A Left-First Search Algorithm for Planar Graphs

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Abstract. We give an $O(|V(G)|)$ time algorithm to assign vertical and horizontal segments to the vertices of any bipartite plane graph G so that (i) no two segments have an interior point in common, (ii) two segments touch each other if and only if the corresponding vertices are adjacent. As a corollary, we obtain a strengthening of the following theorem of Ringel and Petrovič. The edges of any maximal bipartite plane graph G with outer face $bwb'w'$ can be colored by two colors such that the color classes form spanning trees of $G-b$ and $G-b'$, respectively. Furthermore, such a coloring can be found in linear time. Our method is based on a new linear time algorithm for constructing bipolar orientations of 2-connected plane graphs.

1. Introduction.

Throughout this paper we consider only finite graphs G without loops, but we allow multiple edges. If G has no multiple edges then it is called a *simple* graph. A graph is 2-*connected* if it cannot be disconnected by the removal of a vertex.

Let $\vec{G}$ be a directed graph obtained by orienting the edges of G. A vertex of $\vec{G}$ is said to be a *source* (*sink*) if its in-degree (out-degree) is 0. $\vec{G}$ is *acyclic* if it contains no oriented cycle. For any partition of the vertex set $V(\vec{G}) = V_1 \cup V_2$, the family of edges between V_1 and V_2 is said to form a *cocycle* (or an *oriented cut*) if all of them are oriented toward V_2.

The following concept was introduced by Lempel, Even and Cederbaum [LEC] to design an efficient planarity testing algorithm. It plays a crucial role in many problems about graph drawings, motion planning, visibility and incidence relations between geometric objects, etc. ([FOR], [FPP], [FRU], [OW], [P], [Ri], [Ro], [RT], [T], [Ta], [TT1], [TT2]).

Definition 1.1. Given an edge $\vec{e} = \vec{st}$ of $\vec{G}$, we say that the orientation of $\vec{G}$ is *$\vec{e}$-bipolar* (or defines an *st-ordering*) if

(a) $\vec{G}$ is acyclic,

(b) s and t are the unique source and sink of $\vec{G}$, respectively.

We shall also use another equivalent form of this definition (which can easily be extended to matroids).

Lemma 1.2. *Given an edge $\vec{e}$ of $\vec{G}$, the orientation of $\vec{G}$ is $\vec{e}$-bipolar if and only if*

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(a)′ every edge of $\vec{G}$ belongs to a cocycle,
(b)′ every cocycle of $\vec{G}$ contains $\vec{e}$.

Proof. Obviously, (a)′ implies (a). Conversely, if $\vec{G}$ is acyclic and $\vec{e'} = \vec{s't'}$ is any edge, then let V_2 be the set of all vertices that can be reached from t' by a directed path, and let $V_1 = V(\vec{G}) - V_2$. Then all edges between V_1 and V_2, including $\vec{e'}$, are oriented toward V_2, thus (a)′ holds.

To show that (b) implies (b)′ for any acyclic digraph $\vec{G}$, it is enough to observe that, if a partition $V(\vec{G}) = V_1 \cup V_2$ defines a cocycle, then V_1 and V_2 must contain a source and a sink, respectively. Thus, $s \in V_1$, $t \in V_2$ and $\vec{e} = \vec{st}$ belongs to this cocycle. Conversely, if an acyclic digraph satisfies (b)′ with $\vec{e} = \vec{st}$ then, for any source x (and sink y), the collection of edges incident to x (y) forms a cocycle. Consequently, $\vec{e}$ is incident to both x and y. Hence, $x = s$, $y = t$ and (b) holds. $\square$

Corollary 1.3. *If $\vec{G}$ has an $\vec{e}$-bipolar orientation, then it has no two cocycles such that one contains the other.*

Proof. Let E and E' be two cocycles of $\vec{G}$ defined by the partitions $V_1 \cup V_2$ and $V_1' \cup V_2'$, respectively, where $s \in V_1 \cap V_1'$ and $t \in V_2 \cap V_2'$. Suppose without loss of generality that $W_1 = V_1' \cap V_2 \neq \emptyset$. If $E \subseteq E'$, then W_1 and $W_2 = V(G) - W_1$ define a cocycle which does not contain $\vec{e} = \vec{st}$, contradicting condition (b)′ in Lemma 1.2. $\square$

If $\vec{G}$ has an $\vec{e}$-bipolar orientation, then its underlying graph G (obtained by disregarding the orientation of the edges) is obviously 2-connected. Indeed, if G fell into two components G_1 and G_2 by the removal of a vertex x, then, by the acyclicity of $\vec{G}$, both parts of $\vec{G}$ induced by $V(G_1) \cup \{x\}$ and $V(G_2) \cup \{x\}$ would contain a source and a sink, contradicting condition (b) of Definition 1.1.

On the other hand, it is easy to see that, given any 2-connected graph G and any edge $e = st$, there exists an $\vec{e}$-bipolar orientation of the edges of G with $\vec{e} = \vec{st}$. Moreover, Even and Tarjan [ET] devised a linear time algorithm to find such an orientation, which has been subsequently simplified by Ebert [E] and Tarjan [Ta].

In section 2 of this paper we propose an equally fast but much simpler greedy algorithm based on Whitney's theorem [W] to find bipolar orientations of 2-connected plane graphs.

A *plane graph* is a planar graph embedded in the plane (or in the sphere) so that its edges are represented by simple non-crossing Jordan arcs. If a plane graph G with at least three vertices is 2-connected, then its *dual graph* G^* can be defined as follows. Put a vertex of G^* in each face of G and, if two faces meet along an edge f, then connect the corresponding two vertices by an arc f^* crossing f. (It is well known and easy to see that this construction can be carried out so that we obtain a plane graph G^*. By the

2-connectedness of G, G^* has no loops, but it may have multiple edges.) Any orientation of G induces a *dual orientation* of G^* in a natural way: we obtain the orientation of f^* from that of f by a clockwise turn.

Theorem 1.4. *Let $\vec{G}$ be an $\vec{e}$-bipolar orientation of a 2-connected plane graph with at least three vertices, and let $\vec{G}^*$ denote its dual graph with the dual orientation. Then the directed graph $\vec{G}^*_-$ obtained from $\vec{G}^*$ by reversing the orientation of $\vec{e}^*$ is $\vec{e}^*_-$-bipolar oriented, where $\vec{e}^*_-$ and $\vec{e}^*$ are opposite orientations of the same edge.*

Proof. We will show that $\vec{G}^*_-$ satisfies conditions (a)$'$ and (b) of 1.2 and 1.1, respectively.

Let $\vec{G}_-$ denote the digraph obtained from $\vec{G}$ by changing the orientation of $\vec{e} = \vec{st}$ to $\vec{e}_- = \vec{ts}$. Any edge of $\vec{G}$ can be extended to a directed path in $\vec{G}$ connecting s to t. Thus, any edge $\vec{f} \in E(\vec{G}_-)$ belongs to a (simple) cycle of $\vec{G}_-$ passing through $\vec{e}_-$. The edges of $\vec{G}^*_-$ crossing this cycle form a cocycle containing $\vec{f}^*$ (and $\vec{e}^*_-$), which proves (a)$'$.

Suppose, for contradiction, that $\vec{G}^*_-$ does not satisfy (b). Let s^* and t^* denote the endpoints of $\vec{e}^*_-$ ($\vec{e}^*_- = \vec{s^*t^*}$), and assume without loss of generality that $\vec{G}^*_-$ has a source x different from s^*. Clearly, $x \neq t^*$. Those edges of $\vec{G}_-$ which cross an edge incident to x form a cycle. Since this cycle does not use the arc e, this would also be a cycle in $\vec{G}$, a contradiction. $\square$

As any graph which has a bipolar orientation is 2-connected and vice versa, Theorem 1.4 immediately implies that the dual of a 2-connected plane graph is also 2-connected.

In section 3 we shall apply the above concepts and results to obtain the following theorem.

Theorem 1.5. *There exists a linear time algorithm which assigns vertical and horizontal segments to the vertices of any bipartite plane graph G so that*

(i) no two segments have an interior point in common,

(ii) two segments touch each other if and only if the corresponding vertices are adjacent in G.

Note that, if the black and white vertices of a bipartite (2-colored) graph G can be represented by vertical and horizontal segments, respectively, satisfying conditions (i) and (ii), then G is necessarily planar.

We say that a graph G has a *segment representation*, if its vertices can be represented by segments in the plane so that two segments cross each other if and only if the corresponding vertices are adjacent. It was shown in [HNZ] that any bipartite planar graph can be represented in such a way. Note that this fact is an immediate corollary to Theorem 1.5. However, it is not known whether every planar graph admits a segment representation.

Definition 1.6. A bipartite plane graph is called a *quadrilateralization*, if it contains no multiple edges and each of its faces has four edges.

It is easy to see that every quadrilateralization is 2-connected.

Given a bipartite plane graph, in linear time we can remove all multiple edges (by lexicographically bucket-sorting all edges with respect to their endpoints). Then we can use any naive linear time algorithm to extend the remaining graph to a quadrilateralization, by adding edges and vertices. Thus, it is sufficient to prove Theorem 1.5 for quadrilateralizations.

Definition 1.7. [R] Let H be a connected plane graph. Triangulate every face f of H from one of its interior points x_f (by connecting x_f to the vertices of f), and delete all edges belonging to H. The resulting graph $A(H)$ is called the *angle graph* of H.

Remark 1.8. Let H be a connected plane graph. Then $A(H)$ is a quadrilateralization if and only if H is 2-connected. $\square$

On the other hand, every quadrilateralization can be obtained as the angle graph of some 2-connected plane graph.

Lemma 1.9. *Let G be a quadrilateralization, whose vertices are colored with black and white. For every face f of G, connect its two black (white) vertices by an edge within f. The graph G_b (G_w) formed by these edges is called the graph of black (white) diagonals of G. Then,*

$$A(G_b) = A(G_w) = G \ . \qquad \square$$

Corollary 1.10. *G_b and G_w are 2-connected plane graphs, dual to each other.*

Proof. Immediately follows from Remark 1.8. $\square$

For some related results, see [DLR] and [TT3].

In section 4 of this paper we shall apply our technique to give a simple alternative proof of a theorem of Ringel [R] and Petrovič [Pe] on quadrilateralizations (cf. Theorem 4.1).

2. Greedy algorithm for bipolar orientation

2.1. The general scheme Let G be any 2-connected graph with n vertices and m edges. For any edge st of G, G has a *Whitney decomposition* into handles, i.e., there is a nested sequence of subgraphs $G_0 = \{st\} \subset G_1 \subset G_2 \subset \cdots \subset G_k = G$ such that G_{i+1} can be obtained from G_i by the addition of a simple path P_{i+1} which has only its endpoints in common with G_i.

First we present a simple general algorithm which maintains a total ordering of the vertices of G_i such that every P_j ($j \leq i$) forms a monotone chain. Directing every edge of G toward its larger endpoint in the final ordering, we obtain an st-ordering (bipolar orientation) of G. In fact, our algorithm will also maintain the orientation of the edges of G_i compatible with the ordering of its vertices.

Suppose that we have already found a sequence of subgraphs G_j ($j \leq i$) with the above properties, and that the vertices of G_i are totally ordered by a linked list called "LINK". Assume further that all edges of G_i are oriented toward their higher endpoints in this order, and every P_j ($j \leq i$) forms an oriented path. A vertex $x \in V(G_i)$ is said to be *saturated*, if all edges of G incident to x belong to G_i. Step i ($i \geq 0$) of our algorithm consists of three parts.

(1) Find the first unsaturated vertex $x \in V(G_i)$ on the list LINK.

(2) Find a simple path P_{i+1} in $G - G_i$ connecting x to some other vertex $y \in V(G_i)$ such that no internal point of P_{i+1} belongs to $V(G_i)$. Orient the edges of P_{i+1} from x toward y.

(3) Insert the internal vertices of P_{i+1} in the list LINK between x and LINK[x], i.e., immediately after x.

If we cannot execute (1), i.e., all vertices of G_i are saturated, then $G_i = G$ and our algorithm ends. Otherwise, let x' be any neighbor of x such that the edge xx' does not belong to G_i. If $x' \in V(G_i)$, then x' is also unsaturated, so x precedes x' on the list LINK. In this case, P_{i+1} consists of the single edge xx' oriented from x to x', and (3) is void. If $x' \notin V(G_i)$, then it follows from the 2-connectedness of G that it can be connected to some $y \in V(G_i)$, $y \neq x$ by a simple path in $G - G_i$. Obviously, x precedes y on the list LINK, so in this case we can execute (2) and (3) without adding any edge oriented backwards with respect to the revised LINK list, and we can pass to the next step.

Note that in the last part of Step i, when we revise the LINK list, we do not add any elements below x. Since all elements preceding x have already been saturated in G_i, when we come to part (1) of Step $i+1$, we do not have to check any member of LINK before x.

To execute part (2) of Step i, we can use depth-first search on $G - G_i$, starting at x and stopping when we hit the first vertex $y \in V(G_i)$. We put every edge e of the tree visited during the search in a (last-in, first-out) stack, and remove e if we have to "backtrack" along it. At the end of the search, the edges remaining in our stack will form a simple path $P_{i+1} \subseteq G - G_i$ meeting the requirements. However, unless we are lucky, this procedure may take $\Omega(|E(G - G_i)|)$ time, and summing over all $0 \leq i \leq k$, the total running time of our algorithm can be as large as $\Omega(km)$.

2.2. The planar case. If G is a plane graph, then the above algorithm can be implemented in $O(m)$ time. Suppose that, for every vertex v, we are given the clockwise circular order of the edges incident to v. We follow the same scheme as in the general case. To execute part (2) of Step i, we use the following method that can be called "left-first search".

Whenever we orient a new edge f toward one of its endpoints z, then we set IN$[z]$=f. Furthermore, let IN$[s]$=st.

Assume that we have already finished part (1) of Step i, i.e., we have found the first unsaturated vertex $x \in V(G_i)$ on the list LINK. Starting from $x_0 = x$, we shall construct the path $P_{i+1} = x_0x_1x_2\ldots$, as follows. For every $j \geq 0$, let x_jx_{j+1} be the first unoriented edge incident to x_j, which follows IN$[x_j]$ in the clockwise order. Orient x_jx_{j+1} from x_j toward x_{j+1}. If $x_{j+1} \in V(G_i)$, then it is the last point of P_{i+1}, and part (2) of Step i has been completed.

To prove that this construction is correct, it is enough to check that $x_{j+1} \neq x_h$ for any $h \leq j$. But this is true, otherwise $x_hx_{h+1}\cdots x_{j+1}$ would bound a face of G, and $x_jx_{j+1} = x_hx_j$ would precede x_hx_{h+1} in the cyclic order of edges around x_h, contradicting the choice of x_{h+1}.

It remains to show that our algorithm can be implemented in linear time. To this end, whenever we orient an edge $\vec{zz'}$, then we introduce a pointer NEXT$[z]$ pointing to the edge that follows immediately after zz' in the clockwise order of edges incident to z.

Let $x = x_0 \in V(G_i)$ be the first unsaturated vertex on the list LINK at the beginning of Step i of the algorithm. According to the above rule, next we have to find the first unoriented edge x_0x_1 which comes after IN$[x_0]$ in the clockwise order of edges incident to x_0. However, this can be accomplished in constant time, because x_0x_1 =NEXT$[x_0]$. To prove this, it is enough to notice that the edges oriented toward x_0 at the beginning of Step i form a single block in the clockwise order of edges incident to x_0, whose last element is the edge along which x_0 has been visited for the first time.

Note that the same algorithm can be used to find a Whitney decomposition (and a bipolar orientation) of any cellular graph.

Definition 2.3. A graph G that can be embedded in an oriented 2-manifold Σ is called *cellular*, if it divides Σ into connected components so that each of them is topologically equivalent to a disk.

3. Bipartite plane graphs.

In this section we prove Theorem 1.5. As we have pointed out in the Introduction, we can assume that G is a quadrilateralization (cf. Definition 1.6), whose vertices are colored with black and white. Let G_b and G_w be the graph of black diagonals and the graph of white diagonals of G, respectively (cf. Lemma 1.9). Furthermore, let s_b, s_w, t_b, t_w denote the vertices of the outer face of G, listed in clockwise order ($s_b, t_b \in V(G_b)$; $s_w, t_w \in V(G_w)$).

By Corollary 1.10, G_b is 2-connected, so we can use the algorithm described in 2.2 to find an s_bt_b-ordering (bipolar orientation) of G_b. That is, in linear time we can number the black vertices $b_1 = s_b, b_2, b_3, \dots, b_p = t_b$ so that, orienting every edge of G_b toward its endpoint of larger index, we obtain an s_bt_b-bipolar orientation $\vec{G}_b$.

$\vec{G}_b$ induces a dual orientation $\vec{G}_w$ on G_w. By Theorem 1.4, reversing the orientation of the edge $\vec{t_w s_w} \in \vec{G}_w$, we obtain an s_wt_w-bipolar orientation $\vec{G}_w^-$. Using topological sorting, one can easily find a numbering of the white vertices $w_1 = s_w, w_2, w_3, \dots, w_q = t_w$ such that every edge of $\vec{G}_w^-$ is oriented toward its endpoint of larger index ($p + q = |V(G)|$).

For any black point b_i ($1 \le i \le p$), let V_i be a vertical segment in the plane, whose endpoints are $(i, \min_{b_iw_j \in G} j)$ and $(i, \max_{b_iw_j \in G} j)$. Similarly, to any white vertex w_j ($1 \le j \le q$), we assign a horizontal segment H_j, whose endpoints are $(\min_{b_iw_j \in G} i,\, j)$ and $(\max_{b_iw_j \in G} i,\, j)$. We claim that this collection of segments meets the requirements of Theorem 1.5.

It is clear by the definition that all segments are contained in the rectangle enclosed by V_1, H_1, V_p, H_q, and that each of these four segments is in contact with exactly those segments which correspond to its neighbors.

Let us fix now a black point b_k, $1 < k < p$, and let $B_1 = \{b_i \mid i < k\}$, $B_2 = \{b_i \mid i > k\}$. Clearly, the edges connecting B_1 to $B_2 \cup \{b_k\}$ form a cocycle E_1 in $\vec{G}_b$, and the edges connecting $B_1 \cup \{b_k\}$ to B_2 form another cocycle E_2. Since all cocycles of $\vec{G}_b$ are minimal (by Corollary 1.3), the edges of $\vec{G}_w$ intersecting some element of E_1 (E_2) form a (minimal) oriented cycle C_1 (C_2) passing through $\vec{t_w s_w}$. Deleting the edge $\vec{t_w s_w}$ from C_1 and C_2, we obtain two simple oriented paths P_1 and P_2, respectively, connecting s_w to t_w in $\vec{G}_w^-$. It is easy to see that b_k is the only black vertex enclosed by P_1 and P_2. Indeed, if there were another vertex b_i ($i < k$, say) with this property, then all vertices along an oriented path connecting $b_1 = s_b$ to b_i in $\vec{G}_b$ would belong to B_1, hence this path could intersect neither P_1 nor P_2, contradiction. On the other hand, since P_1 and P_2 are not identical, they must enclose at least one black vertex.

Thus, starting from s_w, P_1 and P_2 are identical up to a point s'_w. Then they split up, and meet again at some point t'_w, from which they run together to their common endpoint t_w. Let P'_1 and P'_2 denote the parts of P_1 and P_2, respectively, connecting s'_w to t'_w. Since all edges of $\vec{G}_b$ intersecting some edge of P'_1 (P'_2) must end (start) at b_k, we obtain that all vertices of $P'_1 \cup P'_2$ are adjacent to b_k in G. Moreover, b_k does not have any other neighbor not belonging to $P'_1 \cup P'_2$.

Let W_1 (W_2) denote the set of white points, all of whose black neighbors are in B_1 (B_2). If a white point w does not belong to $W_1 \cup W_2$, then it must be a vertex of P_1 or

P_2. Indeed, if $w \notin W_1 \cup W_2$ then it has two consecutive neighbors b and b' such that, say, $b \in B_1$ and $b' \notin B_1$. But then $\vec{bb'}$ belongs to the cocycle E_1 in $\vec{G}_b$, so the edge of $\vec{G}_w$ crossing $\vec{bb'}$ belongs to P_1, and one of its endpoints is w.

Let w_j $(1 < j < q)$ be a white vertex, and let H_j be the corresponding horizontal segment.

Case 1: $b_k w_j \notin G$.

Then $w_j \in W_1 \cup W_2$ or w_j is an internal vertex of $P_1 \cap P_2$.

If w_j belongs to (say) W_1, then $\max_{b_i w_j \in G} i < k$. So H_j is to the left of V_k, and $H_j \cap V_k = \emptyset$.

Suppose next that w_j belongs to (say) the portion of $P_1 \cap P_2$ lying strictly betweeen s_w and s'_w. Then j is smaller than the index of any white neighbor of b_k, because all of these neighbors belong to $P'_1 \cup P'_2$ and can be reached from w_j by an oriented path in $\vec{G}_w^-$ (along P_1 or P_2). Thus, H_j is below V_k, and $H_j \cap V_k = \emptyset$.

Case 2: $b_k w_j \in G$.

Then w_j belongs to $P'_1 \cup P'_2$.

If $w_j = s_w$ (or t_w), then w_j has the smallest (largest) index among all white neighbors of b_k, so the lower (upper) endpoint of V_k lies on H_j. Moreover, V_k has to touch H_j at one of its interior points, because w_j must be adjacent to at least one black vertex whose index is smaller than k and to another one whose index is larger than k.

Suppose next that w_j is an internal point of (say) P'_1. Then the right endpoint of H_j is an interior point of V_k.

This shows that the vertical and horizontal segments assigned to the vertices of G satisfy the conditions of Theorem 1.5. We have also proved that the only pairs of segments that share an endpoint are $\{V_1, H_1\}$, $\{H_1, V_p\}$, $\{V_p, H_q\}$ and $\{H_q, V_1\}$. Consequently, the segments V_i and H_j $(1 \le i \le p,\ 1 \le j \le q)$ determine a tiling of the rectangle bounded by V_1, H_1, V_p, H_q with smaller rectangles. In a forthcoming paper [FOP], we prove Theorem 1.5 by induction.

4. Partition into trees.

The aim of this section is to show that the following theorem of G. Ringel [R] and V. Petrovič [Pe] can be easily deduced from the above results.

Corollary 4.1. *Let G be a quadrilateralization, and let s_b, s_w, t_b, t_w denote the vertices of the outer face of G, listed in clockwise order. Then the edge set of G can be partitioned into two parts, forming a spanning tree of $G - s_b$ and $G - t_b$, respectively.*

Consider the $\vec{s_b t_b}$-bipolar orientation of G_b and the $\vec{s_w t_w}$-bipolar orientation of G_w

constructed in the previous section.

Lemma 4.2. *There exists an ordering $v_1 = s_b, v_2, v_3, \dots, v_n = t_b$ of the vertex set of G, which is compatible with the above $\vec{s_b t_b}$- and $\vec{s_w t_w}$-bipolar orientations and satisfies the condition that every v_i $(1 < i < n)$ is adjacent to at least one larger and one smaller element.*

Proof. As in the previous section, assign to every vertex $v \in V(G)$ a vertical or horizontal segment with endpoints $(x_1(v), y_1(v)$ and $x_2(v), y_2(v)$, where $x_1(v) \leq x_2(v), y_1(v) \leq y_2(v)$. For any pair of adjacent vertices $v, v' \in V(G)$ different from s_b and t_b, let $v \leq v'$ if and only if $x_2(v) \leq x_1(v')$ or $y_2(v) \leq y_1(v')$. Furthermore, let s_b be smaller and let t_b be larger than any other element of $V(G)$. It is not hard to check that this relation defines a partial order on $V(G)$ compatible with the partial orders on $V(G_b)$ and $V(G_w)$ induced by the corresponding bipolar orientations. Therefore, these three relations have a common extension into a total order of $V(G)$. (It can also be shown that this total order is uniquely determined.) $\square$

Now we can finish the proof of Corollary 4.1, as follows. For any black vertex $v_i \in V(G_b)$, let S_i denote the vertical segment whose endpoints are $(i, \min_{v_i v_j \in G} j)$ and $(i, \max_{v_i v_j \in G} j)$. For any white vertex $v_j \in V(G_w)$, let S_j denote the horizontal segment with endpoints $(\min_{v_i v_j \in G} i, j)$ and $(\max_{v_i v_j \in G} i, j)$. Clearly, two such segments touch each other if and only if the corresponding vertices are adjacent. Moreover, by Lemma 4.2, every segment $S_i (1 < i < n)$ will cross the line $y = x$.

Color every edge $v_i v_j \in G$ by red or green according to whether the point of incidence of S_i and S_j lies above or below the line $y = x$. Then the red and green edges form a spanning tree of $G - v_1$ and $G - v_n$, respectively. This completes the proof of Corollary 4.1.

Assume that there is an enumeration $v_1, v_2, \dots, v_n$ of the vertices of a graph G and a coloring of its edges with c colors such that each color class can be "drawn on a page", that is, there are no two edges of the same color, $v_h v_j$ and $v_i v_k$, with $h < i < j < k$. The smallest number c for which such a representation exists is called the *page number* of G. Notice that the total order of $V(G)$ described in Lemma 4.2 and the red–green coloring defined above also yield the following result.

Corollary 4.3.. *The page number of any quadrilateralization G is at most two. Moreover, the edges of G can always be drawn on two pages so that each page contains a tree.*

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