions. Finally, we will show that H_n

meets the requirements of Theorem 1.3.

We need the following simple technical lemma. A key ingredient that is used in the proof is the Chernoff bound for negatively associated random variables (see, e.g., [15]). It implies that if $A_1, \ldots, A_n$ are n mutually negatively correlated events in an arbitrary probability space such that A_i has probability p_i, then the probability that the number of A_i which occur exceeds the expected number $p_1 + \cdots + p_n$ by at least a is at most $e^{-2a^2/n}$.

LEMMA 3.1. *Suppose that $\delta > 0$, and let $b = \delta^{-3}$, $\beta = 2e^{-2\delta^2 b}$, $r = 4\beta^{-2}b$, $t = r\delta^{-1}$. If n is a sufficiently large multiple of b, then there exist t partitions $\mathcal{P}_1, \ldots, \mathcal{P}_t$ of $[n]$, each consisting of n/b parts of size b, with the following two properties:*

1. *any two parts of size b in different partitions have at most* two *elements in common,*
2. *for every subset $S \subseteq [n]$, there are fewer than r partitions $\mathcal{P}_i$ for which at least $\beta n/b$ parts contain at least $\left(\frac{|S|}{n} + \delta\right) b$ elements of S.*

Proof. We verify that t *randomly* selected partitions of $[n]$ into parts of equal size b almost surely have the desired properties. Fix a set $S \subseteq [n]$, and consider a random partition $\mathcal{P}$ of $[n]$ into parts $I_1, \ldots, I_{n/b}$ of size b. For any $1 \leqslant i \leqslant n/b$, let A_i denote the event that $|I_i \cap S| \geqslant \left(\frac{|S|}{n} + \delta\right) b$. For any $1 \leqslant j \leqslant b$, let $A_{i,j}$ denote the event that the jth element of I_i is in S. The events $A_{i,1}, \ldots, A_{i,b}$ are mutually negatively correlated and each of them has probability $|S|/n$. Thus, by Chernoff's bound [15], we have

$$\Pr[A_i] \leqslant e^{-2(\delta b)^2/b} = e^{-2\delta^2 b} = \frac{\beta}{2}.$$

Let X denote the event that at least $\beta n/b$ of the events $A_1, \ldots, A_{n/b}$ occur. Since the events $A_1, \ldots, A_{n/b}$ are also mutually negatively correlated and each has probability at most $\beta/2$, we can again apply the Chernoff bound [15] to obtain

$$\Pr[X] \leqslant e^{-2(\frac{\beta n}{2b})^2/(n/b)} = e^{-\frac{1}{2}\beta^2 n/b}.$$

Take t independent random partitions of $[n]$, $\mathcal{P}_1, \ldots, \mathcal{P}_t$, each consisting of n/b parts of size b. The probability that a given pair of parts of size b have at least 3 elements in common is at most $\binom{b}{3}\left(\frac{b}{n}\right)^3 \leqslant \frac{b^6}{6n^3}$. Since there are $\binom{tn/b}{2}$ such pairs, by linearity of expectation, the probability that there is a pair sharing at least 3 elements is at most $\binom{tn/b}{2}\frac{b^6}{6n^3} < \frac{t^2 b^4}{12n}$. Hence, by our choice of parameters, almost surely condition (1) will be satisfied.

For a fixed $S \subseteq [n]$, the probability that for at least r of the partitions $\mathcal{P}_1, \ldots, \mathcal{P}_t$, at least $\beta n/b$ of the b-element subsets of the partition have at least $\left(\frac{|S|}{n} + \delta\right) b$ elements in S is at most

$$\binom{t}{r}(\Pr[X])^r \leqslant \binom{t}{r} e^{-r\frac{1}{2}\beta^2 n/b} = \binom{t}{r} e^{-2n} \leqslant e^{-n}.$$

The number of subsets S of $[n]$ is 2^n. Hence, by linearity of expectation, the expected number of subsets S with property (2) is $o(1)$. We conclude that there are t such partitions with the desired properties.

Let $\delta = \varepsilon/50$ and $k = t\binom{b-1}{2}$. Consider the 3-uniform hypergraph H_n with $V(H_n) = [n]$, the hyperedges of which are those triples that lie in the same part in at least one (hence, precisely one) of the partitions $\mathcal{P}_1, \ldots, \mathcal{P}_t$ meeting the requirements of Lemma 3.1. Clearly, in H_n, each vertex belongs to $k = t\binom{b-1}{2}$ hyperedges.

The proof of Theorem 1.3 can now be completed by adapting the idea of Bukh [9]. Consider an embedding of the vertices of H_n in the plane. We shall use the following lemma of Ceder [12]:

LEMMA 3.2. (CEDER [12]) *Assume that n is divisible by 6. Given any set of n points in the plane, there are three concurrent lines that divide the plane into 6 angular regions, each containing roughly the same number of points. More precisely, there are disjoint $\frac{n}{6}$-element point sets $S_1, \ldots, S_6$ such that S_i is contained in the closure of region i.*

We shall assume throughout the n is divisible by 6. Let $S_1, \ldots, S_6$ be the sets from Lemma 3.2, and let p denote the intersection point of the *three* lines from Lemma 3.2. By a simple case analysis, Bukh [9] showed that, for every choice of *six* points, one from each S_i, at least 8 of the $\binom{6}{3} = 20$ triangles induced by them contain p.

Let $I \subseteq [n]$ be a b-element set such that $|I \cap S_i| \leqslant (\frac{|S_i|}{n} + \delta)b = (1 + 6\delta)\frac{b}{6}$, for $1 \leqslant i \leqslant 6$. Obviously, we have

$$|I \cap S_i| \geqslant b - 5(1 + 6\delta)\frac{b}{6} \geqslant (1 - 30\delta)\frac{b}{6},$$

for every i. Each of the

$$\prod_{i=1}^{6} |I \cap S_i| \geqslant (1 - 30\delta)^6 \left(\frac{b}{6}\right)^6$$

6-element sets with one vertex from each $I \cap S_i$ induces at least 8 triangles that contain point p. Each of these triangles belongs to at most $(1+6\delta)^3(\frac{b}{6})^3$ such 6-element sets. Thus, there are at least

$$8\frac{(1-30\delta)^6\left(\frac{b}{6}\right)^6}{(1+6\delta)^3(\frac{b}{6})^3} \geqslant \frac{1}{27}(1-200\delta)b^3 > (1-200\delta)\frac{2}{9}\binom{b}{3}$$

triangles induced by *three* vertices in I which contain p.

According to part 2 of Lemma 3.1, for every i, $1 \leqslant i \leqslant 6$, fewer than r partitions $\mathcal{P}_j$ have the property that at least $\beta\frac{n}{b}$ of their parts contain at least $(1+6\delta)\frac{b}{6}$ elements of S_i. Hence, the total number of b-element parts I in all t partitions, for which $|I \cap S_i| > (1+6\delta)\frac{b}{6}$ for some i, $1 \leqslant i \leqslant 6$, is smaller than

$$6r\frac{n}{b} + 6t\beta\frac{n}{b} = 6\delta t\frac{n}{b} + 6\beta t\frac{n}{b} \leqslant 10\delta t\frac{n}{b}.$$

It follows that the fraction of the $t\frac{n}{b}\binom{b}{3}$ hyperedges of H_n that contain point p in this embedding is at least

$$(1-10\delta)(1-200\delta)\frac{2}{9} \geqslant (1-210\delta)\frac{2}{9} \geqslant \frac{2}{9} - \varepsilon,$$

which completes the proof of Theorem 1.3. $\square$

3.2 Higher dimensions—Proof of Theorem 1.4

As in the proof of Theorem 1.3, we establish Theorem 1.4 using Lemma 3.1. We may assume that $c'_d = 1/m$ with m an integer, where c'_d is the constant in Theorem 1.2, and let n be a multiple of m. Set $\delta = \frac{1}{2m(m-1)}$ and apply Lemma 3.1. Consider now the $(d+1)$-uniform hypergraph H_n with $V(H_n) = [n]$, the hyperedges of which are those $(d+1)$-element sets that lie in the same part in at least one (hence, precisely one) of the partitions $\mathcal{P}_1, \dots, \mathcal{P}_t$ meeting the requirements of Lemma 3.1. Clearly, in H_n, each vertex belongs to $k_d = t\binom{b-1}{d}$ hyperedges.

Consider now any embedding of $V(H_n)$ into $\mathbb{R}^d$, and let P denote the image of $V(H_n)$. By Theorem 1.2, one can find disjoint $c'_d n$-element subsets $P_1, P_2, \dots, P_{d+1} \subseteq P$ and a point q such that picking one element from each subset P_i, their convex hull always contains q. We extend this to a partition $P = P_1 \cup \dots \cup P_m$ into subsets of size n/m by picking the P_i for $d+1 < i \leqslant m$ of size n/m arbitrarily.

Let $I \subseteq [n]$ be a b-element set such that for all $1 \leqslant i \leqslant m$,

$$|I \cap P_i| \leqslant \left(\frac{|P_i|}{n} + \delta\right) b = \left(1 + \frac{1}{2(m-1)}\right)\frac{b}{m}.$$

Obviously, we have

$$|I \cap P_i| \geqslant b - (m-1)\left(1 + \frac{1}{2(m-1)}\right)\frac{b}{m} = \frac{b}{2m},$$

for every $1 \leqslant i \leqslant m$. Each of the

$$\prod_{i=1}^{d+1} |I \cap P_i| \geqslant \left(\frac{b}{2m}\right)^{d+1}$$

$(d+1)$-element sets with one vertex from each $I \cap P_1, \dots, I \cap P_{d+1}$ induces a closed simplex containing point q. Hence, the fraction of $(d+1)$-element subsets of I which induce a closed simplex that contains point q is at least

$$\left(\frac{b}{2m}\right)^{d+1}\binom{b}{d+1}^{-1} \geqslant (d+1)!\left(\frac{c'_d}{2}\right)^{d+1}.$$

According to part (2) of Lemma 3.1, for every $1 \leqslant i \leqslant m$, fewer than r partitions $\mathcal{P}_j$ have the property that at least $\beta\frac{n}{b}$ of their parts contain at least $(\frac{|P_i|}{n}+\delta)b$ elements of P_i. Hence, the total number of b-element parts I in all t partitions, for which $|I \cap P_i| > (\frac{|P_i|}{n}+\delta)b$ for some $1 \leqslant i \leqslant m$, is smaller than

$$mr\frac{n}{b} + mt\beta\frac{n}{b} = m\delta t\frac{n}{b} + m\beta t\frac{n}{b} \leqslant \frac{3}{4}\cdot t \cdot \frac{n}{b}.$$

Hence, the fraction of the $t\frac{n}{b}\binom{b}{d+1}$ hyperedges of H_n that contain the point q in this embedding is at least $\frac{1}{4}(d+1)!\left(\frac{c'_d}{2}\right)^{d+1}$. $\square$

4 Deterministic constructions using expander graphs

In the next two subsections, we present deterministic constructions based on expander graphs, to provide alternative proofs of Theorem 1.3 and Theorem 1.4. These proofs yield significantly better bounds on $k(\varepsilon)$ and k_d in Theorem 1.3 and Theorem 1.4, respectively. As in the previous section, the proof gives a nearly optimal bound in the plane, but not in higher dimension.

4.1 Highly overlapping triple systems—second proof of Theorem 1.3

Fix integers $k, n \in \mathbb{N}$, with n divisible by 6, and let $G = (\{1, \dots, n\}, E)$ be a k-regular graph on the vertex set $\{1, \dots, n\}$. Let $k = \lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n$ be the eigenvalues of the adjacency matrix of G in decreasing order, and write $\lambda = \max_{i \in \{2,\dots,n\}} |\lambda_i|$. For any $S, T \subseteq \{1, \dots, n\}$ let $E(S,T)$ denote the number of ordered pairs $(i,j) \in S \times T$ such that $ij \in E$. The expander mixing lemma (see Corollary 9.2.5 in [1]) states that

$$\left| E(S,T) - \frac{k|S|\cdot|T|}{n} \right| \leqslant \lambda\sqrt{|S|\cdot|T|}. \tag{4.1}$$

For every $i \in \{1, \dots, n\}$ let $N_G(i) \stackrel{\text{def}}{=} \{j \in \{1, \dots, n\} : ij \in E\}$ denote its neighborhood in G.

Define a hypergraph H on the vertex set $\{1,\dots,n\}$ by letting $E(H)$ consist of those triples $\{i,j,\ell\}$ for which there exists $r\in\{1,\dots,n\}$ such that $ir,jr,\ell r\in E$, i.e., $i,j,\ell\in N_G(r)$. Assume from now on that the graph G is quadrilateral-free. This implies that the hyperedges in H corresponding to three vertices $i,j,\ell\in N_G(r)$ cannot arise from neighborhoods of vertices of G other than r itself. Hence the 3-uniform hypergraph corresponding to H is $k\binom{k-1}{2}$-regular and $|E(H)|=\binom{k}{3}n$.

Fix $\varepsilon,\delta\in(0,1)$. Let $\{P_i\}_{i=1}^6$ be a partition of $\{1,\dots,n\}$ such that $|P_j|=\frac{n}{6}$ for all $1\leqslant j\leqslant 6$. Write

$$A_j=\left\{i\in\{1,\dots,n\}:\ |N_G(i)\cap P_j|<\frac{(1-\delta)k}{6}\right\}.$$

Then, by definition, we have $E(A_j,P_j)<|A_j|\frac{(1-\delta)k}{6}$. An application of (4.1) yields the inequality:

$$\begin{aligned}(4.2)\quad |A_j|\frac{(1-\delta)k}{6}&\geqslant\frac{k|A_j|\cdot|P_j|}{n}-\lambda\sqrt{|A_j|\cdot|P_j|}\\&=\frac{k|A_j|}{6}-\lambda\sqrt{\frac{n|A_j|}{6}},\end{aligned}$$

which simplifies to

$$|A_j|\leqslant\frac{6\lambda^2 n}{\delta^2k^2}.$$

Thus, if we define

$$(4.3)\quad A=\left\{i\in\{1,\dots,n\}:\ |N_G(i)\cap P_j|\geqslant\frac{(1-\delta)k}{6}\quad\forall j\in\{1,\dots,6\}\right\},$$

then

$$(4.4)\qquad A\geqslant n-\sum_{j=1}^{6}|A_j|\geqslant n\left(1-\frac{36\lambda^2}{\delta^2k^2}\right).$$

We shall assume from now on that $\frac{36\lambda^2}{\delta^2k^2}<1$. We also note that for every $i\in A$ and $j\in\{1,\dots,6\}$ we have

$$\begin{aligned}(4.5)\quad |N_G(i)\cap P_j|&\leqslant k-\sum_{r\in\{1,\dots,6\}\smallsetminus\{j\}}|N_G(i)\cap P_r|\\&\leqslant k-5\frac{(1-\delta)k}{6}=\frac{(1+5\delta)k}{6}.\end{aligned}$$

Let $x_1,\dots,x_n\in\mathbb{R}^2$ be an embedding of $\{1,\dots,n\}$ in the plane. Let $S_1,\dots,S_6$ be a partition of $\{x_1,\dots,x_n\}$, as in the first proof of Theorem 1.3, which corresponds to the three concurrent lines from Lemma 3.2, whose common intersection point is $p\in\mathbb{R}^2$. We shall use the above reasoning (and notation) for the partition $P_1,\dots,P_6$ of $\{1,\dots,n\}$ given by $P_j=\{i\in\{1,\dots,n\}:\ x_i\in S_i\}$.

Fix $i\in A$, where A is as in (4.3). For every $(j_1,\dots,j_6)\in\prod_{r=1}^{6}(N_G(i)\cap P_r)$ at least 8 of the 20 triangles induced by the points $\{x_{j_1},\dots,x_{j_6}\}$ contain p. By the definition of A, there are at least $\left(\frac{(1-\delta)k}{6}\right)^6$ such 6-tuples, while, using (4.5), each of these triangles that contains p belongs to at most $\left(\frac{(1+5\delta)k}{6}\right)^3$ such 6-tuples. Observe also that by the definition of H, since all of these triangles correspond to neighbors of i, their corresponding triples of indices belong to $E(H)$, and since G is quadrilateral-free, they cannot arise from the above reasoning with i replaced by any other vertex. Thus, the number of triangles that are images of hyperedges of H and contain p is at least

$$\begin{aligned}(4.6)\quad 8\cdot\frac{\left(\frac{(1-\delta)k}{6}\right)^6}{\left(\frac{(1+5\delta)k}{6}\right)^3}\cdot|A|&\overset{(4.4)}{\geqslant}\frac{(1-\delta)^6k^3}{27(1+5\delta)^3}n\left(1-\frac{36\lambda^2}{\delta^2k^2}\right)\\&=\left(1-O\left(\delta+\frac{\lambda^2}{\delta^2k^2}+\frac{1}{k}\right)\right)\cdot\frac{2}{9}\binom{k}{3}n.\end{aligned}$$

For arbitrarily large n, we can choose the graph G so that it is quadrilateral-free and $\lambda\leqslant 2\sqrt{k}$ (e.g., Ramanujan graphs work—see [22, 19]). By choosing $\delta\asymp\varepsilon$ and $k\asymp\frac{1}{\varepsilon^3}$ in (4.6), we get that p is in at least $\left(\frac{2}{9}-\varepsilon\right)|E(H)|$ of the triangles in that are images of hyperedges of H. Note that the degree of H is $O(k^3)=O\left(\frac{1}{\varepsilon^9}\right)$. This proves Theorem 1.3 with the bound $k(\varepsilon)=O\left(\frac{1}{\varepsilon^9}\right)$. $\square$

4.2 Higher dimensions—second proof of Theorem 1.4 Here we shall use a variant of the construction in Section 4.1, to give an alternative proof of Theorem 1.4. We use the notation from Section 4.1, and we assume that $k\geqslant d$. Fix n vectors $x_1,\dots,x_n\in\mathbb{R}^d$. Define a set of d-dimensional simplices H' whose vertices are in $\{x_1,\dots,x_n\}$ by taking the simplex whose vertices are the distinct vectors $\{x_{j_1},x_{j_2},\dots,x_{j_{d+1}}\}$ if and only if we have $j_1j_2,j_2j_3,\dots j_dj_{d+1}\in E$. In other words, the simplices in H' correspond to non-returning walks of length d in G. Thus, $|H'|\leqslant k^dn$.

Let $P_1,\dots,P_{d+1}\subseteq\{x_1,\dots,x_n\}$ be the disjoint subsets from Theorem 1.2, i.e., $|P_i|\geqslant c_d'n$, and all the closed simplices with one vertex in each of the sets $\{P_1,\dots,P_{d+1}\}$ have a point in common. Set $Q_i=\{j\in\{1,\dots,n\}:\ x_j\in P_i\}$. Define $\widetilde{Q}_{d+1}=Q_{d+1}$ and inductively for $i \in \{2, \ldots, d+1\}$,

$$\widetilde{Q}_{i-1} = \left\{ j \in Q_{i-1} : \exists \ell \in \widetilde{Q}_i \; j\ell \in E \right\}.$$

By definition, there are no edges between $Q_{i-1} \smallsetminus \widetilde{Q}_{i-1}$ and $\widetilde{Q}_i$. It follows from (4.1) that

$$\frac{k}{n} \left| Q_{i-1} \smallsetminus \widetilde{Q}_{i-1} \right| \cdot \left| \widetilde{Q}_i \right| \leqslant \lambda \sqrt{\left| Q_{i-1} \smallsetminus \widetilde{Q}_{i-1} \right| \cdot \left| \widetilde{Q}_i \right|}.$$

Thus, we have

$$\frac{\lambda^2 n^2}{k^2} \geqslant \left(|Q_{i-1}| - \left| \widetilde{Q}_{i-1} \right| \right) \left| \widetilde{Q}_i \right| \geqslant \left(c'_d n - \left| \widetilde{Q}_{i-1} \right| \right) \left| \widetilde{Q}_i \right|,$$

or

$$\left| \widetilde{Q}_{i-1} \right| \geqslant c'_d n - \frac{\lambda^2 n^2}{k^2 \left| \widetilde{Q}_i \right|}. \tag{4.7}$$

Assuming that $\lambda \leqslant \frac{c'_d}{2} k$, inequality (4.7) implies by induction that for all $i \in \{1, \ldots, d+1\}$ we have

$$\left| \widetilde{Q}_{i-1} \right| \geqslant \frac{c'_d}{2} n$$

(for $i = d+1$ this follows from our assumption, arising from Theorem 1.2, on the cardinality of P_{d+1}). Thus, $\left| \widetilde{Q}_1 \right| \geqslant \frac{c'_d}{2} n$, and by construction any point $j \in \widetilde{Q}_1$ can be completed to a walk in G of length d whose ith vertex is in Q_i. Each such walk corresponds to a simplex in H', and by Theorem 1.2, all of these simplices have a common point. Thus, the number of simplices in H' which have a common point is at least $\frac{c'_d}{2} n \geqslant \frac{c'_d}{2k^d} |H'|$. Since there exist arbitrarily large graphs G with $\lambda \leqslant \frac{c'_d}{2} k$ and $k \leqslant k_d$ (e.g., for Ramanujan graphs we can take $k_d \asymp \frac{1}{(c'_d)^2}$), this completes our deterministic proof of Theorem 1.4. $\square$

References

[1] N. Alon and J. H. Spencer. *The probabilistic method*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., Hoboken, NJ, third edition, 2008. With an appendix on the life and work of Paul Erdős.

[2] F. Aurenhammer. Voronoi diagrams - a survey of a fundamental geometric data structure. *ACM Comput. Surv.*, 23(3):345–405, 1991.

[3] C. M. Ballantine. Ramanujan type buildings. *Canad. J. Math.*, 52(6):1121–1148, 2000.

[4] I. Bárány. A generalization of Carathéodory's theorem. *Discrete Math.*, 40(2-3):141–152, 1982.

[5] I. Bárány, Z. Füredi, and L. Lovász. On the number of halving planes. *Combinatorica*, 10(2):175–183, 1990.

[6] B. Bekka, P. de la Harpe, and A. Valette. *Kazhdan's property (T)*, volume 11 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008.

[7] P. Blagojević, B. Matschke, , and G. Ziegler. bounds for a colorful Tverberg–Vrećica type problem. Preprint, 2009. Available at `http://arxiv.org/abs/0911.2692`.

[8] E. Boros and Z. Füredi. The number of triangles covering the center of an n-set. *Geom. Dedicata*, 17(1):69–77, 1984.

[9] B. Bukh. A point in many triangles. *Electron. J. Combin.*, 13(1):Note 10, 3 pp. (electronic), 2006.

[10] B. Bukh, J. Matoušek, and G. Nivasch. Stabbing simplices by points and flats. 2006. Preprint availabel at arXiv:0804.4464, to appear in Discrete Comput. Geom.

[11] D. I. Cartwright, P. Solé, and A. Żuk. Ramanujan geometries of type $\tilde{A}_n$. *Discrete Math.*, 269(1-3):35–43, 2003.

[12] J. G. Ceder. Generalized sixpartite problems. *Bol. Soc. Mat. Mexicana (2)*, 9:28–32, 1964.

[13] B. Chazelle and F. P. Preparata. Halfspace range search: an algorithmic application of k-sets. *Discrete Comput. Geom.*, 1(1):83–93, 1986.

[14] G. Davidoff, P. Sarnak, and A. Valette. *Elementary number theory, group theory, and Ramanujan graphs*, volume 55 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2003.

[15] D. Dubhashi and D. Ranjan. Balls and bins: a study in negative dependence. *Random Structures Algorithms*, 13(2):99–124, 1998.

[16] P. Erdős, L. Lovász, A. Simmons, and E. G. Straus. Dissection graphs of planar point sets. In *A survey of combinatorial theory (Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1971)*, pages 139–149. North-Holland, Amsterdam, 1973.

[17] J. Fox, M. Gromov, V. Lafforgue, A. Naor, and J. Pach. Overlap properties of geometric expanders. Preprint available at `http://arxiv.org/abs/1005.1392`, 2010.

[18] M. Gromov. Singularities, expanders and topology of maps, part 2: From combinatorics to topology via algebraic isoperimetry. 2008. To appear in Geom. Funct. Anal.

[19] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc. (N.S.)*, 43(4):439–561 (electronic), 2006.

[20] W.-C. W. Li. Ramanujan hypergraphs. *Geom. Funct. Anal.*, 14(2):380–399, 2004.

[21] L. Lovász. On the number of halving lines. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 14:107–108 (1972), 1971.

[22] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.

[23] A. Lubotzky, B. Samuels, and U. Vishne. Explicit constructions of Ramanujan complexes of type $\tilde{A}_d$. *European J. Combin.*, 26(6):965–993, 2005.

[24] A. Lubotzky, B. Samuels, and U. Vishne. Ramanujan complexes of type $\tilde{A}_d$. *Israel J. Math.*, 149:267–299, 2005. Probability in mathematics.

[25] G. A. Margulis. Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators. *Problemy Peredachi Informatsii*, 24(1):51–60, 1988.

[26] J. Pach. A Tverberg-type result on multicolored simplices. *Comput. Geom.*, 10(2):71–76, 1998.

[27] M. S. Pinsker. On the complexity of a concentrator. In *7th International Telegraffic Conference*, pages 318/1–318/4. 1973.

[28] A. Sarveniazi. Explicit construction of a Ramanujan $(n_1, n_2, \ldots, n_{d-1})$-regular hypergraph. *Duke Math. J.*, 139(1):141–171, 2007.

[29] T. Steger. Local fields and buildings. In *Harmonic functions on trees and buildings (New York, 1995)*, volume 206 of *Contemp. Math.*, pages 79–107. Amer. Math. Soc., Providence, RI, 1997.

[30] U. Wagner. On the number of corner cuts. *Adv. in Appl. Math.*, 29(2):152–161, 2002.

[31] P. Yamamoto, K. Kato, K. Imai, and H. Imai. Algorithms for vertical and orthogonal L_1 linear approximation of points. In *Proceedings of the Fourth Annual Symposium on Computational Geometry (Urbana, IL, 1988)*, pages 352–361, New York, 1988. ACM.

[32] R. T. Živaljević and S. T. Vrećica. The colored Tverberg's problem and complexes of injective functions. *J. Combin. Theory Ser. A*, 61(2):309–318, 1992.