Drawing cubic graphs with at most five slopes

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Abstract

We show that every graph G with maximum degree three has a straight-line drawing in the plane using edges of at most five different slopes. Moreover, if G is connected and has at least one vertex of degree less than three, then four directions suffice.

1 Introduction

A planar layout of a graph G is called a *straight-line drawing* if the vertices of G are represented by distinct points in the plane and every edge is represented by a straight-line segment connecting the corresponding pair of points and not passing through any other point representing a vertex. If it leads to no confusion, in notation and terminology we make no distinction between a vertex and the corresponding point and between an edge and the corresponding segment. The *slope* of an edge of the layout is the slope of the segment representing it. Layouts with few slopes and few bends have been extensively studied in "graph drawing" [2]. In particular, Ungar proved that every three-connected *cubic* planar graph (i.e., every vertex has degree three) can be drawn using only vertical and horizontal straight-line edges and altogether at most three bends on the outer-face [8].

Wade and Chu [9] introduced the following graph parameter: The *slope number* of a graph G is the smallest number s with the property that G has a straight-line drawing with edges of at most s distinct slopes and with no bends. Obviously, if G has a vertex of degree d, then its slope number is at least $\lceil d/2 \rceil$, because, according to the above definitions, in a proper drawing two edges are not allowed to partially overlap. The question arises whether the slope number can be bounded from above by any function of the maximum degree d (see [3]). Barát, Matoušek, and Wood [1] and, independently, Pach and Pálvölgyi [7] proved that the answer is no for $d \geq 5$. Trivially, every graph of maximum degree two has slope number at most three. What happens if $d = 3$ or 4?

*Supported by NSF grant CCR-0514079 and grants from NSA, PSC-CUNY, Hungarian Research Foundation, and BSF

†Supported by OTKA-T-038397 and OTKA-T-046246.

The aim of this note is to establish the following theorem.

Theorem 1. *Every graph of maximum degree at most three has slope number at most five.*

Our terminology is somewhat unorthodox: by the *slope* of a line ℓ, we mean the angle α modulo π such that a counterclockwise rotation through α takes the x-axis to a position parallel to ℓ. The slope of an edge (segment) is the slope of the line containing it. In particular, the slopes of the lines $y = x$ and $y = -x$ are $\pi/4$ and $-\pi/4$, and they are called *Northeast* (or Southwest) and *Northwest* (or Southeast) lines, respectively.

For any two points $p_1 = (x_1, y_1), p_2 = (x_2, y_2) \in \mathbf{R}^2$, we say that p_2 is *to the North (or to the South)* of p_1 if $x_2 = x_1$ and $y_2 > y_1$ (or $y_2 < y_1$). Analogously, we say that p_2 is to the Northeast (to the Northwest) of p_1 if $y_2 > y_1$ and p_1p_2 is a Northeast (Northwest) line. Directions are often abbreviated by their first letters: N, NE, E, SE, etc. These four directions are referred to as *basic*. That is, a line ℓ is said to be of one of the four basic directions if ℓ is parallel to one of the axes or to one of the NE and NW lines $y = x$ and $y = -x$.

The main tool of our proof is the following result of independent interest.

Theorem 2. *Let G be a connected graph that is not a cycle and whose every vertex has degree at most three. Suppose that G has at least one vertex of degree less than three, and denote these vertices by $v_1, v_2, \ldots, v_m$ $(m \geq 1)$.*

Then, for any sequence $x_1, x_2, \ldots, x_m$ of real numbers, linearly independent over the rationals, G has a straight-line drawing with the following properties:

(1) *Vertex v_i is mapped into a point with x-coordinate $x(v_i) = x_i$ $(1 \leq i \leq m)$;*

(2) *The slope of every edge is $0, \pi/2, \pi/4$, or $-\pi/4$.*

(3) *No vertex is to the North of any vertex of degree two.*

(4) *No vertex is to the North or to the Northwest of any vertex of degree one.*

2 Embedding cycles

Let C be a straight-line drawing of a cycle in the plane. A vertex v of C is said to be a *turning point* if the slopes of the two edges meeting at v are not the same.

We start with two simple auxiliary statements.

Lemma 2.1. *Let C be a straight-line drawing of a cycle such that the slope of every edge is 0, $\pi/4$, or $-\pi/4$. Then the x-coordinates of the vertices of C are not independent over the rational numbers.*

Moreover, there is a vanishing linear combination of the x-coordinates of the vertices, with as many nonzero (rational) coefficients as many turning points C has.

Proof. Let $v_1, v_2, \ldots, v_n$ denote the vertices of C in cyclic order ($v_{n+1} = v_1$). Let $x(v_i)$ and $y(v_i)$ be the coordinates of v_i. For any i ($1 \leq i \leq n$), we have $y(v_{i+1}) - y(v_i) = \lambda_i (x(v_{i+1}) - x(v_i))$, where $\lambda_i = 0, 1$, or -1, depending on the slope of the edge $v_i v_{i+1}$. Adding up these equations for all i, the left-hand sides add up to zero, while the sum of the right-hand sides is a linear combination of the numbers $x(v_1), x(v_2), \ldots, x(v_n)$ with integer coefficients of absolute value at most *two*.

Thus, we are done with the first statement of the lemma, unless all of these coefficients are zero. Obviously, this could happen if and only if $\lambda_1 = \lambda_2 = \ldots = \lambda_n$, which is impossible, because then all points of C would be collinear, contradicting our assumption that in a proper *straight-line drawing* no edge is allowed to pass through any vertex other than its endpoints.

To prove the second statement, it is sufficient to notice that the coefficient of $x(v_i)$ vanishes if and only if v_i is not a turning point. □

Lemma 2.1 shows that Theorem 2 does not hold if G is a cycle. Nevertheless, according to the next claim, cycles satisfy a very similar condition.

Lemma 2.2. *Let C be a cycle with vertices $v_0, v_1, \ldots, v_m$, in this cyclic order.*

Then, for any real numbers $x_1, x_2, \ldots, x_m$, linearly independent over the rationals, C has a straight-line drawing with the following properties:

(1) *Vertex v_i is mapped into a point with x-coordinate $x(v_i) = x_i$ ($1 \leq i \leq m$);*

(2) *The slope of every edge is $0, \pi/4$, or $-\pi/4$.*

(3) *No vertex is to the North of any other vertex.*

(4) *No vertex has a larger y-coordinate than $y(v_0)$.*

Proof.

We can assume without loss of generality that $x_2 > x_1$. Place v_1 at any point $(x_1, 0)$ of the x-axis. Assume that for some $i < m$, we have already determined the positions of $v_1, v_2, \ldots v_i$, satisfying conditions (1)–(3). If $x_{i+1} > x_i$, then place v_{i+1} at the (unique) point Southeast of v_i, whose x-coordinate is x_{i+1}. If $x_{i+1} < x_i$, then put v_{i+1} at the point West of x_i, whose x-coordinate is x_{i+1}. Clearly, this placement of v_{i+1} satisfies (1)–(3), and the segment $v_i v_{i+1}$ does not pass through any point v_j with $j < i$.

After m steps, we obtain a noncrossing straight-line drawing of the path $v_1 v_2 \ldots v_m$, satisfying conditions (1)–(3). We still have to find a right location for v_0. Let R_W and R_{SE} denote the rays (half-lines) starting at v_1 and pointing to the West and to the Southeast. Further, let R be the ray starting at v_m and pointing to the Northeast. It follows from the construction that all points $v_2, \ldots, v_m$ lie in the convex cone below the x-axis, enclosed by the rays R_W and R_{SE}.

P lace v_0 at the intersection point of R and the x-axis. Obviously, the segment $v_m v_0$ does not pass through any other vertex v_j $(0 < j < m)$. Otherwise, we could find a drawing of the cycle $v_j v_{j+1} \dots v_m$ with slopes $0, \pi/4$, and $-\pi/4$. By Lemma 2.1, this would imply that the numbers $x_j, x_{j+1}, \dots, x_m$ are *not* independent over the rationals, contradicting our assumption. It is also clear that the horizontal segment $v_0 v_1$ does not pass through any vertex different from its endpoints because all other vertices are below the horizontal line determined by $v_0 v_1$. Hence, we obtain a proper straight-line drawing of C satisfying conditions (1),(2), and (4).

It remains to verify (3). The only thing we have to check is that $x(v_0)$ does not coincide with any other $x(v_i)$. Suppose it does, that is, $x(v_0) = x(v_i) = x_i$ for some $i > 0$. By the second statement of Lemma 2.1, there is a vanishing linear combination

$$\lambda_0 x(v_0) + \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_m x_m = 0$$

with rational coefficients λ_i, where the number of nonzero coefficients is at least the number of turning angles, which cannot be smaller than *three*. Therefore, if in this linear combination we replace $x(v_0)$ by x_i, we still obtain a nontrivial rational combination of the numbers $x_1, x_2, \dots, x_m$. This contradicts our assumption that these numbers are independent over the rationals. $\Box$

3 The embedding procedure: Proof of Theorem 2

The proof is by induction on the number of vertices of G. The statement is trivial if the number of vertices is at most *two*. Suppose that we have already established Theorem 2 for all graphs with fewer than n vertices.

Let $v_1, v_2, \dots, v_m$ be the vertices of G with degree less than *three*, and let the x-coordinates assigned to them be $x_1, x_2, \dots, x_m$.

We distinguish several cases.

Case 1: *G has a vertex of degree* one.

Assume, without loss of generality, that v_1 is such a vertex. If G has no vertex of degree *three*, then it consists of a simple path $P = v_1 v_2 \dots v_m$, say. Place v_m at the point $(x_m, 0)$. In general, assuming that v_{i+1} has already been embedded for some $i < m$, and $x_i < x_{i+1}$, place v_i at the point West of v_{i+1}, whose x-coordinate is x_i. If $x_i > x_{i+1}$, then put v_i at the point Northeast of v_{i+1}, whose x-coordinate is x_i. The resulting drawing of $G = P$ meets all the requirements of the theorem. To see this, it is sufficient to notice that if v_j would be Northwest of v_m for some $j < m$, then we could apply Lemma 2.1 to the cycle $v_j v_{j+1} \dots v_m$, and conclude that the numbers $x_j, x_{j+1}, \dots, x_m$ are dependent over the rationals. This contradicts our assumption.

Assume next that v_1 is of degree *one*, and that G has at least one vertex of degree *three*. Suppose without loss of generality that $v_1 v_2 \dots v_k w$ is a path in G, whose internal vertices are of degree *two*, but the degree of w is *three*. Let G' denote the graph obtained from G by removing the vertices $v_1, v_2, \ldots, v_k$. Obviously, G' is a connected graph, in which the degree of w is *two*.

If G' is a *cycle*, then apply Lemma 2.2 to $C = G'$ with w playing the role of the vertex v_0 which has no preassigned x-coordinate. We obtain an embedding of G' with edges of slopes $0, \pi/4$, and $-\pi/4$ such that $x(v_i) = x_i$ for all $i > k$ and there is no vertex to the North, to the Northeast, or to the Northwest of w. By Lemma 2.1, the numbers $x(w), x_{k+1}, \ldots, x_m$ are not independent over the rationals. Therefore, $x(w) \neq x_k$, so we can place v_k at the point to the Northwest or to the Northeast of w, whose x-coordinate is x_k, depending on whether $x(w) > x_k$ or $x(w) < x_k$. After this, embed $v_{k-1}, \ldots, v_1$, in this order, so that v_i is either to the Northeast or to the West of v_{i+1} and $x(v_i) = x_i$. According to property (4) in Lemma 2.1, the path $v_1 v_2 \ldots v_k$ lies entirely above G', so that no point of G can lie to the North or to the Northwest of v_1.

If G' is *not a cycle*, then use the induction hypothesis to find an embedding of G' that satisfies all conditions of Theorem 2, with $x(w) = x_k$ and $x(v_i) = x_i$ for every $i > k$. Now place v_k very far from w, to the North of it, and draw $v_{k-1}, \ldots, v_1$, in this order, in precisely the same way as in the previous case. Now if v_k is far enough, then none of the points $v_k, v_{k-1}, \ldots, v_1$ is to the Northwest or to the Northeast of any vertex of G'. It remains check that condition (4) is true for v_1, but this follows from the fact that there is no point of G whose y-coordinate is larger than that of v_1.

From now on, we can and will assume that G has *no vertex of degree one*.

A graph with *four* vertices and *five* edges between them is said to be a Θ-*graph*.

Case 2: *G contains a Θ-subgraph.*

Suppose that G has a Θ-subgraph with vertices a, b, c, d, and edges ab, bc, ac, ad, bd. If neither c nor d has a third neighbor, then G is identical to this graph, which can easily be drawn in the plane with all conditions of the theorem satisfied.

If c and d are connected by an edge, then all four points of the Θ-subgraph have degree *three*, so that G has no other vertices. So G is a complete graph of four vertices, and it has a drawing that meets the requirements.

Suppose that c and d have a common neighbor $e \neq a, b$. If e has no further neighbor, then a, b, c, d, e are the only vertices of G, and again we can easily find a proper drawing. Thus, we can assume that e has a third neighbor f. By the induction hypothesis, $G' = G \setminus \{a, b, c, d, e\}$ has a drawing satisfying the conditions of Theorem 2. In particular, no vertex of G' is to the North of f (and to the Northwest of f, provided that the degree of f in G' is *one*). Further, consider a drawing H of the subgraph of G induced by the vertices a, b, c, d, e, which satisfies the requirements. We distinguish two subcases.

If the degree of f in G' is *one*, then take a very small *homothetic* copy of H (i.e., similar copy in parallel position), and rotate it about e in the clockwise direction through $3\pi/4$. There is no point of this drawing, denoted by H', to the Southeast of e, so that we can translate it into a position in which e is to the Northwest of $f \in V(G')$ and very close to it. Connecting now e to f, we obtain a drawing of G satisfying the conditions. Note that it was important to make H' very small and to place it very close to f, to make sure that none of its vertices is to the North of any vertex of G' whose degree is at most *two*, or to the Northwest of any vertex of degree *one* (other than f).

If the degree of f in G' is *two*, then we follow the same procedure, except that now H' is a small copy of H, rotated by π. We translate H' into a position in which e is to the North of f, and connect e to f by a vertical segment. It is again clear that the resulting drawing of G meets the requirements in Theorem 2. Thus, we are done if c and d have a common neighbor e.

Suppose now that only one of c and d has a third neighbor, different from a and b. Suppose, without loss of generality, that this vertex is c, so that the degree of d is *two*. Then in $G' = G \setminus \{a, b, d\}$, the degree of c is *one*. Apply the induction hypothesis to G' so that the x-coordinate originally assigned to d is now assigned to c (which had no preassigned x-coordinate in G). In the resulting drawing, we can easily reinsert the remaining vertices, a, b, d, by adding a very small square whose lowest vertex is at c and whose diagonals are parallel to the coordinate axes. The highest vertex of this square will represent d, and the other two vertices will represent a and b.

We are left with the case when both c and d have a third neighbor, other than a and b, but these neighbors are different. Denote them by c' and d', respectively. Create a new graph G' from G, by removing a, b, c, d and adding a new vertex v, which is connected to c' and d'. Draw G' using the induction hypothesis, and reinsert a, b, c, d in a small neighborhood of v so that they form the vertex set of a very small square with diagonal ab. (See Figure 1.) As before, we have to choose this square sufficiently small to make sure that a, b, c, d are not to the North of any vertex $w \neq c', d', v$ of G', whose degree is at most *two*, or to the Northwest of any vertex of degree *one*. Thus, we are done if G has a Θ-subgraph.

¿From now on we can and will assume that G has *no* Θ*-subgraph*.

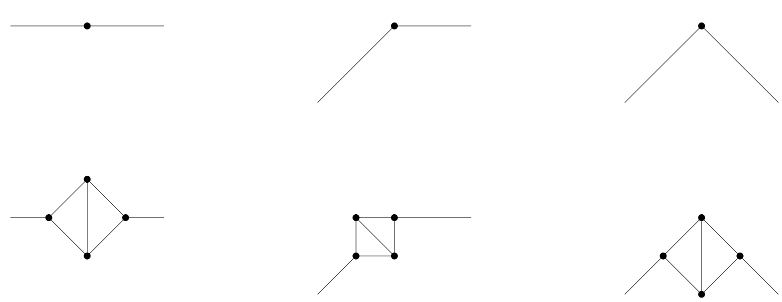

Figure 1: Replacing v by Θ

Case 3: *G has no cycle that passes through a vertex of degree* two*.*

Since G is not three-regular, it contains at least one vertex of degree *two*. Consider a decomposition of G into two-connected blocks and edges. If a block contains a vertex of degree *two*, then it consists of a single edge. The block decomposition has a treelike structure, so that there is a vertex w of degree *two*, such that G can be obtained as the union of two graphs, G_1 and G_2, having only the vertex w in common, and there is no vertex of degree *two* in G_1.

By the induction hypothesis, for any assignment of rationally independent x-coordinates to all vertices of degree less than *three*, G_1 and G_2 have proper straight-line embeddings (drawings) satisfying conditions (1)–(4) of the theorem. The only vertex of G_1 with a pre-assigned x-coordinate is w. Applying a vertical translation, if necessary, we can achieve that in both drawings w is mapped into the same point. Using the induction hypothesis, we obtain that in the union of these two drawings, there is no vertex in G_1 or G_2 to the North or to the Northwest of w, because the degree of w in G_1 and G_2 is *one* (property (4)). This is stronger than what we need: indeed, in G the degree of w is *two*, so that we require only that there is no point of G to the North of w (property (3)).

The superposition of the drawings of G_1 and G_2 satisfies all conditions of the theorem. Only two problems may occur:

1. A vertex of G_1 may end up at a point to the Northwest of a vertex of G_2 with degree *two*.

2. The (unique) edges in G_1 and G_2, incident to w, may partially overlap.

Notice that both of these events can be avoided by enlarging the drawing of G_1, if necessary, from the point w, and rotating it about w by $\pi/4$ in the clockwise direction. The latter operation is needed only if problem 2 occurs. This completes the induction step in the case when G has no cycle passing through a vertex of degree *two*.

It remains to analyze the last case.

Case 4: *G has a cycle passing through a vertex of degree* two.

By assumption, G itself is not a cycle. Therefore, we can also find a *shortest* cycle C whose vertices are denoted by $v, u_1, \ldots, u_k$, in this order, where the degree of v is *two* and the degree of u_1 is *three*. The length of C is $k+1$.

It follows from the minimality of C that u_i and u_j are not connected by an edge of G, for any $|i-j|>1$. Moreover, if $|i-j|>2$, then u_i and u_j do not even have a common neighbor ($1 \le i \ne j \le k$). This implies that any vertex $v \in V(G \setminus C)$ has at most *three* neighbors on C, and these neighbors must be consecutive on C. However, *three* consecutive vertices of C, together with their common neighbor, would form a Θ-subgraph in G (see Case 2). Hence, we can assume that every vertex belonging to $G \setminus C$ is joined to at most *two* vertices on C.

Let B_i denote the set of all vertices of $G \setminus C$ that have precisely i neighbors on C ($i = 0, 1, 2$). Thus, we have $V(G \setminus C) = B_0 \cup B_1 \cup B_2$. Further, $B_1 = B_1^2 \cup B_1^3$, where an element of B_1 belongs to B_1^2 or B_1^3, according to whether its degree in G is *two* or *three*.

Consider the list $v_1, v_2, \ldots, v_m$ of all vertices of G with degree *two*. (Recall that we have already settled the case when G has a vertex of degree *one*.) Assume without loss of generality that $v_1 = v$ and that v_i belongs to C if and only if $1 \leq i \leq j$.

Let $\mathbf{x}$ denote the *assignment* of x-coordinates to the vertices of G with degree *two*, that is, $\mathbf{x} = (x(v_1), x(v_2), \ldots, x(v_m)) = (x_1, x_2, \ldots, x_m)$. Given G, C, $\mathbf{x}$, and a real parameter L, we define the following so-called EMBEDDING PROCEDURE$(G, C, \mathbf{x}, L)$ to construct a drawing of G that meets all requirements of the theorem, and satisfies the additional condition that the y-coordinate of every vertex of C is at least L higher than the y-coordinates of all other vertices of G.

STEP 1: Construct recursively a drawing of $G' := G \setminus C$ satisfying the conditions of Theorem 2 with the assignment $\mathbf{x}'$ of x-coordinates $x(v_i) = x_i$ for $j < i \leq m$, and $x(u_1') = x_1$, where u_1' is the unique vertex in $G \setminus C$, connected by an edge to $u_1 \in V(C)$.

STEP 2: For each element of $B_1^2 \cup B_2$, take two rays starting at this vertex, pointing to the Northwest and to the North. Further, take a vertical ray pointing to the North from each element of B_1^3 and each element of the set $B_{\mathbf{x}} := \{(x_2, 0), (x_3, 0), \ldots, (x_j, 0)\}$. Let $\mathcal{R}$ denote the set of all of these rays. Choose the x-axis above all points of G' and all intersection points between the rays in $\mathcal{R}$.

For any u_h ($1 \leq h \leq k$) whose degree in G is *three*, define $N(u_h)$ as the unique neighbor of u_h in $G \setminus C$. If u_h has degree *two* in G, then $u_h = v_i$ for some $1 \leq i \leq j$, and let $N(u_h)$ be the point $(x_i, 0)$.

STEP 3: Recursively place $u_1, u_2, \ldots u_k$ on the rays belonging to $\mathcal{R}$, as follows. Place u_1 on the vertical ray starting at $N(u_1) = u_1'$ such that $y(u_1) = L$. Suppose that for some $i < k$ we have already placed $u_1, u_2, \ldots u_i$, so that $L \leq y(u_1) \leq y(u_2) \leq \ldots \leq y(u_i)$ and there is no vertex to the West of u_i. Next we determine the place of u_{i+1}.

If $N(u_{i+1}) \in B_1^2$, then let $r \in \mathcal{R}$ be the ray starting at $N(u_{i+1})$ and pointing to the Northwest. If $N(u_{i+1}) \in B_1^3 \cup B_{\mathbf{x}}$, let $r \in \mathcal{R}$ be the ray starting at $N(u_{i+1})$ and pointing to the North. In both cases, place u_{i+1} on r: if u_i lies on the left-hand side of r, then put u_{i+1} to the Northeast of u_i; otherwise, put u_{i+1} to the West of u_i.

If $N(u_{i+1}) \in B_2$, then let $r \in \mathcal{R}$ be the ray starting at $N(u_{i+1})$ and pointing to the North, or, if we have already placed a point on this ray, let r be the other ray from $N(u_{i+1})$, pointing to the Northwest, and proceed as before.

STEP 4: Suppose we have already placed u_k. It remains to find the right position for $u_0 := v$, which has only two neighbors, u_1 and u_k. Let r be the ray at u_1, pointing to the North. If u_k lies on the left-hand side of r, then put u_0 on r to the Northeast of u_k;

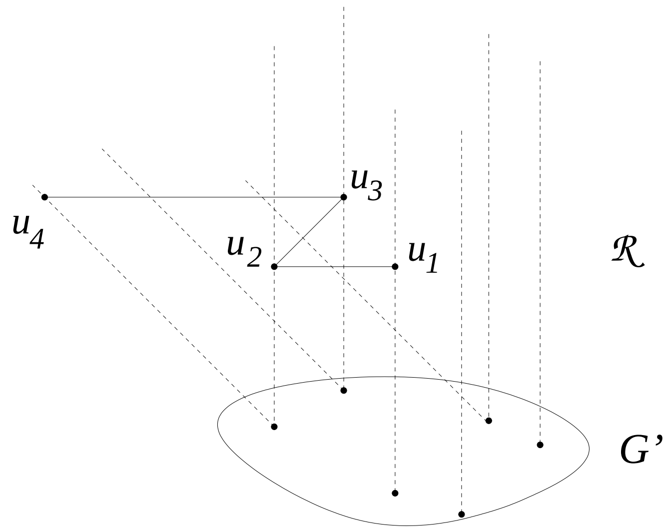

Figure 2: Recursively place $u_1, u_2, \dots u_k$ on the rays belonging to $\mathcal{R}$

otherwise, put u_0 on r, to the West of u_k.

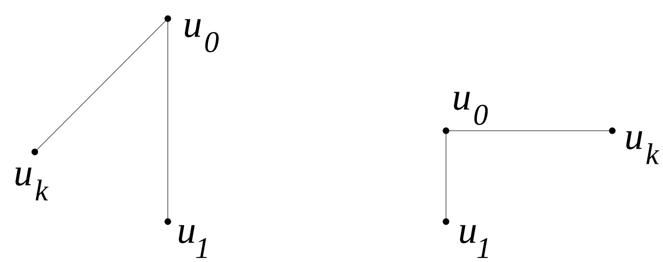

Figure 3: Find the right position for u_0

During the whole procedure, we have never placed a vertex on any edge, and all other conditions of Theorem 2 are satisfied $\square$.

Remark that the y-coordinates of the vertices $u_0 = v, u_1, \dots, u_k$ are at least L higher than the y-coordinates of all vertices in $G \backslash C$. If we fix G, C, and $\mathbf{x}$, and let L tend to infinity, the coordinates of the vertices given by the above Embedding Procedure$(G, C, \mathbf{x}, L)$ change continuously.

4 Proof of Theorem 1

We are going to show that any graph G permits a straight-line drawing using only the four basic directions (of slopes $0, \pi/2, \pi/4$, and $-\pi/4$), and perhaps one further direction, which is almost vertical and is used for at most one edge in each connected component of G.

Denote the connected components of G by $G_1, G_2, \ldots, G_t$. If a component G_s is not three-regular, or if it is a complete graph with *four* vertices, then, by Theorem 2, it can be drawn using only the four basic directions. If G_s has a Θ-subgraph, one can argue in the same way as in Case 2 of the proof of Theorem 2: Embed recursively the rest of the graph, and attach to it a small copy of this subgraph such that all edges of the Θ-subgraph, as well as the edges used for the attachment, are parallel to one of the four basic directions. Actually, in this case, G_s itself can be drawn using the four basic directions, so the fifth direction is not needed.

Thus, in the rest of the proof we can assume that G_s is three-regular, it has more than *four* vertices, and it contains no Θ-subgraph. For simplicity, we drop the subscript and we write G instead of G_s. Choose a shortest cycle $C = u_0u_1 \ldots u_k$ in G. Each vertex of C has precisely one neighbor in $G \setminus C$. On the other hand, as in the proof of the last case of Theorem 2, all vertices in $G \setminus C$ have at most two neighbors in C.

Let u_0' denote the neighbor of u_0 in $G \setminus C$. Since G has no Θ-subgraph, u_0' cannot be joined to both u_1 and u_k. Assume without loss of generality that u_0' is not connected to u_1. Let u_1' denote the neighbor of u_1 in $G \setminus C$. Let G' denote the graph obtained from G after the removal of the edge u_0u_0'. Fix a rationally independent assignment $\mathbf{x}$ of x-coordinates to the vertices of degree at most *two* in $G' \setminus C$, such that $x(u_0') = 1$, $x(u_1') = \sqrt{2}$, and the x-coordinates of the other vertices are all *negative*. Consider a drawing of $G' \setminus C$, meeting the requirements of Theorem 1. In view of the remark following Theorem 1, for any sufficiently small $\varepsilon > 0$, EMBEDDING PROCEDURE$(G', C, \mathbf{x}, L)$ gives a drawing of G', in which the slope of the line connecting u_0 and u_0' is $\frac{\pi}{2} - \varepsilon$. It follows from the construction, that $x(u_0) = x(u_1') = \sqrt{2}$.

We want to add the segment u_0u_0' to this drawing. If ε was sufficiently small, then the line of slope $\frac{\pi}{2} - \varepsilon$ passing through $u_0' \in V(G' \setminus C)$ cannot pass through any vertex of $G' \setminus C$. Nor can it pass through any vertex belonging to C (other than u_0), because every u_i $(1 \leq i \leq k)$ lies either on a vertical line whose x-coordinate is negative, or on a line of slope $-\frac{\pi}{4}$ that passes through a vertex of $G' \setminus C$. However, all of these points lie above the x-axis and have x-coordinates less than $x(u_0') = 1$.

Summarizing: if ε is sufficiently small (that is, if L is sufficiently large), then each component of the graph has a proper drawing in which all edges are of one of the four basic directions, with the exception of at most one edge whose slope is $\frac{\pi}{2} - \varepsilon$. If we choose an $\varepsilon > 0$ that works for all components, then the whole graph can be drawn using only at most *five* directions. This concludes the proof of Theorem 1. $\square$

5 Algorithm and concluding remarks

Based on the proof, it is not hard to design an algorithm to find a proper drawing, in quadratic time.

First, if our graph is a circle, we have no problem drawing it in $O(n)$ steps. If our graph has a vertex of degree *one* then the procedure of Case 1 of the proof of Theorem 2 requires at most $O(m)$ time when we reinsert $v_1, \ldots, v_m$.

We can check if our graph has any Θ-subgraph in $O(n)$ time. If we find one, we can proceed by induction as in Case 2 of the proof of Theorem 2 We can reinsert the Θ-subgraph as described in Case 2 in $O(1)$ time.

Now assume that we have a vertex v of degree *two*. Execute a breadth first search from any vertex, and take a *minimal* vertex of degree two, that is, a vertex v of degree two, all of whose descendants are of degree three. If there is an edge in the graph connecting a descendant of v with a non-descendant, then there is a cycle through v; we can find a minimal one with a breadth first search from it and proceed as in Case 4. Otherwise, v can play the role of w in Case 3, and we can proceed recursively.

Finally, if the graph is 3-regular, then we draw each component separately, except the last step, when we have to pick an ϵ small enough simultaneously for all components, this takes $O(n)$ steps. We only have to find the greatest slope and pick an ε such that $\frac{\pi}{2} - \varepsilon$ is even steeper.

We believe that this algorithm is far from being optimal. It may perform a breadth first search for each induction step, which is probably not necessary. One may be able to replace this step by repeatedly updating the results of the first search. We cannot even rule out the problem can be solved in linear time.

It was shown by Dujmović at al. [3] that every planar graph with maximum degree three has a drawing with noncrossing straight-line edges of at most three different slopes, except that three edges of the outer-face may have a bend.

Eppstein [6], Duncan et al. [4], and Barát et al. [1] studied another parameter, the *geometric thickness* of a graph, which is closely related to the slope number.

Max Engelstein [5], a student from Stuyvesant High School, New York has shown that every graph of maximum degree *three* that has a Hamiltonian cycle can be drawn with edges of at most *five* different slopes.

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