Monotone crossing number

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Abstract

The *monotone crossing number* of G is defined as the smallest number of crossing points in a drawing of G in the plane, where every edge is represented by an x-monotone curve, that is, by a connected continuous arc with the property that every vertical line intersects it in at most one point. It is shown that this parameter can be strictly larger than the classical crossing number $\textsc{cr}(G)$, but it is bounded from above by $2\textsc{cr}^2(G)$. This is in sharp contrast with the behavior of the rectilinear crossing number, which cannot be bounded from above by any function of $\textsc{cr}(G)$.

1 Introduction

Let $G = (V(G), E(G))$ be a graph with no loops and multiple edges, and let $V(G)$ and $E(G)$ denote its vertex set and edge set. A *drawing* of G is an embedding of G in the plane, where each vertex $v \in V(G)$ is mapped to a point and each edge $uv \in E(G)$ is mapped into a simple continuous arc connecting the images of its endpoints, but not passing through the image of any other vertex of G. The arcs representing the edges of G are allowed to cross, but we assume for simplicity that any two arcs have finitely many points in common and no three arcs pass through the same point. A common interior point p of two arcs is said to be a *crossing* if in a small neighborhood of p one arc passes through one side of the other arc to the other side. If it leads to no confusion, the vertices and their images, as well as the edges and the arcs representing them, will be denoted by the same symbols.

In the special case where G is a complete bipartite graph, the problem of minimizing the number of crossings in a drawing of G was first studied by Turán [T77]. The question became known as the *brick factory problem*. It was generalized to all graphs by Erdős and Guy [ErG73]. In two previous papers [PaT00a], [PaT00b], the authors of the present note pointed out some inconsistencies between various definitions of crossing numbers implicitly used in early publications on the subject. To distinguish between these notions, they introduced some new terminology and notation. The *crossing number* of G, denoted by $\textsc{cr}(G)$, is the smallest number of crossings in a drawing of G in the plane. The *pairwise crossing number*, $\textsc{pair-cr}(G)$, is the smallest number of crossing pairs of edges in a drawing of G. If two edges cross several times, they still count as a single crossing pair, so that we have $\textsc{pair-cr}(G) \leq \textsc{cr}(G)$ for every graph G. It is one of the most tantalizing open problems in this area to decide whether these two parameters coincide or at least $\textsc{cr}(G) = O(\textsc{pair-cr}(G))$ holds for all graphs G. It was shown in [PaT00a] that $\textsc{cr}(G) = O(\textsc{pair-cr}^2(G))$, which was successively improved in [Va05], [To08], and [To11] to $\textsc{cr}(G) = O(\textsc{pair-cr}^{7/4}(G)/\log^{3/2}\textsc{pair-cr}(G))$. It is not easy to make

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any conjecture in this respect or even to experiment with concrete graphs. The computation of $\text{CR}(G)$ and $\text{PAIR-CR}(G)$ are both NP-hard problems [GaJS76], [GaJ83], [PaT00a].

On the other hand, there is another natural parameter that can be much larger than the above two crossing numbers. $\text{LIN-CR}(G)$, the *rectilinear crossing number* of G, is the smallest number of crossings in a rectilinear drawing of G, that is, in a drawing where every edge is represented by a straight-line segment. We have $\text{CR}(G) \leq \text{LIN-CR}(G)$. Bienstock and Dean [BiD93] constructed a series of graphs with crossing number 4, whose rectilinear crossing numbers are arbitrarily large.

An *x-monotone curve* is a connected, continuous arc with the property that every straight-line parallel to the y-axis intersects it in at most one point. A drawing of G is called *x-monotone* (or *monotone*, for short) if every edge of G is represented by an x-monotone curve. We define $\text{MON-CR}(G)$, the *monotone crossing number* of G, as the smallest number of crossings in a monotone drawing of G. Obviously, every rectilinear drawing of G, in which no two vertices share the same x-coordinate, is a monotone drawing. Therefore, we have

$$\text{CR}(G) \leq \text{MON-CR}(G) \leq \text{LIN-CR}(G),$$

for every graph G.

Monotone drawings and rectilinear drawings share many interesting properties. In particular, it was shown in [PaT04] that every crossing-free monotone drawing of a (planar) graph G can be "stretched" without changing the x-coordinates of the vertices. In other words, there is a crossing-free rectilinear drawing of G, isomorphic to the original one, in which the vertices have the same x-coordinates. Another example, for drawings with many crossings, is related to Conway's famous *thrackle conjecture* [Wo69], which says that if a graph can be drawn in the plane such that any two edges have *exactly one* common points (either a common endpoint, or a crossing) then the number of edges cannot exceed the number of vertices. (The conjecture has been verified for monotone drawings [PaS11].) In sharp contrast to these analogies, there are no graphs with bounded crossing numbers that have arbitrarily large monotone crossing numbers. In the present note, we answer a question of Fulek, Pelsmajer, Schaefer, and Štefankovič [FuPS11] by establishing the following results.

Theorem 1. *Every graph G satisfies the inequality*

$$\text{MON-CR}(G) < 2\text{CR}^2(G).$$

Theorem 2. *There are infinitely many graphs G with arbitrarily large crossing numbers such that*

$$\text{MON-CR}(G) \geq \frac{7}{6}\text{CR}(G) - 6.$$

The proof of Theorem 1 is algorithmic. It is based on a recursive procedure to redraw a plane graph without changing its combinatorial structure so that in the resulting drawing any pair of vertices of the same cell can be connected by an x-monotone curve. See Theorem 2.2. One of the key ideas of the construction proving Theorem 2, the use of "weighted" edges or repeated paths, goes back to the paper of Bienstock and Dean [BiD93] mentioned above. This idea was further developed and applied to related problems by Pelsmajer, Schaefer, and Štefankovič [PeSS08] and by Tóth [To08].

2 Proof of Theorem 1

Two crossing-free (plane) drawings of a planar graph are said to be *isomorphic* if there is a homeomorphism of the plane which maps one to the other. In particular, it takes the unbounded cell of the first drawing to the unbounded cell of the second.

Definition 2.1. Let $\mathcal{D}$ be a crossing-free drawing of a planar graph G, and let $v \in V(G)$. We say that $\mathcal{D}$ is *v-spinal* if

1. $\mathcal{D}$ is a monotone drawing;
2. v is the leftmost vertex;
3. any two vertices belonging to the same (bounded or unbounded) cell C can be connected by an x-monotone curve that lies in the interior of C (with the exception of its endpoints);
4. every vertical ray starting at a boundary vertex of the unbounded cell C_0 and pointing downwards lies in the interior of C_0 (with the exception of its endpoint).

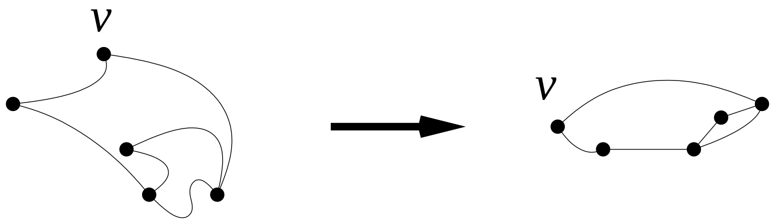

Figure 1: A plane drawing and a v-spinal drawing.

Theorem 1 is an easy corollary of the following result.

Theorem 2.2. *For any crossing-free drawing $\mathcal{D}$ of a planar graph and for any vertex v of the unbounded cell, there is a v-spinal drawing isomorphic to $\mathcal{D}$.*

It follows from the result of [PaT04] mentioned in the introduction that every v-spinal drawing can be "stretched" without changing the x-coordinates of the vertices. That is, we can assume without loss of generality that the drawing whose existence is guaranteed by Theorem 2.2 is rectilinear. However, in the recursive argument proving Theorem 2.2, we will not need this fact. It will be sufficient to assume that the edges are represented by x-monotone polygonal paths, so that in a small neighborhood of their endpoints it will make sense to talk about the *slopes* of these paths.

Before turning to the proof of Theorem 2.2, we show how Theorem 2.2 implies Theorem 1.

Proof of Theorem 1 (using Theorem 2.2). Let G be any graph, and let $\mathcal{D}$ be a drawing of G with $\text{CR}(G)$ crossings. Let $G' \subseteq G$ denote the subgraph consisting of all vertices of G and all edges not crossed by any other edge in this drawing. Clearly, G' is a planar graph. Let $\mathcal{D}'$ stand for the corresponding crossing-free subdrawing of $\mathcal{D}$.

Let v be a vertex of the unbounded cell. By Theorem 2.2, there is a v-spinal drawing $\mathcal{D}''$ of G', isomorphic to $\mathcal{D}'$. Consider now an edge $v_1v_2 \in E(G) \setminus E(G')$. In $\mathcal{D}$, this edge was represented by a curve that, with the exception of its endpoints, lied in the interior of a single cell C' in the subdrawing $\mathcal{D}'$. Let C'' denote the cell in $\mathcal{D}''$, which corresponds to C'. In view of condition 3 in Definition 2.1, the points representing v_1 and v_2 can be connected by an x-monotone curve within the cell C''. Let us choose such an x-monotone connecting curve for each edge in $E(G) \setminus E(G')$, so that the total number of crossings between them is as small as possible. Observe that any two such curves can cross at most once, otherwise by swapping their sections between two consecutive crossing points and slightly separating them, we could reduce the total number of crossings by 2. During this transformation, both curves remain x-monotone.

Therefore, in the resulting x-monotone drawing of G, the total number of crossings is at most $\binom{|E(G)|-|E(G')|}{2}$. This yields that

$$\textsc{mon-cr}(G) \leq \binom{|E(G)| - |E(G')|}{2}.$$

On the other hand, taking into account that every edge in $E(G) \setminus E(G')$ participates in at least one crossing in $\mathcal{D}$, we have

$$|E(G)| - |E(G')| \leq 2\textsc{cr}(G).$$

Comparing the last two inequalities, the theorem follows. $\Box$

Proof of Theorem 2.2. We proceed by induction on the number of vertices of $\mathcal{D}$. The theorem is obviously true for graphs with one or two vertices. Suppose now that $\mathcal{D}$ has n vertices and that the theorem has already been proved for all drawings of graphs with fewer than n vertices. Let v be a vertex of the unbounded cell in $\mathcal{D}$.

Case 1: $\mathcal{D}$ is not connected. Suppose for simplicity that it has two connected components, $\mathcal{D}_1$ and $\mathcal{D}_2$; the other cases can be treated analogously. Assume without loss of generality that $v \in \mathcal{D}_1$.

Subcase 1.1: $\mathcal{D}_2$ has a vertex v' that belongs to the unbounded cell in $\mathcal{D}$. Take a v-spinal drawing isomorphic to $\mathcal{D}_1$, and place a v'-spinal drawing isomorphic $\mathcal{D}_2$ completely to the right of it, so that every vertex of the latter has a larger x-coordinate than any vertex of the former. The resulting drawing meets the requirements.

Subcase 1.2: $\mathcal{D}_2$ does not have a vertex that belongs to the boundary of the unbounded cell in $\mathcal{D}$. Let C denote the cell in $\mathcal{D}_1$ that contains $\mathcal{D}_2$, and fix a vertex w of C. Let v' be a vertex of the unbounded cell in $\mathcal{D}_2$. Take a v-spinal drawing isomorphic to $\mathcal{D}_1$, and place a very small copy of a v'-spinal drawing isomorphic to $\mathcal{D}_2$ in the cell C' of $\mathcal{D}_1$ that corresponds to C, in a small neighborhood of the vertex that corresponds to w.

The resulting drawing $\mathcal{D}$ obviously satisfies conditions 1, 2, and 4 in Definition 2.1. As for condition 3, we have to verify only that any two vertices, v_1 and v_2, that belong to the union of the boundary of C' and the outer boundary of the small v'-spinal drawing isomorphic to $\mathcal{D}_2$ can be connected by an x-monotone curve that does not cross $\mathcal{D}$. This readily follows by the induction hypothesis, unless v_1 belongs to the boundary of C' and v_2 belongs to the outer boundary of the small drawing isomorphic to $\mathcal{D}_2$. In the latter case, move slightly downward from v_2 and then closely follow the x-monotone curve connecting w to v_1.

Case 2: $\mathcal{D}$ has a cut vertex v'. Suppose that $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, where the only point that $\mathcal{D}_1$ and $\mathcal{D}_2$ have in common is v'. Assume without loss of generality that v is a vertex of $\mathcal{D}_1$. Note that v and v' may be identical.

Let C denote the cell in $\mathcal{D}_1$ that contains $\mathcal{D}_2$. In particular, v' is a vertex of C. In $\mathcal{D}_2$, the vertex v' belongs to the unbounded cell.

Take a v-spinal drawing isomorphic to $\mathcal{D}_1$, and fix a very short non-vertical segment s, which is incident to the point $p(v')$ representing v' and which lies in the cell C' that corresponds to C. In the special case where $v' = v$ and C' is the unbounded cell, make sure that the x-coordinates of the points of s are larger than the x-coordinate of $p(v')$. In addition, take a very small v'-spinal drawing isomorphic to $\mathcal{D}_2$ such that the point representing v' coincides with $p(v')$. Applying a suitable orientation preserving linear transformation to this second drawing, it can be achieved that it becomes very "flat" and small, and lies in a very small neighborhood of the segment s, within C'. Putting these two drawings together, the resulting drawing meets the requirements.

Note that, if the x-coordinates of the points of s are smaller than the x-coordinate of $p(v')$, then the above linear transformation reverses the order of the x-coordinates in the v'-spinal drawing isomorphic to $\mathcal{D}_2$. In order to preserve the combinatorial structure of the cell decomposition, we have to make sure that we use a linear transformation that preserves the orientation of the plane.

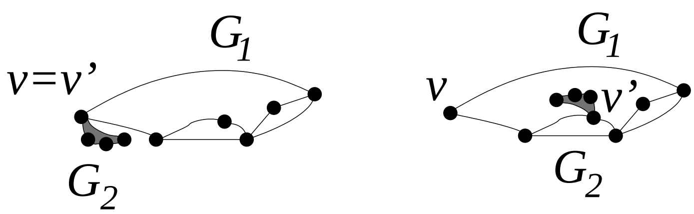

Figure 2: Case 2. $\mathcal{D}$ has a cut vertex v'.

CASE 3: $\mathcal{D}$ is 2-connected. We need the following well known result.

Lemma 2.3. [Di05] *For every 2-connected graph other than a cycle, there exists a path such that removing all edges and all internal vertices of this path, the remaining graph is still 2-connected.*

Let $\mathcal{D}$ be a drawing of a cycle with vertices $v = v_1, v_2, \dots, v_n$, in counterclockwise order. Then the rectilinear drawing induced by the points $p(v_i) = (i, i^2)$ is v-spinal and isomorphic to $\mathcal{D}$.

If $\mathcal{D}$ is not a cycle, then, according to the lemma, it can be obtained from a 2-connected drawing $\mathcal{D}_0$, by adding a path P between two vertices, u and w, of $\mathcal{D}_0$), which, with the exception of its endpoints, lies in the interior of a cell C. We distinguish two subcases.

Subcase 3.1: v is a vertex of $\mathcal{D}_0$. Take a v-spinal drawing isomorphic to $\mathcal{D}_0$. Let C' denote the cell that corresponds to C in this drawing. The vertices u and w belong to the boundary of this cell. Therefore, by condition 3 in Definition 2.1, u and w can be connected by an x-monotone curve within C'. Put all internal vertices of P along this curve, very close to u. The resulting drawing meets the requirements.

Subcase 3.2: v is an internal vertex of P. Since v is a vertex of the unbounded cell in $\mathcal{D}$, the cell C in $\mathcal{D}_0$ that contains P, must be the unbounded cell.

Let $P = uu_1 \cdots u_m v w_1 w_2 \cdots w_k w$. Assume without loss of generality that in $\mathcal{D}$ the unbounded cell lies on the *left-hand side* of P, as we traverse it from u to w. Take a u-spinal drawing $\mathcal{D}_1$ isomorphic to $\mathcal{D}_0$. Place v to the left and w_1 to the right of all vertices of $\mathcal{D}_0$.

Connect u and v by an x-monotone curve in $\mathcal{D}_1$, and place the vertices $u_1, \dots, u_m$ on this curve, in this order. Then connect v to w_1 by an x-monotone curve running above all previously drawn vertices and edges. Finally, connect w_1 to w by an x-monotone curve which does not cross any previously drawn edges, and place the vertices $w_2, \ldots, w_k$ on this curve, in this order, very close to w_1. Adding these three curves that represent P to $\mathcal{D}_1$, we obtain a v-spinal drawing isomorphic to $\mathcal{D}$, as required. $\square$

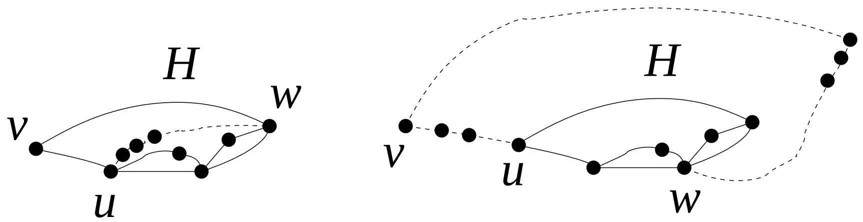

Figure 3: Case 3. $\mathcal{D}$ is two-connected.

3 Proof of Theorem 2

Throughout this section, let k be a fixed positive integer. We construct a graph G_k with $\text{CR}(G_k) = 6k + 6$ and $\text{MON-CR}(G) = 7k + 6$, as follows.

First, we define an auxiliary graph on the vertex set $V(H) = \{u, w, v_1, \ldots, v_9\}$ such that each of its edges is red, blue, or black. Let w be connected to every element of $v_1, \ldots, v_9$ by a red edge. Let $v_1, \ldots, v_9$ form a red cycle, in this order. Finally, let H have three blue edges, uv_2, uv_5, and uv_8, and three black edges, v_1v_6, v_7v_3, v_4v_9. See Figure 4. Let H' be a colored graph isomorphic to H with $V(H') = \{u', w', v'_1, \ldots, v'_9\}$ and $V(H') \cap V(H) = \emptyset$.

Let H_k denote the graph obtained from H by substituting each of its red edges by $10k$ paths of length two and each of its blue edges by k paths of length two such that the middle vertices of these paths are disjoint from one another and from all previously listed vertices. We will refer to these paths as *red paths* and *blue paths*, respectively. Let H'_k denote the graph with $V(H'_k) \cap V(H_k) = \emptyset$ which can be obtained from H' in exactly the same way as H_k was constructed from H.

Finally, connect u to u' by a red edge, and replace this edge by $10k$ vertex disjoint red paths of length two, as above. Denote the resulting graph by G_k.

We start with the following simple observation.

Claim 3.1. $\text{CR}(G_k) \le 6k + 6$ *and* $\text{MON-CR}(G_k) \le 7k + 6$.

Proof. A drawing of G_k with $6k + 6$ crossings and a monotone drawing with $7k + 6$ crossings are depicted on Figure 5, and Figure 6, respectively. The thick edges and the dotted edges represent bundles consisting of $10k$ red paths and k blue paths, respectively. The paths representing the same colored edge run very close to one another and do not cross. The only difference between the two drawings is that in the first one v_4v_9 crosses uv_2, while in the second it crosses uv_5 and uv_8. $\square$

A drawing of a graph G is called CR-*optimal* if the number of crossings in it is $\text{CR}(G)$. Analogously, a MON-CR-optimal drawing is a monotone drawing in which the number of crossings is $\text{MON-CR}(G)$.

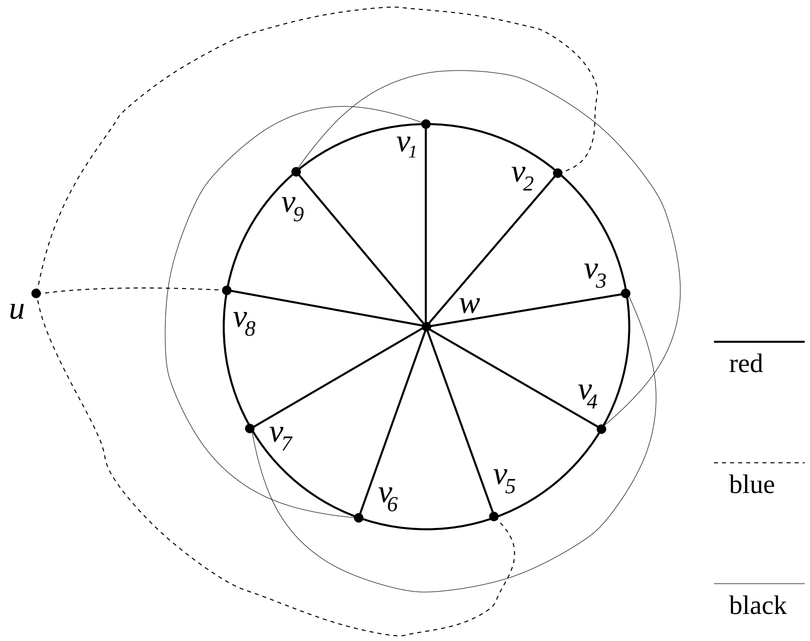

Figure 4: Graph H.

Claim 3.2. *Each of the graphs G_k, H_k, and H'_k has a* CR*-optimal drawing and a* MON-CR*-optimal drawing satisfying the the following conditions. (i) The red paths substituting the same red edge run very close to one another and do not cross any edge. (ii) The blue paths substituting the same blue edge run very close to one another, do not cross one another, and cross exactly the same edges.*

Proof. Let G stand for one of the graphs G_k, H_k, or H'_k. Let $P_1, \dots, P_m$ ($m = 10k$ or k) denote the paths substituting the same red or blue edge. Consider a CR-optimal or a MON-CR-optimal drawing of G. Suppose without loss of generality that among all P_is the path P_1 participates in the smallest number of crossings. Redraw $P_2, \dots, P_m$ so that they run "parallel" to P_1 and very close to it. Clearly, this transformation does not increase the total number of crossings, so that the resulting drawing remains optimal.

Suppose that $P_1, \dots, P_m$ ($m = 10k$) are *red paths* that substitute the same red edge and run parallel to one another. If any of them crosses an edge, then all of them do. This alone creates a total of at least $10k$ crossings, which contradicts the assumption the drawing was optimal. □

Claim 3.3. $\text{CR}(H_k) = \text{MON-CR}(H_k) = 3k + 3$. *Consequently, we have* $\text{CR}(G_k) = 6k + 6$.

Proof. The right part of Figure 6 shows a monotone drawing of H. ¿From this one can easily construct a monotone drawing of H'_k with $3k + 3$ crossings. Therefore, we have $\text{CR}(H_k) \leq \text{MON-CR}(H_k) = \text{MON-CR}(H'_k) \leq 3k + 3$. As before, the thick and the dotted edges represent bundles of $10k$ parallel red paths and bundles of k parallel blue paths.

Consider a CR-optimal drawing of H_k which satisfies the conditions in Claim 3.2. Replace now the red paths substituting the same red edge by a single red edge running along any one of those paths. The red cycle $C = v_1 v_2 \cdots v_9$ divides the rest of the plane into a bounded and an unbounded region. All points that belong to the bounded (unbounded) region are said to be *inside (outside)* of C. Assume without loss of generality that the vertex w lies *inside* of C. Since no red edge is allowed to cross any other edge, the edges $v_3 v_7$, $v_1 v_6$, and $v_4 v_9$, as well as the vertex u with all edges incident

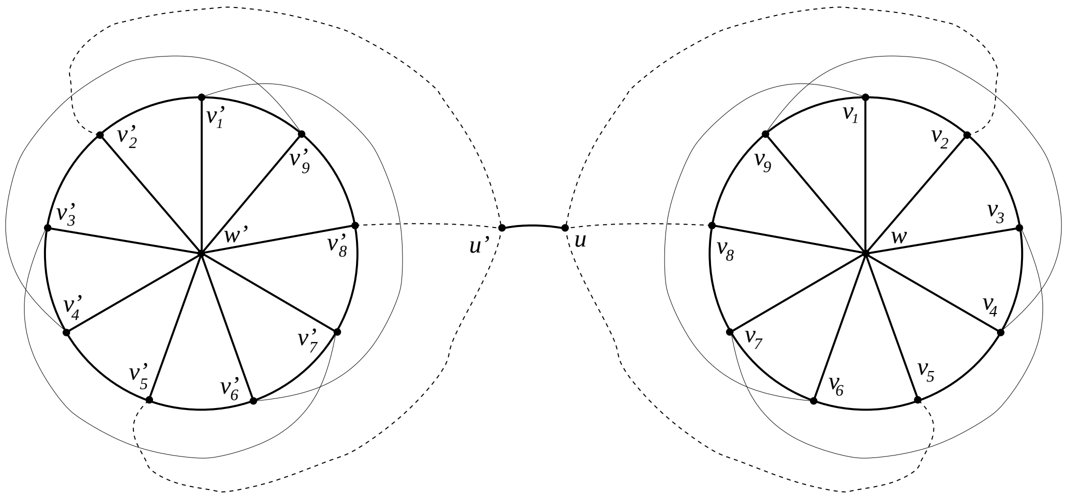

Figure 5: A CR-optimal drawing of G.

to it, must lie *outside* of C. Thus, the edges v_3v_7, v_1v_6, and v_4v_9 are pairwise crossing. Moreover, the path v_2uv_5 must cross the edges v_3v_7 and v_4v_9, and the path v_2uv_8 must cross the edge v_1v_6. This already guarantees the existence of $3k+3$ crossings, so that we have $\text{CR}(H_k) = \text{MON-CR}(H_k) = 3k+3$. □

To complete the proof of Theorem 2, it remains to verify the following.

Claim 3.4. $\text{MON-CR}(G_k) \geq 7k+6$.

Proof. Fix a MON-CR-optimal drawing of G_k, satisfying the conditions in Claim 3.2. As in the proof of Claim 3.3, replace every bundle of red paths substituting the same red edge by a single red edge. Let C and C' denote the red cycles induced by the vertices $v_1, v_2, \ldots, v_9$ and $v'_1, v'_2, \ldots, v'_9$. Both of them divide the plane into a bounded and an unbounded region, so that it makes sense to say that a point is inside or outside of C or C'.

By Claim 3.2, in the original drawing of G_k, the red edges cannot cross any other edge. Suppose that a blue edge belonging to $H_k \subset G_k$ crosses an edge belonging to $H'_k \subset G_k$. Then the number of crossings is at least $k + \text{MON-CR}(H_k) + \text{MON-CR}(H'_k) = 7k+6$, and we are done. Thus, we can assume that in the drawing of G_k, the blue edges of H_k do not cross any edge of H'_k, and analogously, the blue edges of H'_k do not cross any edge of H_k.

Let v be the vertex of G_k with the smallest x-coordinate, and suppose without loss of generality that $v \in V(H'_k)$. Consider now separately the drawing of H_k and the induced cell decomposition. By definition, v lies in the unbounded cell. Observe, that if we remove edges $v'_1v'_6$, $v'_7v'_3$, $v'_4v'_9$ from H'_k, that is, if we keep only the red and blue edges, we still have a connected graph. The red and blue edges of H'_k cannot cross any edge of H_k. Hence, *all* vertices of H'_k must lie in the unbounded cell of the cell decomposition induced by H_k.

The vertices u and u' are connected by a red edge in G_k. Hence, u must lie on the boundary of the unbounded cell of the cell decomposition induced by H_k. In particular, u is *outside* of the cycle C. Since w is connected to each edge of C by a red edge, u and w lie on different sides of C. Thus, w must be *inside* of C. Therefore, the edges v_3v_7, v_1v_6, v_4v_9, as well as the vertex u together with all edges incident to it, must lie *outside* of C. Consequently, the edges v_3v_7, v_1v_6, v_4v_9 must be pairwise

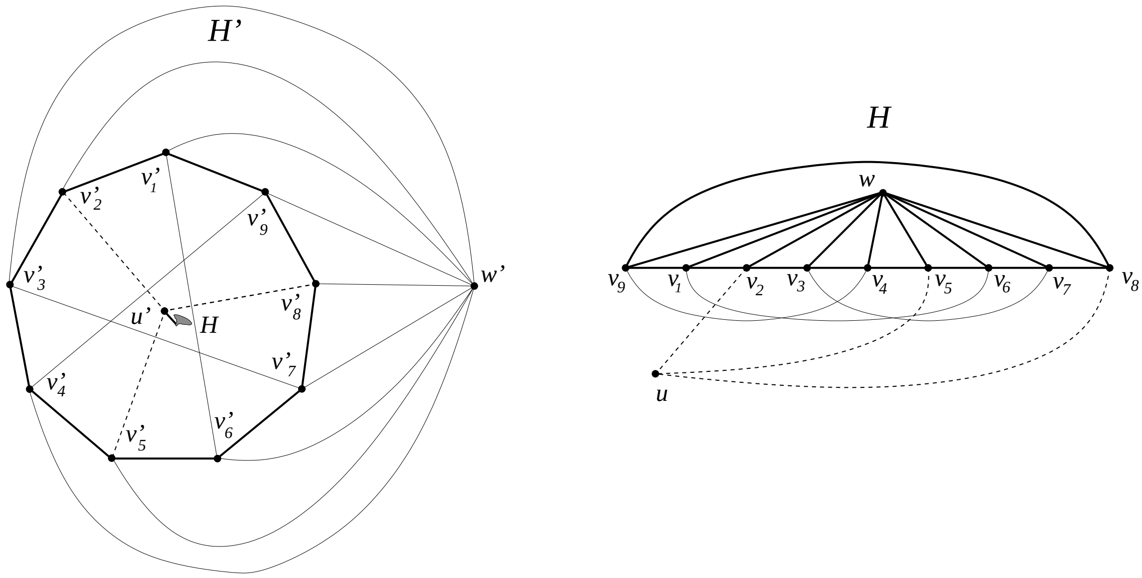

Figure 6: A MON-CR-optimal drawing of G.

crossing. The edges v_3v_7, v_1v_6, v_4v_9 together with C divide the plane into *eight* cells, one of which is unbounded, and u must belong to this cell Γ.

Let v_i be the vertex of C with the smallest x-coordinate. Since v_3v_7, v_1v_6, v_4v_9 are represented by monotone curves, v_i has to lie on the boundary of the unbounded cell Γ. We can assume without loss of generality that $1 \leq i \leq 3$. (If this is not the case, we can add 3 or 6 to all indices modulo 9.) So, v_i is on the boundary of the unbounded cell, and u is in the unbounded cell. Using the fact that the edges v_1v_2 and v_2v_3 do not cross any other edge, we can conclude that v_1, v_2, and v_3 all lie on the boundary of the unbounded cell Γ. See Figure 6. Since we started with a MON-CR-optimal drawing, the edge uv_2 does not cross v_4v_9. The path v_2uv_5 crosses v_4v_9, so that uv_5 must cross v_4v_9. Analogously, v_2uv_8 crosses v_4v_9, so that uv_8 crosses v_4v_9. Moreover, the path v_2uv_5 crosses v_3v_7, and v_2uv_8 crosses v_1v_6. Recall from the previous paragraph that the edges v_3v_7, v_1v_6, and v_4v_9 are pairwise crossing. Summarizing, there are at least $4k+3$ crossings between edges of H_k. By Claim 3.3, $\text{MON-CR}(H'_k) \geq 3k+3$, so that altogether $\text{MON-CR}(G_k) \geq (4k+3)+(3k+3) \geq 7k+6$, as required. $\square$

4 Concluding remarks

1. Another important parameter of a graph, the *odd-crossing number*, was introduced implicitly by Tutte [Tu70]. It is defined as the minimum number $\text{ODD-CR}(G)$ of all pairs of edges that cross an odd number of times, over all drawings of G. Clearly, for any graph G, we have $\text{ODD-CR}(G) \leq \text{PAIR-CR}(G) \leq \text{CR}(G) \leq \text{MON-CR}(G) \leq \text{LIN-CR}(G)$. Theorem 1 can be strengthened as follows.

Corollary 4.1. *Every graph G satisfies the inequality*

$$\text{MON-CR}(G) < 2\text{ODD-CR}^2(G).$$

Proof. Let $\mathcal{D}$ be a drawing of G, in which the number of pairs of edges that cross an odd number of times is $\text{ODD-CR}(G)$. Let $G' \subseteq G$ denote the subgraph consisting of all vertices of G and all edges that do not cross any other edge an odd number of times. It was shown in [PaT00a] that G has another drawing, $\mathcal{D}'$, in which the edges belonging to G' do not participate in any crossing, and hence they form a plane graph. Every edge in $E(G) \setminus E(G')$ is represented by a curve that lies entirely in a cell of this plane graph. According to our Theorem 2.2, this plane graph admits a v-spinal (monotone) drawing for some $v \in V(G)$. By definition, we can add to this drawing all edges in $E(G) \setminus E(G')$, so that all of them are represented by monotone curves, and they do not cross any edge of G'. Among all such monotone drawings of G, consider one that minimizes the total number of crossings. In this drawing, any two edges cross at most once. Thus, we have

$$\text{MON-CR}(G) \leq \binom{|E(G)| - |E(G')|}{2}.$$

On the other hand, taking into account that every edge in $E(G) \setminus E(G')$ participates in at least one pair of edges in $\mathcal{D}$ which cross an odd number of times, we obtain that

$$|E(G)| - |E(G')| \leq 2\text{ODD-CR}(G).$$

Comparing the last two inequalities, the corollary follows. $\square$

In [PaT00b], we introduced the following variant of the odd-crossing number. Two edges of a graph G are called *independent* if they do not share a vertex. Let $\text{ODD-CR}_{-}(G)$ denote the smallest number of pairs of independent edges that cross an odd number of times, over all drawings of G. That is, we do not count those pairs of edges that are incident to the same vertex, even if they cross an odd number of times. Pelsmajer, Schaefer, and Štefankovič [PeSS10] managed to strengthen the result of [PaT00a], used in the proof of Corollary 4.1. They established the following result. Consider a drawing of G in the plane. An edge $e \in E(G)$ is called *independently even* if it crosses every other edge of G which is independent of e an even number of times. Then G has another drawing in which no independently even edge crosses any edge. Plugging this result into the above proof, we obtain the following strengthening of Corollary 4.1.

Corollary 4.1'. *Every graph G satisfies the inequality*

$$\text{MON-CR}(G) \leq 2\text{ODD-CR}_{-}^{2}(G).$$

2. As mentioned in the Introduction, Tóth [To11] proved that every graph G satisfies the inequality

$$\text{CR}(G) = O(\text{PAIR-CR}^{7/4}(G)/\log^{3/2} \text{PAIR-CR}(G)).$$

Restricting the notion of pair-crossing number to monotone drawings, we obtain another closely related graph parameter. The *monotone pair-crossing number* of G, $\text{MON-PAIR-CR}(G)$, is defined as the smallest number of crossing pairs of edges over all *monotone* drawings of G. Obviously, we have that $\text{ODD-CR}(G) \leq \text{PAIR-CR}(G) \leq \text{MON-PAIR-CR}(G)$, for any graph G. Valtr [Va05] proved that every graph G satisfies the inequality $\text{MON-CR}(G) = O(\text{MON-PAIR-CR}^{4/3}(G))$.

References

[BiD93] D. Bienstock and N. Dean, Bounds for rectilinear crossing numbers, *J. Graph Theory* **17** (1993), 333–348.

[Di05] R. Diestel, *Graph Theory (3rd ed.)*, Graduate Texts in Mathematics **173**, Springer-Verlag, Berlin, 2005.

[ErG73] P. Erdős and R. K. Guy, Crossing number problems, *Amer. Math. Monthly* **80** (1973), 52-58.

[Fa48] I. Fáry, On straight line representation of planar graphs, *Acta Univ. Szeged. Sect. Sci. Math.* **11** (1948), 229–233.

[FuPS11] R. Fulek, M. J. Pelsmajer, M. Schaefer, and D. Štefankovič, Hanani-Tutte, monotone drawings, and level-planarity, to appear.

[GaJ83] M. R. Garey and D. S. Johnson, Crossing number is NP-complete, *SIAM J. Alg. Disc. Meth.* **4** (1983), 312–316.

[GaJS76] M. R. Garey, D. S. Johnson, and L. J. Stockmeyer, Some simplified NP-complete graph problems, *Theoretical Computer Science* **1** (1976), 237–267.

[Gu69] R. K. Guy, The decline and fall of Zarankiewicz's theorem, in: *Proof Techniques in Graph Theory*, Academic Press, New York, 1969, 63–69.

[PaS11] J. Pach and E. Sterling, Conways conjecture for monotone thrackles, *Amer. Math. Monthly* **118** (2011), 544–548.

[PaT00a] J. Pach and G. Tóth, Which crossing number is it, anyway? *J. Combin. Theory Ser. B* **80** (2000), 225-246.

[PaT00b] J. Pach and G. Tóth, Thirteen problems on crossing numbers, *Geombinatorics* **9** (2000), 194-207.

[PaT04] J. Pach and G. Tóth, Monotone drawings of planar graphs, *J. Graph Theory* **46** (2004), 39-47. Also in: arXiv:1101.0967v1, (2010).

[PeSS08] M. J. Pelsmajer, M. Schaefer, and D. Štefankovič, Odd crossing number and crossing number are not the same, *Discrete Comput. Geom.* **39** (2008), 442–454.

[PeSS10] M. J. Pelsmajer, M. Schaefer, and D. Štefankovič, Removin independently even crossings, *SIAM J. Discrete Math.* **24** (2010), 379–393.

[To08] G. Tóth, Note on the pair-crossing number and the odd-crossing number, *Discrete Comput. Geom.* **39** (2008), 791–799.

[To11] G. Tóth, A better bound for the pair-crossing number, manuscript.

[T77] P. Turán, A note of welcome, *J. Graph Theory* **1** (1977), 7–9.

[Tu70] W. T. Tutte, Toward a theory of crossing numbers, *J. Combinatorial Theory* **8** (1970), 45–53.

[Va05] P. Valtr, On the pair-crossing number, In: Combinatorial and computational geometry, Math. Sci. Res. Inst. Publ., **52**, 569-575, Cambridge Univ. Press, Cambridge, 2005.

[Wo69] D. R. Woodall, Thrackles and deadlock, in: Welsh, D. J. A., *Combinatorial Mathematics and Its Applications* (D. J. A. Welsh, ed.), Academic Press, 1969, 335-348.