Opposite-quadrant depth in the plane

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Abstract

Given a set S of n points in the plane, the *opposite-quadrant depth* of a point $p \in S$ is defined as the largest number k such that there are two opposite axis-aligned closed quadrants (NW and SE, or SW and NE) with apex p, each containing at least k elements of S. We prove that S has a point with opposite-quadrant depth at least $n/8$. If the elements of S are in convex position, then we can guarantee the existence of an element whose opposite-quadrant depth is at least $n/4$. Both results are asymptotically best possible.

1 Introduction

Let S be a set of n points in the plane. Given any point p, the horizontal and vertical lines through p define four *closed* quadrants $\mathrm{NW}(p)$, $\mathrm{NE}(p)$, $\mathrm{SE}(p)$, and $\mathrm{SW}(p)$. We define the *opposite-quadrant depth* $\mathrm{opp}(p, S)$ of a point p with respect to S as the largest number k such that there are two opposite quadrants $\mathrm{NW}(p)$ and $\mathrm{SE}(p)$ or $\mathrm{NE}(p)$ and $\mathrm{SW}(p)$, each containing at least k elements of S. In notation:

$$\begin{aligned} \mathrm{opp}(p, S) &= \max\left(\min\left(|\mathrm{NW}(p) \cap S|, |\mathrm{SE}(p) \cap S|\right),\right. \\ &\qquad \left.\min\left(|\mathrm{NE}(p) \cap S|, |\mathrm{SW}(p) \cap S|\right)\right), \\ \mathrm{opp}(S) &= \max_{p \in \mathbb{R}^2} \mathrm{opp}(p, S). \end{aligned}$$

A point p is said to be α-*deep*, for some $\alpha > 0$, if its opposite-quadrant depth is at least αn. A (not necessarily unique) point of maximum opposite-quadrant depth is called a *deepest* point, and a point which is α-deep is also called an α-*centerpoint*. If we do not require the centerpoint to be in S, then there always exists a point that is $\frac{1}{4}$-deep at the intersection of the vertical and horizontal

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halving lines. In fact, $\frac{1}{4}$ is the best general bound that can be established on the maximum depth for any point in the plane, as shown by a set of n points uniformly distributed in a square, a disk, or even on a circle. There are also sets with deeper points, e.g., any set of n collinear points has a point of depth $\lceil n/2 \rceil$.

In what follows, we require the centerpoint to belong to the set S. In this case, we cannot guarantee that there always exists a $\frac{1}{4}$-deep point in S. However, we can show that S always contains a $\frac{1}{8}$-centerpoint.

Theorem 1 *Any set of n points in the plane has an element with opposite-quadrant depth at least $n/8$. Moreover, such a point can be found in $O(n)$ time.*

The proof is inspired by the "catline" of Hubert and Rousseuw [6].

Somewhat surprisingly, Theorem 1 is essentially tight.

Theorem 2 *For infinitely many values of n, there are n-element sets S_n with $\max_{p \in S_n} \mathrm{opp}(p, S_n) = (n+4)/8$.*

If the points of S are in *convex position*, that is, if they form the vertex set of a convex polygon, we can also guarantee the existence of a $\frac{1}{4}$-centerpoint in S. In fact, this statement remains true under the weaker condition that S is an *exposed* set, that is, for any $p \in S$, at least one of the four quadrants $\mathrm{NW}(p)$, $\mathrm{NE}(p)$, $\mathrm{SE}(p)$, and $\mathrm{SW}(p)$ contains no element of S in its interior. With a little twist of terminology, in this case we say that such a quadrant is *empty* (although it necessarily contains the point p and may perhaps contain some other boundary points that belong to S).

Theorem 3 *Any exposed set of n points in the plane has an element with opposite-quadrant depth at least $n/4$. Moreover, such a point can be found in $O(n)$ time.*

The above notions are closely related to *half-space depth*, *multivariate regression depth*, and other measures of statistical depth [7]. For instance, the half-space depth of a point p with respect to an n-element set S in d-space is the minimum number of points of S covered by a closed half-space containing p. It is well-known that one can always find a point of half-space depth at least $\left\lfloor \frac{n}{d+1} \right\rfloor$ with respect to S [2]. In sharp contrast to the opposite-quadrant depth, however, now we cannot guarantee that S has an element of large half-space depth. Indeed, if the points of S are in strictly convex position, none of them has half-space depth larger than one.

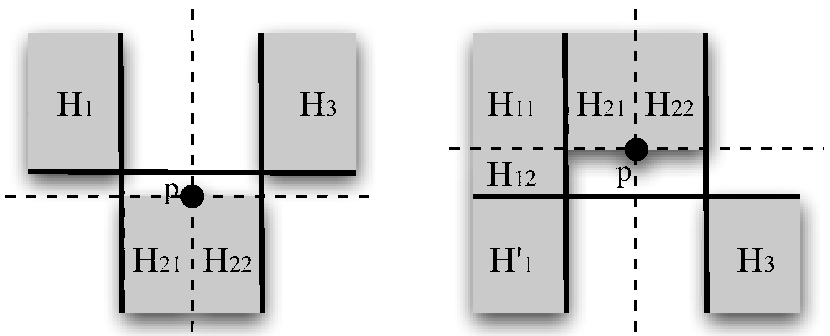
2 Proofs of Theorems 1 and 3

First, notice that having points on the same vertical or horizontal line does not create a problem, since the quadrants were defined to be *closed*. In fact, we can slightly perturb our original point set S by assigning to each element $p_i \in S$ a vector $\mathbf{v}_i = (i, i^2)$. Then there exists a sufficiently small $\varepsilon > 0$ such that the modified sets $S_\lambda = \{p_i + \lambda \mathbf{v}_i\}$ are in *general position* for any $\lambda \in (0, \varepsilon]$, in the sense that no pair of points have the same x or y coordinate. Consequently, varying λ in this small interval will not change the horizontal or vertical order of the elements of S_λ. If our theorems hold for sets in general position, we can apply them to S_ε, and find an α-centerpoint $p_i + \varepsilon \mathbf{v}_i$. Then $p_i + \lambda \mathbf{v}_i$ is α-deep with respect to S_λ for all $\lambda \in (0, \varepsilon]$. Bringing back our points to their original positions in S only adds points on the boundary of the two opposite quadrants of p_i. Therefore, p_i is also an α-centerpoint for S.

Thus, from now on we assume that S is in general position. To simplify the presentation, we also assume that $n = |S|$ is divisible by 8, so that we do not have to deal with upper and lower integer parts. The proof of Theorem 1 proceeds by first splitting the set into three parts by two vertical lines, not passing through any element of S, such that in the left and right half-planes there are precisely $n/4$ points and the vertical strip in the middle contains $n/2$ points. Next, we find a horizontal line, not passing through any point of S, that divides S into two equal halves. In this way, we obtain a 3×2 "grid," whose six regions will be called "cells." We say that a cell is *heavy* if it contains at least as many points as the other cell in the same column, that is, the cell below or above it. Note that if every cell in a row is heavy, then each of them contains precisely as many elements as the cell below or above it. Otherwise, there could not be $n/2$ points in the other row. Thus, in this case, all six cells must be heavy. Therefore, we can always pick one heavy cell per column so that not all of them belong to the same row. We distinguish two cases up to symmetries (reflections about the coordinate axes).

(a) Bitonic configuration: The upper cells H_1 and H_3 of the first and third columns, respectively, and the lower cell H_2 of the second column are all heavy. (See Figure 1(a).)

(b) Monotone configuration: The upper cells H_1 and H_2 of the first two columns and the lower cell H_3 of the third column are all heavy. (See Figure 1(b).)



(a) Bitonic configuration (b) Monotone configuration

Figure 1: Proof of Theorem 1. We let H_1' be the lower cell in the left column, $H_{11} := H_1 \cap \mathrm{NW}(p)$, $H_{12} := H_1 \cap \mathrm{SW}(p)$, $H_{21} := H_2 \cap \mathrm{NW}(p)$, and $H_{22} := H_2 \cap \mathrm{NE}(p)$. Case (a): H_1, H_3, and one of H_{21} or H_{22} have $\geq n/8$ points. Case (b): H_3 and one of $\mathrm{NW}(p)$ or $\mathrm{SW}(p)$ have $\geq n/8$ points. If $\mathrm{NW}(p)$ does not, both $\mathrm{SW}(p)$ and H_{22} have $\geq n/8$ points.

First we prove Theorem 1 for bitonic configurations. Let p be the highest point of the lower (heavy) cell H_2 of the middle column. The quadrants $\mathrm{NW}(p)$ and $\mathrm{NE}(p)$ contain the heavy cells H_1 and H_3, therefore both of them have at least $n/8$ points of S. All the at least $n/4$ points in the heavy cell H_2 that belong to S lie in the half-plane $\mathrm{SW}(p) \cup \mathrm{SE}(p)$. Thus, at least one of the two quadrants, $\mathrm{SW}(p)$ or $\mathrm{SE}(p)$, must contain at least $n/8$ points. This quadrant together with the opposite one forms a good pair, concluding the proof for bitonic configurations.

Next we establish Theorem 1 for monotone configurations. We have to use a slightly different argument, because we have no information about the relative vertical order of the points of H_1 and H_2. Choose p to be the lowest point in S that belongs to H_2, the upper cell in the middle column. Since $\mathrm{SE}(p)$ contains the heavy cell H_3, it has at least $n/8$ points of S. If $\mathrm{NW}(p)$, the